<u>December 24, 1968:</u> More Motives

[this is mostly incorrect ~~and~~ worthwhile to remember exact sequence ☆ on page 6]

Motives over a base manifold S:

Let $\mathcal{V}/S$ be the category whose objects are C^∞ maps $f: X \longrightarrow S$ and let $\mathcal{V}_S$ be the full subcategory of those objects for which the map f is smooth ~~submersion~~ ("smooth" as Grothendieck says in alg geom.) (a submersion as with Bourbaki.). Let $\mathcal{M}_S$ be the category whose objects are the same as those of $\mathcal{V}_S$ and with

(1) $\mathrm{Hom}_{\mathcal{M}_S}((X),(Y)) =$ bordism classes of ~~manifolds~~ maps $Z \longrightarrow X \times_S Y$ proper over X and oriented over Y,

where (X) denotes the object of $\mathcal{M}_S$ corresponding to the object X of $\mathcal{V}_S$. (Definition of composition: Given $\alpha: U \longrightarrow X \times_S Y$, $\beta: V \longrightarrow Y \times_S Z$ we can move α, β so that $\alpha \times \beta$ is transversal to γ whence

$$\begin{tikzcd}
U \times_Y V \arrow[r] \arrow[d] & X \times_S Y \times_S Z \arrow[r] \arrow[d, "\gamma"] & Y \arrow[d] \\
U \times V \arrow[r, "\alpha \times \beta"] & (X \times_S Y) \times (Y \times_S Z) \arrow[r, "\substack{\text{trans. as } X\to S,\ Z \to S \\ \text{smooth.}}", "\text{sm}"'] & Y \times Y
\end{tikzcd}$$

one sees that fibre product $U \times_Y V$ can be defined. Then $\beta \circ \alpha$ is defined to be $U \times_Y V \longrightarrow X \times_S Y \times_S Z \longrightarrow X \times_S Z$. This is usual composition

$$F(X \times_S Y) \times F(Y \times_S Z) \longrightarrow F((X \times_S Y) \times (Y \times_S Z)) \xrightarrow{j^*} F(X \times_S Y \times_S Y \times_S Z) \xrightarrow{(id \times \Delta \times id)^*} F(X \times_S Y \times_S Z) \xrightarrow{(pr_{13})_*} F(X \times_S Z)$$

so associative, etc.).

Remark:

1. We think of (X) as being ~~the cohomology of X with~~ the motive-theoretic $f_! \mathbb{Z}_X \in D(S)$ where $f: X \to S$ is the structural map. The formula (1) is motivated by the following duality calculation: Let $f: X \to S$ and $g: Y \to S$ be maps of nice locally compact spaces. ~~which~~ Then

$$\mathrm{Hom}^*_{D(S)}(f_! \mathbb{Z}_X, g_* \mathbb{Z}_Y) = \mathrm{Hom}^*_{D(Y)}(g^* f_! \mathbb{Z}_X, \mathbb{Z}_Y)$$

$$= \mathrm{Hom}^*_{D(Y)}(f'_! g'^* \mathbb{Z}_X, \mathbb{Z}_Y) \qquad \text{proper base change}$$

$$= \mathrm{Hom}^{*+d}_{D(X \times_S Y)}(\mathbb{Z}_{X \times_S Y}, f'^! \mathbb{Z}_Y) \qquad \text{duality for } f'$$

$$= H^{*+d}(X \times_S Y, f'^! \mathbb{Z}_Y)$$

where $f': X \times_S Y \to Y$ is the projection. Now if f and g are transversal, then $f'^!$ is the base change of $f^!$, so if f is oriented, then $f'^! \mathbb{Z}_Y = \mathbb{Z}_{X \times_S Y}$, so the last thing is just $H^{*+d}(X \times_S Y)$.

2. It seems perhaps reasonable to enlarge $\mathcal{M}_S$ so as to include objects corresponding to motive-theoretic $f_* \mathbb{Z}_X$ where $f: X \to S$ is smooth. Must then introduce objects corresponding to a family of supports UGH.

Variance of M_S with S:

Inverse image f^*: If $f: T \to S$ and X is smooth over S then set $(T \times_S X) = f^*(X)$. Given $\alpha: Z \to X \times_S Y$ representing an element of $B_{p/X, \sigma/Y}(X \times_S Y)$ move α slightly to be transversal to f

(Lemma: Given

$$\begin{tikzcd} & Z \arrow[d, "\alpha"] \\ & X \arrow[d, "f"] \\ W \arrow[r, "g"] & Y \end{tikzcd}$$

with f ~~smooth~~ trans. to g, one can always move α slightly so that g is transversal to $f\alpha$.

Proof: As f ~~smooth~~ (trans. to g) can form

$$\begin{tikzcd} & Z \arrow[d, "\alpha"] \\ W \times_Y X \arrow[r, "g'"] & X \end{tikzcd}$$

and ~~if~~ α transversal to g' ~~then~~ iff $f\alpha$ transversal to g. (Last assertion follows from fact that transversal cartesian squares ~~can be compound~~. behave like cartesian squares, e.g.

$$\boxed{A}\boxed{B}$$

A, B tr. cart. $\Rightarrow$ AB tr. cart

AB, B tr. cart. $\Rightarrow$ A tr. cart.

Note that ~~a transversal~~ cartesian square of vector spaces is same as a bicartesian squares).) [This lemma shows that on preceding page to compose we need only jiggle NO $\alpha: U \to X \times_S Y$ provided $X \times_S Y \to Y \leftarrow Y \times_S Z$ are transversal, which by proof of lemma is equivalent to $Z \to S$ being transversal to $X \times_S Y \to S$.

$$\begin{tikzcd} & X \times_S Y \arrow[d] \\ Y \times_S Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{cart}"] & Y \arrow[d] \\ Z \arrow[r] & S \end{tikzcd}$$

This last condition we encountered yesterday. In effect to say X, Y, Z are transversal over S is like saying X, Y, Z are independent. Not the same as X, Y, Z being pairwise transversal, but instead X, Y being transv., Z being trans. to $X \times_S Y$. NO Y might be $\emptyset$

The situation at the moment is the following: Can define morphisms from (X) to (Y) as $\mathcal{B}_{p/X,\, or/Y}(X \times_S Y)$ when $X \to S$ and $Y \to S$ are transversal and we can define composition $\mathrm{Hom}((X),(Y)) \times \mathrm{Hom}((Y),(Z)) \longrightarrow \mathrm{Hom}((X),(Z))$ whenever the family $\{X, Y, Z\}$ is transversal over S. The way out is the following

PROB. FALSE

<u>Theorem</u>: For each X/S the functor on $\mathcal{M}_S$

$$(Y) \longmapsto h_{(X)}(Y) = \mathcal{B}_{p/X,\, or/Y}(X \times_S Y)$$

is pro-representable. Moreover if X/S and Y/S are transversal, then

$$\mathrm{Hom}(h_{(Y)}, h_{(X)}) \simeq \mathcal{B}_{p/X,\, or/Y}(X \times_S Y).$$

where composition of morphisms of functors corresponds to the composition defined above when $\{X, Y, Z\}$ is a transversal family.

~~Proof: Given $f: X \to S$ factor it $X \xrightarrow{i} E \xrightarrow{p} S$ where p is smooth and i is the inclusion of the zero section in a vector bundle. Consider the directed set of nbds. of X in E. Claim that~~

<u>Proof</u>: Given $f: X \to S$, one factors f in the form $X \xrightarrow{i} E \xrightarrow{p} S$ where p is smooth and where i is an <u>oriented</u> imbedding. (e.g. let $j: S \to R^n$ be an embedding and let ~~i be the composition $X \to$~~ V be a tubular nbd for j with $\pi: V \to S$ a smooth retraction. Then have

$$\begin{tikzcd}
X \arrow[r, "\Gamma_f"] \arrow[dr, "f"'] & X \times S \arrow[r, "id \times j"] \arrow[d, "pr_2"] & X \times V \arrow[d, "pr_2"] \\
 & S \arrow[r, "j"] \arrow[dr, "id"'] & V \arrow[d, "\pi"] \\
 & & S
\end{tikzcd}$$

Take i to be $X \xrightarrow{\Gamma_{jf}} X \times V$ which is framed since V is parallelizable.)

Next consider the directed set of open neighborhoods U of X in E. If $U \subset U'$, then we have a canonical morphism $(U) \to (U')$ in $\mathcal{M}_S$ represented by

$$\begin{tikzcd}
U \arrow[r, "\text{inclusion}", "\text{with canon. orient.}"'] \arrow[d, "id"] & U' \arrow[d] \\
U \arrow[r] & S
\end{tikzcd}$$

such that we get a pro-object $U \mapsto (U)$ in $\mathcal{M}_S$. Will show this pro-object represents $h_{(X)}$. Recall that $i_U: X \to U$ inherits an orientation from $i: X \to E$ so that we obtain a morphism

$$\mathcal{B}_{\substack{pr/U \\ or/Y}}(U \times_S Y) \xrightarrow{\ \iota_U^*\ } \mathcal{B}_{\substack{pr/X \\ or/Y}}(X \times_S Y)$$

assuming X, Y are <u>transversal.</u>

Check: orient.

$$\begin{tikzcd} X \times_U V \arrow[r, "or"] \arrow[d, "pr"'] & V \arrow[r, "or"] \arrow[d, "pr."] & Y \\ X \arrow[r, "or"'] & U & \end{tikzcd}$$

well-defd. Given $V \longrightarrow U \times_S Y$ want to jiggle so trans. ~~to~~ $X \to U$.

$$\begin{tikzcd} & V \arrow[d] & \\ X \times_S Y \arrow[r] \arrow[d] \arrow[dr, phantom, "tr"] & U \times_S Y \arrow[r] \arrow[d] \arrow[dr, phantom, "tr."] & Y \arrow[d] \\ X \arrow[r] & U \arrow[r] & S \end{tikzcd}$$

OKAY.

Now clearly compatible with inclusions $U \subset U'$ so get map

$$\varinjlim_{U} \mathcal{B}_{\substack{pr/U \\ or/Y}}(U \times_S Y) \longrightarrow \mathcal{B}_{\substack{pr/X \\ or/Y}}(X \times_S Y)$$

which we will show is an isomorphism!

Method is to use old long exact sequence

(A)

$$\cdots \xrightarrow{\ \partial\ } \mathcal{B}_{\substack{pr/U \\ or/Y}}((U-X)\times_S Y) \longrightarrow \mathcal{B}_{pr/U,\, or/Y}(U \times_S Y) \xrightarrow{\ \iota_U^*\ } \mathcal{B}_{\substack{pr/X \\ or/Y}}(X \times_S Y) \xrightarrow{\ \partial\ } \cdots$$

~~and the for We take inductive limit over U. Given Z proper over X, its image is a closed set in~~

~~in~~ Take inductive limit over U. Given $Z \xrightarrow{\alpha} (U-X)$ proper over U, then $\alpha(Z)$ is closed in U not meeting X, so we get $V = U - \alpha(Z)$ ~~[illegible]~~ a nbd. of X in E such that $Z = \phi$ over V. Thus

$$\varinjlim_{U} B_{\substack{pr/U \\ or/Y}} \left((U-X) \times_S Y \right) = 0$$

This gives first assertion of thm. (modulo exact sequence $*$).

For second represent (Y) as $\{(V)\}$ where $Y \xrightarrow{i'} E' \xrightarrow{p'} S$ is analogous. Then

$$\begin{aligned} \mathrm{Hom}(\{(U)\}, \{(V)\}) &= \varprojlim_{V} \varinjlim_{U} \mathrm{Hom}((U),(V)) \\ &= \varprojlim_{V} B_{pr/X,\, or/V} (X \times_S V) \end{aligned}$$

Here, however, we have the homotopy axiom at our disposal.

Claim: 1) For a cofinal family of V's, we have that $X \times_S Y \longrightarrow X \times_S V$ is a homotopy equivalence proper over X

2) Consequently for such V

$$B_{pr/X,\, or/Y}(X \times_S Y) \xrightarrow{\sim} B_{pr/X,\, or/V}(X \times_S V).$$

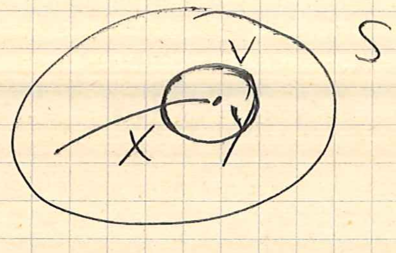
$\boxed{\text{false}}$

$$\begin{tikzcd} X \times_S Y \arrow[d] & Y \arrow[d, "\text{tub. nbd.}"] \\ X \times_S V \arrow[dd] & V \arrow[d, phantom, "\cap \ \text{op.}"] \\ & E' \arrow[d, "\text{sm}"] \\ X \arrow[r] & S \end{tikzcd}$$

If V sufficiently small tube around Y, then $X \times_S V$ is a tube around $X \times_S Y$ by transversality.

NO

~~Coun~~Example:



Y in closure of X. Then can take V to be nbds. of Y in S (assuming $Y \to S$ oriented). But $X \times_S Y = \phi$, $X \times_S V = X \cap V$ which has cohomology with compact support? Want

$\varprojlim B_{pr/X}(X \cap V) \neq 0$. Maybe OKAY here. Thus take a

sequence of $\underbrace{\text{nbds.}}_{}$ (tubular) of Y $V_1 \supset V_2 \supset \cdots$ and suppose can find $Z_n \longrightarrow X \cap V_n$ proper over X. ~~Then can past.~~ Assume also these represent same class in $X \cap V_n$. So then can bound $Z_n \cup Z_{n+1}$, $Z_{n+1} \cup Z_{n+2}$, etc and so each Z_n bounds ~~by~~ something proper over X.

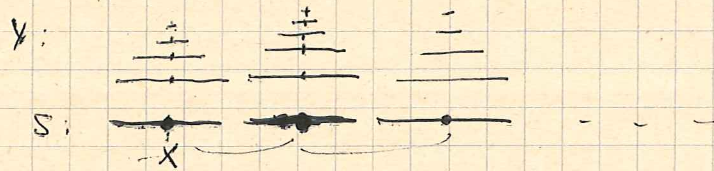
~~So maybe works if . . .~~ So maybe its OKAY if ~~denu~~ the systems of V's is essentially denumerable although one expects even then only an exact sequence

$$0 \longrightarrow R^1 \varprojlim_{V_n} B^{k-1}_{\substack{pr/X \\ or/V_n}}(X \times V_n) \longrightarrow B^k_{pr/X,\, or/Y}(X \times_S Y) \longrightarrow \varprojlim_{V_n} B^k_{pr/X,\, or/V_n}(X \times V_n) \longrightarrow 0$$

In any case $X \times_S Y \longrightarrow X \times_S V$ not a homotopy equivalence making method of proof suspicious.

Example to show that the system of U's required in the theorem is not denumerable. Take $S = \mathbb{N}\times \mathbb{R}^1$ ~~$Y = \mathbb{N}\times\mathbb{N}$~~

$$Y = \coprod_{\substack{n\in\mathbb{N}\\ p\in\mathbb{N}\ p>0}} \{n\}\times\{p\}\times(-p^{-1},p^{-1}) , \qquad X = \mathbb{N}$$



Recall that U's nbds. of X in S, such that given a Z over $X\times_S Y$ proper over X one could find a U and and a Z' ~~proper~~ in $U\times_S Y$ proper over U inducing Z over X. But taking V to be the sequence of points $\coprod \{n\}\times\{p_n\}$ where p_n is any sequence one sees that for any such sequence $\exists$ a $U = \coprod U_n$ where $U_n \subset (p_n^{-1}, p_n)$ for all n. Usual diagonal argument $\Rightarrow$ must use uncountably many U's.

Not yet done: Modulo verification of (A) we know that

$$\varinjlim_{U} B(U\times_S Y) = B(X\times_S Y) \quad \text{if } X, Y \text{ transversal over } S$$

Question: Can we define for arbitrary X, Y over S

$$\mathrm{Hom}((X),(Y)) = \varinjlim_{U} B(U\times_S Y)_{pr/U,\, or/Y} \ ?$$

Check with cohomology.

$$\mathrm{Hom}_{D(S)}\left(f_! \mathcal{O}_X, \underline{g_!} \mathcal{O}_Y\right)$$ if g proper so $g_! = g_*$

$$= \mathrm{Hom}_{D(\cancel{X}_S)}\left(\mathcal{O}_{X \times_S Y}, f'^! \mathcal{O}_Y\right)$$

To calculate $f'^!$ $f' : X \times_S Y \longrightarrow Y$ factor into a closed immersion followed by a smooth thing.

$$\begin{tikzcd} X \times_S Y \arrow[rr, "f'"] \arrow[dr, "i"'] & & Y \\ & U \times_S Y \arrow[ur, "pr_2"'] & \end{tikzcd}$$

forget orient

Then $$\mathrm{Hom}_{D(X \times_S Y)}\left(\mathcal{O}_{X \times_S Y}, \underline{i^! \, pr_2^!} \mathcal{O}_Y\right)$$
$$\|$$
$$\mathrm{Hom}_{D(U \times_S Y)}\left(i_* \mathcal{O}_{X \times_S Y}, \mathcal{O}_{U \times_S Y}\right) \overset{?}{=} \varinjlim_{U} H^*(U \times_S Y)$$

Is it reasonable to expect that if A is closed in B then

$$H^*_A(B) = \varinjlim_{U} H^*(U)$$

where U runs over a basis of nbds of A? Not unless A is pure in B. Thus our proposed defintion is unreasonable, but it gives an idea.

$$\varinjlim_{U} H^*_{X \times_S Y}(U \times_S Y)$$ By excision independent of U.

But $U \times_S Y$ is a manifold so by duality we know this is somehow $H_*(X \times_S Y)$

Dec. 25, 1968

<u>Local cohomology in cobordism theory.</u>

X manifold, A closed subset of X.

<u>Defn</u>:

$H^q_A(X) =$ equivalence classes of triples (V, W, φ) where $V \longrightarrow X$ is proper-oriented (of dim q), $W \longrightarrow X-A$ is proper-oriented with boundary and $\varphi : \partial W \xleftarrow{\sim} V|X-A$ is an isomorphism over A.

a triple (V, W, φ) is equivalent to zero if there exists $(\tilde{V}, \tilde{W}, \alpha, \beta, \gamma)$

$$\tilde{V} \longrightarrow X \quad \text{proper-oriented with boundary}$$

$$\tilde{W} \longrightarrow X-A$$

$$\alpha : \partial\tilde{V} \xleftarrow{\sim} V$$

$$\left.\begin{array}{l} \beta : \tilde{V}|X-A \hookrightarrow \partial\tilde{W} \\ \gamma : W \hookrightarrow \partial\tilde{W} \end{array}\right\} \text{inclusions such that}$$

$$\beta + \gamma : \underbrace{\partial(\tilde{V}|X-A) \cup W}_{V|X-A \xrightarrow[\varphi]{\sim} \partial W} \xrightarrow{\sim} \partial\tilde{W}$$

<u>Proposition 1:</u> There is a long exact sequence

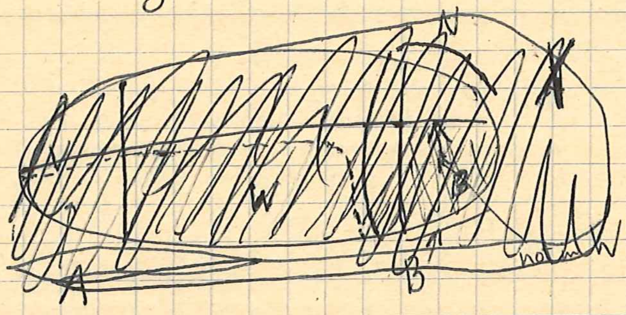
$$\cdots \xrightarrow{\delta} H^q_A(X) \longrightarrow H^q(X) \longrightarrow H^q(X-A) \xrightarrow{\delta} H^{q+1}_A(X) \longrightarrow \cdots$$

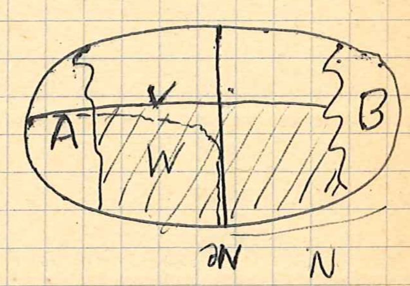
<u>Proposition 2:</u> (Excision) If B is a closed subset of X not meeting A, then

$$H^q_A(X) \xrightarrow{\sim} H^q_A(X-B).$$

Proof of 2: Surjectivity: Given (V, W, φ) where $V \to (X-B)$
$W \to (X-A-B)$ are proper-oriented and $\varphi: V/X-B-A \cong \partial W$.

Let N be a submanifold of X with boundary which is a nbd of B separating A and B and whose boundary is transversal to W (and V understood)

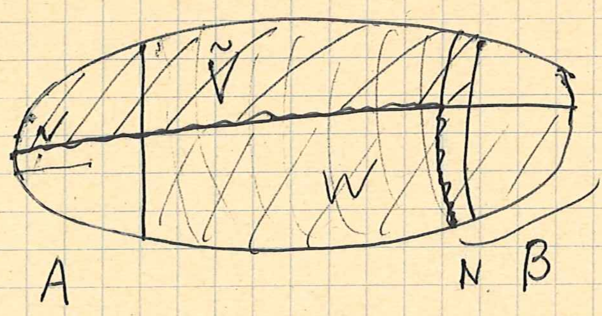




Then one replaces V by $[V-(V \cap \mathrm{Int}\, N)] \cup_{V \cap \partial N} [W \cap N]$

~~which is a closed manifold of X−A~~ and W by $W - W \cap (\mathrm{Int}\, N)$,
or rather the smooth approximation indicated in the figure

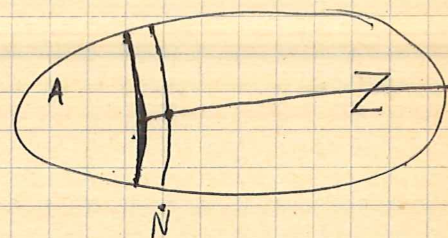
Injectivity: Suppose (V, W, φ) in $H^*_A(X)$ is zero in $X-B$



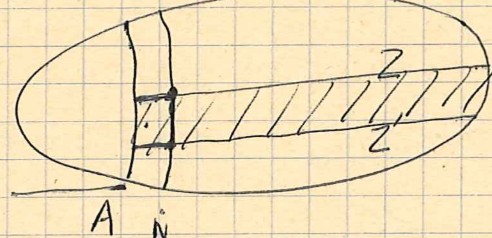
— new V

should be OKAY.

Indication of proof of 1: **Definition of δ.** Given $Z \longrightarrow X - A$ proper.

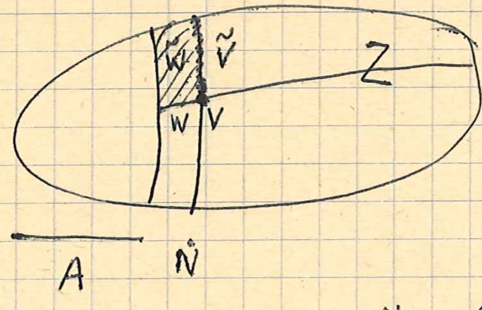


Let N be a nice nbd. of A, submanifold of X with boundary transversal to Z. Then $\delta(Z)$ is $\dot{N} \times_X Z = V$, $N \times_X Z = W$, OKAY because rel. dimension goes up by 1 and V is proper over X. Check independence



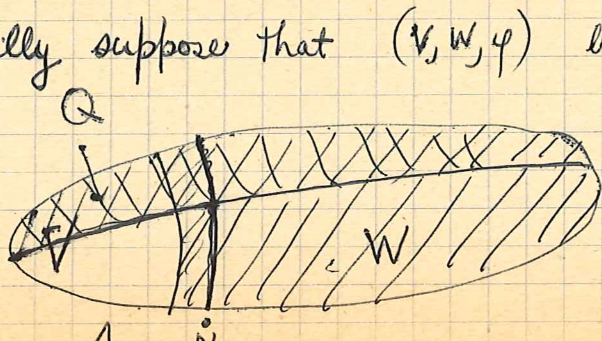
OKAY.

so δ well defined. Suppose $\delta(Z) = 0$



Then Z bordant to $\tilde{V} \cup_V (Z - W)$ which is a variety closed within X.

Finally suppose that (V, W, φ) becomes 0 in $H^*(X)$ e.g. $V = \partial Q$



~~to show that (V, W, φ) bordant to $\partial(V-A)$~~

To show that (V, W, φ) bordant to $W \underset{V/X-A}{\cup} Q/X-A. = V'$

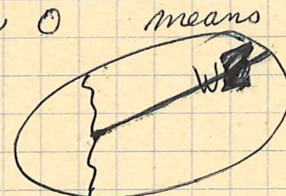
Looks as if you have to rotate V into $\mathring{N} \cap V'$ sweeping through ~~[illegible]~~ $W - (\mathrm{Int}\, N) \cup (Q \cap N)$ joined together along $\mathring{N} \cap Z$ after blowing this up. The sweeping motion moves W into $V' \cap N$. · Seems OKAY

So now if Φ is any family of supports on a manifold X we should be able to form $H^q_\Phi(X) = \varinjlim_{A \in \Phi} H^q_A(X)$.

Elements should be represented by $V \longrightarrow X$ proper oriented together with a trivialization over some member of Φ, e.g. for some $A \subset \Phi$ given $W \longrightarrow X-A$ proper oriented and $\varphi : V/X-A \xrightarrow{\sim} \partial W$.

---

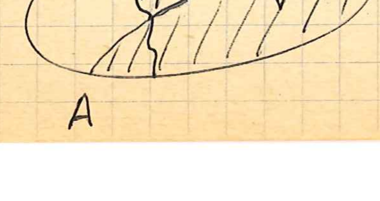
on page 3: better definition of δ: given $Z \rightarrow X-A$ proper-or dim q, take $(V, W, \varphi) = (\phi, Z, \phi)$. exactness at $H(X)$ ✓; at $H(X-A)$: Thus $(\phi, Z, \phi) \sim 0$ means $\exists\ \tilde{W} \longrightarrow X-A$ proper oriented and $\tilde{V} \longrightarrow X$ proper oriented $\ni \partial \tilde{W} = W \cup (\tilde{V} | X-A)$ i.e. that Z comes from X.



A

V

Exactness at $H_A(X)$: given (V, W, φ) and $V = \partial \tilde{V}$ one has $(V, W, \varphi) \sim (\phi, W \cup (\tilde{V}/X-A), \phi)$, where $\tilde{W} = I \times [\tilde{V} | (X-A) \cup W]$



A

Motives over a base ~~variety~~ manifold S.

Given $f : X \to S$, $g : Y \to S$ choose a factorization of f into

$$X \xrightarrow{i} U \xrightarrow{p} S$$

where p is smooth and i is a closed oriented embedding. Now define

$$\mathrm{Hom}_{M(S)}\left(f_! \mathcal{O}_X, g_! \mathcal{O}_Y\right) = \text{bordism classes of } (V, W, \alpha)$$

$$\begin{tikzcd}
\partial W \arrow[d] & j^*V \arrow[l, "\alpha"'] \arrow[r] & V \arrow[r, "or"] \arrow[d, "pr"] & Y \arrow[d] \\
W \arrow[r, "pr"'] \arrow[urrr, bend left, "or"] & U - X \arrow[r, "j"'] & U \arrow[r] & S
\end{tikzcd}$$

where $V \longrightarrow U \times_S Y$ is proper$/U$, oriented$/Y$

$W \longrightarrow (U-X) \times_S Y$ is proper$/U-X$, oriented $/Y$

$\alpha : (U-X) \times_U V \xrightarrow{\sim} \partial W$ isomorphism of manifolds $/(U-X)\times_S Y$ pr$/U-X$, or$/Y$

and where the bordism equivalence relation is as for local cohomology.

$$\boxed{\mathrm{Hom}_{M(S)}\left(f_! \mathcal{O}_X, g_! \mathcal{O}_Y\right) = \dot{B}_{pr/U,\, or/Y}\left(U \times_S Y, (U-X) \times_S Y\right).}$$

Independence of the factorization.

$$\begin{tikzcd}
 & & U \times_S U' \arrow[dl] \arrow[d] \\
X \arrow[r] \arrow[urr, bend left] \arrow[rr, bend right] & U \arrow[d] & U' \arrow[dl] \\
 & S &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & U'' \arrow[d, "sm"] \\
X \arrow[ur, "i''"] \arrow[r, "or"'] & U
\end{tikzcd}$$

cutting down U'' may assume hat i'' is a heq whence can correct the normal bundle of U'' by an embedding

Thus may assume have

$$\begin{tikzcd} & U' \arrow[r, "\text{closed}", "\text{or}"'] \arrow[d] & U \times \mathbb{R}^n \arrow[dl] \\ X \arrow[ur, "\text{or}"] \arrow[r, "\text{or}"'] & U & \end{tikzcd}$$

using excision one ultimately reduces to proving independence wrt

$$\begin{tikzcd} X \arrow[r, "\text{or}"'] & U \arrow[rr, "\text{d. orien}", "\cancel{\text{c.d.o}}"'] \arrow[dr, "p"'] & & \cancel{U \times \mathbb{R}^n}\ U' \arrow[dl, "p\, pr_1"] \\ & & S & \end{tikzcd}$$

~~perhaps only before that U~~

and hence to a local coh. calculation

$$H_X(U) \cong H_X(U' \cancel{\times \mathbb{R}})$$

which is clear ∎

<u>Compositions</u>

$$\begin{tikzcd} & V \arrow[dl, "pr"'] \arrow[dr, "\text{or}"] & & V' \arrow[dl, "pr"'] \arrow[dr, "\text{or}"] & \\ X \arrow[r, hook, "\text{d o}"'] & U \arrow[dr, "sm"'] & Y \arrow[r, hook, "\text{d or}"'] & U' \arrow[dl, "sm"] & Z \\ & & S & & \end{tikzcd}$$

as U, U' smooth over S it follows that the family $\{U, U', Z\}$ is ~~smooth~~ transversal, so we know how to compose

$$\begin{tikzcd}
V \arrow[d, "\alpha"] & V\times_S Z \arrow[l] \arrow[dd] & V\times_{U'} V' \\
U\times_S U' \arrow[d] \arrow[r, phantom, "tr"] & & \\
U' & U'\times_S Z \arrow[l] & V' \arrow[l, "\beta"]
\end{tikzcd}$$

Move α so that can form $V\times_S Z$, then move β so can form $V\times_{U'} V'$ which is the composition

---

$$B(U\times_S Y;\ (U-X)\times_S Y)\otimes B(U'\times_S Z,\ (U'-Y)\times_S Z)$$

$$\downarrow$$

$$B(U\times_S U',\ (U\times_S U')-X\times_S Y)\otimes B(U'\times_S Z,\ U\times_S Z-Y\times_S Z)$$

$$\downarrow$$

$$B\big((U\times_S U')\times(U'\times_S Z),\ (U\times_S U')\times(U'\times_S Z)\ -\ (X\times_S Y)\times(Y\times_S Z)\big)$$

$$\Big\downarrow \text{ pull-back}$$

$$B(U\times_S U'\times_S Z,\ \ U\times_S U'\times_S Z\ -\ X\times_S Y\times_S Z)$$

$$\Big\downarrow \text{ integration over } U'$$

$$B(U\times_S Z,\ U\times_S Z - X\times_S Z)$$

✓

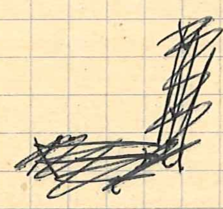
In case X, Y are transversal over S, then $X \times_S Y$ is a submanifold of $U \times_S Y$ (more generally if X, Y intersect cleanly over S) so then

except for orientation

$$i^* : \mathcal{B}(U \times_S Y, (U-X)\times_S Y) \simeq \mathcal{B}(X \times_S Y).$$

not i^*, rather is just gotten by a retraction of $U \times_S Y$ onto $X \times_S Y$

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & Y \\ X \arrow[r, "i"'] & U \end{tikzcd}$$



Remaining checking:

1. define $f_* \mathcal{O}_X$

Question: Is $f_* \mathcal{O}_X$ necessarily a new object of $\mathbf{M}(S)$ or is it expressible in terms of $f_!$'s.

2. calculation by means of spectra — reduction to homotopy theory. e.g. in the framed case you get the stable homotopy category over a base S!!

3. variance in S; have $f_!$ defined at present

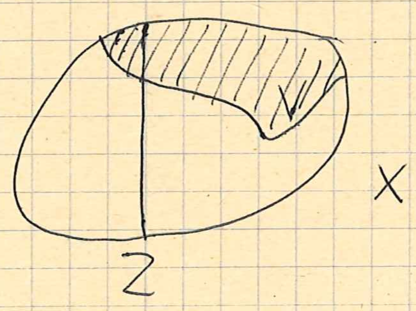
4. Axiomatization and construction of triangulated category.

5. Equivariant bordism theory + Steenrod-Atiyah power operations

<u>Proposition</u>: Let $Z \xrightarrow{i} X$ be an oriented closed embedding of dimension d. Then

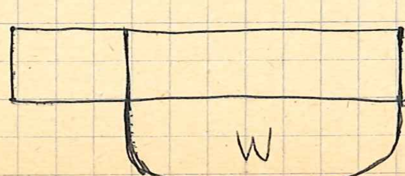
$$H^q(Z) \xrightarrow{\sim} H_Z^{q+d}(X).$$

<u>Proof</u>: An element of $H^q(Z)$ is represented by $U \longrightarrow Z$ proper and oriented of dimension q. As i is closed + oriented, we get $U \longrightarrow X$ proper + oriented of dim $q+d$ and as $U \times_X (X-Z) = \phi$ a canonical trivialization over $X-Z$, hence an element of $H_Z^{q+d}(X)$.
To see map is surjective, suppose given $V \longrightarrow X$ with trivialization $\alpha: V/X-Z \xrightarrow{\sim} \partial W$ over $X-Z$ where $W \longrightarrow X-Z$ is proper-oriented with boundary. By excision may assume i is inclusion of 0-section of a vector bundle in which case we get a retraction $r: X \to Z$ and a homotopy $h: X \times I \longrightarrow X$ joining ir to id_X. Using this homotopy we move $V \xrightarrow{f} X$ into $rf: V \to Z$ and we can remove W completely.



$$\partial W = V \mid X - Z$$

But get $V \times I \longrightarrow X$ so can put $(V \times I \mid X-Z) \cup_{\partial W} W$

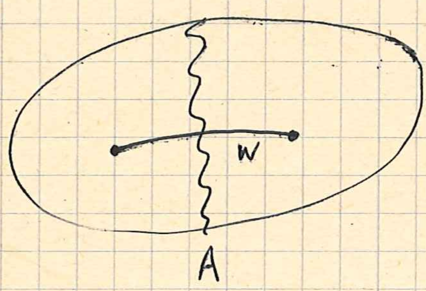


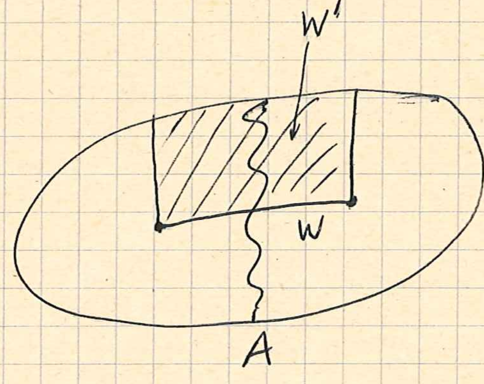
OKAY. ?

seems that if A (closed in X) is a deformation retract of a nbd. then any element in $H_A(X)$ comes from a bordism class in A

December 28, 1968: More motives

Let A be a closed subset of a manifold X. Define motive theoretic **cohomology** $H^q(A)$ as (bordism) equivalence classes of maps $f: W \longrightarrow X$ proper and oriented of dimension q, where W is a manifold with boundary as $f(\partial W) \subset X - A$. We say that $f: W \to X$ is a boundary if $\exists$ $W' \longrightarrow X$ proper-oriented of degree $q-1$ and an embedding $W \longrightarrow \partial W'$ preserving orientation and maps to X, such that $\partial W' - \mathrm{Int}\, W$ is (situated) over $X - A$. Pictures





Properties:

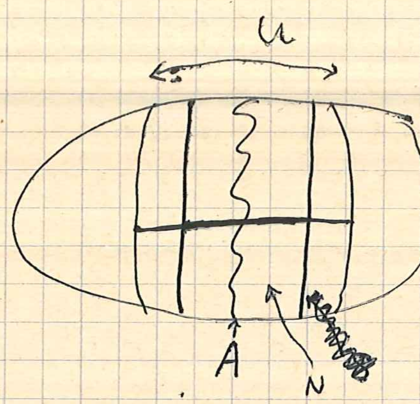
Proposition 1:
$$H^q(A) = \varinjlim_{U} H^q(U)$$
where U runs over the open neighborhoods of A in X.

Proposition 2: If A is a closed submanifold of X, then the two definitions of $H^q(A)$ we have coincide.

Proposition 3: There is a long exact sequence

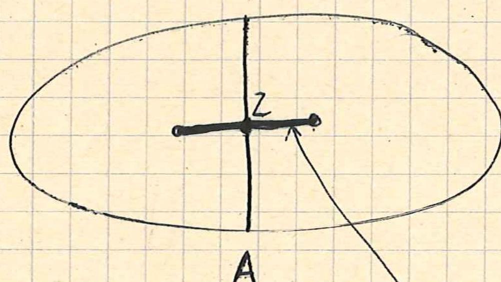
$$\cdots\; H^q_{proper/X}(X-A) \xrightarrow{j_*} H^q(X) \xrightarrow{i^*} H^q(A) \xrightarrow{\delta} H^{q+1}_{proper/X}(X-A) \cdots\cdots$$

<u>Proof:</u> 1. Restriction $H^8(U) \longrightarrow H^8(A)$ defined by choosing a



manifold with boundary N containing A in its interior and contained in U such that $\partial \bar{N}$ is transversal to $Z \to U$, and setting $W = N \times_U Z$. surjectivity ✓ injectivity ✓

2. follows from 1 because then have a cofinal system of U's consisting of open tubes which have same cohomology as A by homotopy axiom. Geometrically picture is



W= normal disk along $Z = A \cap W$.

3. δ defined by $\delta(W) = \partial W$, which is proper and oriented over $X - A$. i^* defined by sending $Z \to X$ into $W = Z \to X$. j_* defined by sending $Z \to X\text{-}A$ into $Z \to X$. exactness follows by passing to limit in

$$H^8_{X-U}(X) \longrightarrow H^8(X) \longrightarrow H^8(U) \longrightarrow \cdots$$

where U runs over the neighborhoods of A in X.

Let A, B be two closed subsets of a manifold X.

$$\begin{tikzcd} & B \arrow[d, "\iota'"] & B-(A\cap B) \arrow[l, "k"'] \arrow[d, "k'"] \\ A \arrow[r, "\iota"] & X \arrow[d, "j'"] & (X-A) \arrow[l, "j"'] \\ & X-B & \end{tikzcd}$$

$$0 \to j'_! \mathcal{O}_{X-B} \to \mathcal{O}_X \to \iota_* \mathcal{O}_A \to 0$$

$$\mathrm{Hom}(\iota_* \mathcal{O}_A, \iota'_* \mathcal{O}_B) = H_A(\iota'_* \mathcal{O}_B)$$

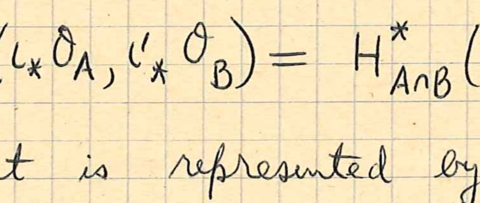
$$\cdots \to H^q_A(\iota'_* \mathcal{O}_B) \to H^q(B) \to H^q(X-A, j^* \iota'_* \mathcal{O}_B)$$

$$= H^q(X-A, k'_* \mathcal{O}_{B-(A\cap B)})$$

$$= H^q(B - A\cap B).$$

$$\therefore \quad \mathrm{Hom}(\iota_* \mathcal{O}_A, \iota'_* \mathcal{O}_B) = H^*_{A\cap B}(B).$$

Such an element is represented by a manifold with bdry V proper-oriented of ~~W~~ dim q over X such that ∂V is ~~situated~~ over $X-B$ together with a trivialization ~~$V/X-A \cong \partial W$ where $W \to X-A$ proper-oriented~~ of the restriction to $B-(A\cap B)$, i.e. a $W \longrightarrow X-A$ proper-oriented of dim $q+1$ and an embedding $V/X-A \hookrightarrow \partial W$ such that $\partial W - \mathrm{Int}(V/X-A)$ is ~~proper~~ situated (hence proper) over $X-(A\cup B)$.



Φ sheaf of family of supports of X/S.

i.e. to each $U \subset S$ get $\Phi(U)$ a family of supports on $f^{-1}U$.

$$\Gamma(U, f_{\Phi}(F)) = \Gamma_{\Phi(U)}(f^{-1}U, F).$$

form topological space

$$X \underset{\Phi}{\cup} S$$

where open sets are those of X

and those of the form $(f^{-1}U - F) \cup U$

where $F \in \Phi(U)$.

finite intersection

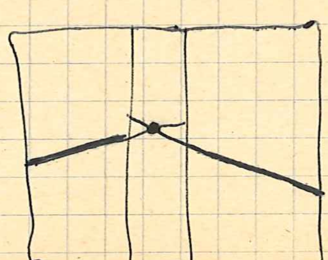
$$\left[(f^{-1}U - F) \cup U\right] \cap \left[(f^{-1}U' - F') \cup U'\right]$$

$$= \left[f^{-1}(U \cap U') - [f^{-1}(U \cap U') \cap F] - [f^{-1}(U \cap U') \cap F']\right] \cup (U \cap U').$$

union:

$$\bigcup (f^{-1}U_i - F_i) \cup U_i = \left\{f^{-1}(\bigcup U_i) - G\right\} \cup (\bigcup U_i)$$

where $G = \{x \mid x \in f^{-1}U_i \Longrightarrow x \in F_i\}$



G closed + $G \cap f^{-1}U_i \subset F_i$

$\therefore$ Φ sheaf $\Rightarrow G \in \Phi(\bigcup U_i)$.

Conversely given

$$\begin{tikzcd}
X \arrow[r, hook, "j", "\text{open}"'] \arrow[dr, bend right, "f"'] & Z \arrow[d, shift left, "p"] \\
& S \arrow[u, shift left, "s"]
\end{tikzcd} \qquad ps = id_S$$

such that $Z = j(X) \amalg s(S)$, for each U in S let

$$\Phi(U) = \{ F \subset f^{-1}U \mid j(F) \text{ closed in } p^{-1}U \}$$

Claim $\Phi(U)$ is a sheaf of supports of X/S.

If $U = \bigcup U_i$ $F \in f^{-1}U$

$$\underset{\text{open}}{p^{-1}U - jF} \Longleftarrow \underset{\text{open}}{\bigcup \left(p^{-1}U_i - j(F \cap f^{-1}U_i)\right)}$$

Homology:

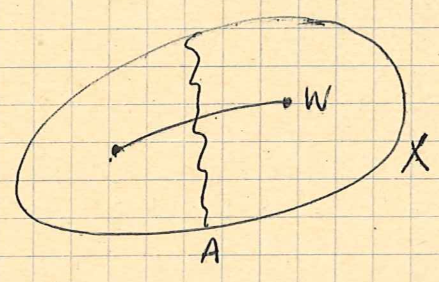
A closed subset of a manifold X, $U = X - A$.

$\boxed{H_q(X, U)}$ = bordisms classes $W \longrightarrow X$ where W is compact oriented of dim q and ∂W lies over U.

exact sequence:

$$\cdots\ H_q(U) \longrightarrow H_q(X) \longrightarrow H_q(X, U) \xrightarrow{\partial} H_{q-1}(U) \cdots$$

where $\partial(W \to X) = \partial W \to X$.



Just like $H^q(A)$ except that W ~~[illegible]~~ is proper-oriented over pt. instead of X.

duality thm: If X is oriented of dimension n and if A is compact, then

$$H^q(A) \simeq H_{n-q}(X, U)$$

excision: ~~[illegible]~~

If B is closed in X and $A \cap B = \phi$, then

$$H_*(X - A, U - A) \simeq H_*(X, U)$$

If X compact-oriented dimension n get isomorphism of exact seq.

$$\begin{array}{ccccc} H^q_c(U) & \longrightarrow & H^q(X) & \longrightarrow & H^q(A) \\ \wr\mid & & \wr\mid & & \wr\mid \\ H_{n-q}(U) & \longrightarrow & H_{n-q}(X) & \longrightarrow & H_{n-q}(X, U) \end{array}$$

$\boxed{H_q(A)}$ = bordism classes $Z \longrightarrow X$ + $W \xrightarrow{g} U$
+ $Z/U \simeq \partial W$

such that Z compact and oriented of dim q and W is oriented of dim $q+1$ and $g^{-1}B$ compact for all B closed in $X \ni B \cap A = \phi$.

$\boxed{H_q(X,A)}$ = bordism classes $Z \xrightarrow{f} U$ where Z is oriented of dimension q and where $f^{-1}B$ compact for any B closed in $X \ni B \cap A = \phi$.

Exact sequence:

$$\cdots\cdots H_q(A) \longrightarrow H_q(X) \longrightarrow H_q(X,A) \xrightarrow{\partial} H_{q-1}(A) \cdots\cdots$$

$$\partial(Z \to U) = \{\phi \to X,\ Z \to U,\ \phi \simeq \partial Z\}.$$

Natural isomorphism if X oriented compact

$$\begin{tikzcd}
\cdots H_q(A) \arrow[r] \arrow[d, "\wr"] & H_q(X) \arrow[r] \arrow[d, "\wr"] & H_q(X,A) \cdots \arrow[d, "\wr"] \\
\cdots H^{n-q}_A(X) \arrow[r] & H^{n-q}(X) \arrow[r] & H^q(X-A) \cdots
\end{tikzcd}$$

The point is that if X is compact, then a proper map $f: Z \to X-A$ is the same as a map $f \ni f^{-1}B$ compact for all B closed in $X \ni B \cap A = \phi$, since such a B is compact.

---

duality thm: If X is oriented of dimn and if $X-A$ is relatively compact in X, then $H_{n-q}(X,A) \simeq H^q(X-A)$. (this $\Rightarrow$ other dual. thm)

From the point of view of homology the basic object is

$$H_g(X, A)$$

since it determines $H_g(A)$ by exact sequence and since

$$\varinjlim_{\substack{B \text{ compact in } X \\ B \subset U}} H_g(X,B) = H_g(X,U).$$

$$\varinjlim_{B \text{ compact in } U} H_g(B) = H_g(U).$$

In fact it seems to be enough to know $H_g(K)$ for K compact.

Proposition: If A is closed in X, then

$$\varinjlim_{K} H_g(X,K) = H_g(X,A)$$

$$\varinjlim_{K} H_g(K) = H_g(A)$$

where K runs over the compact subsets of A.

Proof. By long exact sequence, enough to prove first. So given $f: V \longrightarrow X-A$ where $f^{-1}B$ compact for all B closed in $X \ni B \cap A = \phi$, have to show $K = \overline{f(V)} \cap A$ is compact (in effect then if U is a nbd of K, then $f^{-1}(X-K)$ is compact). Suppose K non-compact. Then it contains an infinite discrete set $\{a_n\}$. Let Q_n be an exhaustion of V by compact sets. ~~and let~~ Will construct a nbd. U of A such that $f^{-1}(X-U)$ not compact. For each n choose $b_n = f(x_n)$ ~~near a_n where~~ sufficiently near

a_n such that $B = \{b_n\}$ is discrete and such that $x_n \notin Q_n$. Then B is closed in X, $B \cap A = \phi$, and $f^{-1}B \not\subset Q_n$ for any n so $f^{-1}B$ is not compact.

Remark: The proof of the proposition shows the equivalence of $f : V \longrightarrow X - A$ being such that $f^{-1}B$ compact for all B closed in $X \ni B \cap A = \phi$, and (i) f proper and (ii) $\overline{f(V)}$ compact.

The proposition enables us to make the following definition If Φ is a family of supports on X, then

$$H_q(\Phi) = \varinjlim_{K \in \Phi} H_q(K)$$

$$H_q(X, \Phi) = \varinjlim_{K \in \Phi} H_q(X, K)$$

where K runs over the compact subsets of Φ. If X is compact oriented of dimension n, then for any family of supports Φ we have

$$H_q(\Phi) \simeq H^{n-q}_{\Phi}(X)$$

December 31, 1968 more motives

Careful construction of the category of motives over a base manifold S.

$M(S)$: $\quad \operatorname{Ob} M(S) =$ manifolds $X \longrightarrow S$. We use (X) to denote the object of $M(S)$ corresponding to $X \in \operatorname{Ob} \mathcal{V}/S$.

To define $\operatorname{Hom}_{M(S)}((X),(Y))$ ~~we~~ first consider ~~the~~ ~~point~~ Case 1: ~~when~~ X and Y are transversal over S

$$\operatorname{Hom}^{q}_{M(S)}((X),(Y)) = \text{bordism classes of maps } (f,g): Z \to X \times_S Y$$

such that f is proper and oriented of dimension q.

~~(This definition should be valid if [illegible] X and Y not transversally [illegible])~~

Case 2: Let $Y \xrightarrow{j} V \xrightarrow{g} S$ be a factorization of $Y \to S$ ~~with~~ j a closed embedding and g smooth. Let

$$\operatorname{Hom}^{q}_{M(S)}((X),(Y)) = \text{bordism classes of triples}$$

$$\begin{cases} Z \xrightarrow{(f,g)} X \times_S V \\ W \xrightarrow{(f',g')} X \times_S (V-Y) \\ \varphi: Z|V-Y \xrightarrow{\sim} \partial W \quad \text{over } X \times_S (V-Y) \end{cases}$$

where f proper + oriented of dim q

f' ——————— $q-1$

for any B closed in V not meeting Y we have $(g')^{-1}B$ is f'-proper.

In other words we want there to be a long exact sequence

$$\longrightarrow \mathrm{Hom}^q_S((X),(Y)) \longrightarrow \mathrm{Hom}^q_S((X),(V)) \longrightarrow \mathrm{Hom}^q_S((X),(V,Y)) \xrightarrow{\ \delta\ } \mathrm{Hom}^{q+1}_S((X),(Y)) \cdots$$

Have to show compatible with definition of case 1, so suppose X, Y meet transversally over S. Given $(f,g): Z \longrightarrow X \times_S Y$ with f proper and oriented of dim q one may take $W = \phi$ and φ the unique isom of $\partial Z \cong \phi$. This gives a map

$$\mathrm{Hom}^q_S((X),(Y))^{①} \longrightarrow \mathrm{Hom}^q_S((X),(Y))^{②}$$

where the superscripts signify cases ① & ②. Claim surjective. Suppose given (Z, W, φ) as before.

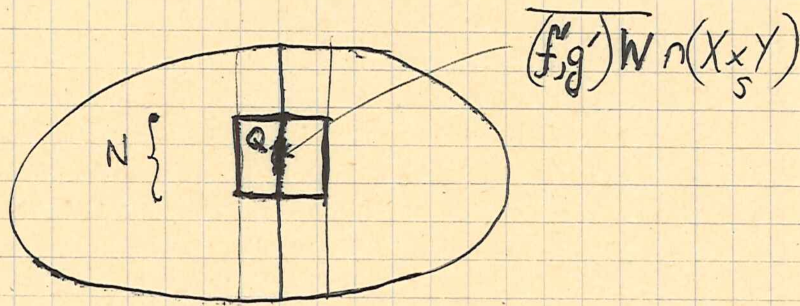
<u>Lemma</u>: $\overline{(f',g')W} \subset X \times_S V$ is proper over X.

<u>Proof</u>: Suppose given $q_n \in \overline{(f',g')W}$ such that $pr_1(q_n)$ $\to x \in X$. To show q_n has a cgt. subsequence. Choose $w_n \in W \ni$ $dist(q_n, (f',g')w_n) \longrightarrow 0$ and $\ni$ $f'w_n \longrightarrow x$. May thus assume $q_n = (f'w_n, g'w_n) \in X \times_S V$, where w_n is a sequence in W. Let $B = \{g'w_n\} \subset V$. If $\xi \in B \cap Y$, then have $g'w_{n_k} \longrightarrow \xi$ whence q_{n_k} converges + done. Otherwise $B \cap Y = \phi$ whence $w_n \in (g')^{-1}B$ is proper over X and hence w_n has a convergent subsequence. **QED.**

$$\begin{tikzcd}
X \times_S Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\pitchfork"] & Y \arrow[d] \\
X \times_S V \arrow[r] \arrow[d] \arrow[dr, phantom, "\pitchfork"] & V \arrow[d, "sm"] \\
X \arrow[r] & S
\end{tikzcd}$$

~~Choose deformation retraction of V onto Y it gives one of $X \times_S V$ back onto $X \times_S Y$ which is not constant over X necessarily.~~

Let Q be a neighborhood of $\overline{(f',g')W} \cap (X \times_S Y)$ which is proper over X and which admits a ~~[illegible]~~ proper deformation retraction $h: Q \times I \longrightarrow Q$ into $Q \cap (X \times_S Y)$ ~~which is also proper over X.~~ (Existence of Q proved using lemma: Start with a neighborhood N of $\overline{(f',g')W} \cap (X \times_S Y)$ in $X \times_S Y$ which is proper over X. Can even assume N is a closed manifold with boundary. Next take a tubular neighborhood Q of $X \times_S Y$ in $X \times_S V$ whose restriction to N is proper over X. Then the deformation of Q into $Q \cap (X \times_S Y)$ is proper over X)



By the usual excision argument we may assume that W, Z lie over Q. Next put

$$\widetilde{W} = \overline{W} \times I \xrightarrow{(f',g') \times id} Q \times I \xrightarrow{h} Q$$
$$\widetilde{Z} = Z \times I \xrightarrow{(f,g) \times id} Q \times I \xrightarrow{h} Q$$

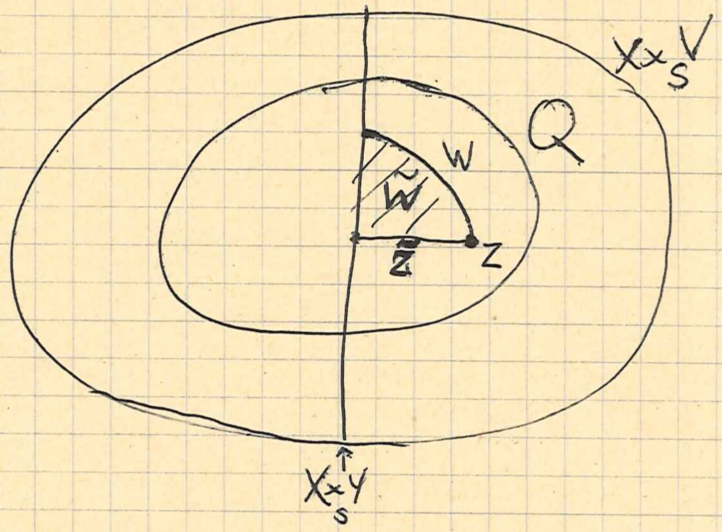
Then $\widetilde{Z}$ proper over X. ~~and $\widetilde{W}$ is such that (the inverse image of any closed subset of N not meeting Y)~~

Here

$$\overline{W} = W \cup \left[\overline{(f',g')W} \cap X \times_S Y \right]$$

with the topology $\ni$ W open and a open nbd of $z \in \overline{(f',g')W} \cap (X \times_S Y)$ is inverse image of a open nbd in $X \times_S V$.

Then $\overline{W}$ is proper over X. (Proof as in lemma: ~~Given~~ Suppose $q_n \in \overline{W}$ with $f'(q_n)$ converging to x in X. To prove convergence of a subsequence in $\overline{W}$ may assume $q_n \in W$; in effect we may choose w_n in W so that $\text{dist}(f'(q_n), f'(w_n))$ and $\text{dist}(g'(q_n), g'(w_n))$ go to zero, and $w_n = q_n$ if $q_n \in W$; in this way if w_n has a convergent subsequence so does q_n. Set $B = \overline{\{g' w_n\}} \subset Y$. If $B \cap Y = \phi$, then $\{w_n\} \in g'^{-1} B$ proper over X $f' w_n$ converges so w_n has a convergent ~~subsequence~~. If $B \cap Y \neq \phi$ then $g' w_{n_k} \longrightarrow y$ so $w_{n_k} \longrightarrow (x, y) \in \overline{W}$.). In particular proper over Q so $\widetilde{W}$ is proper over X and removing inverse image of Y we get something $(f'', g''): \widetilde{W} \longrightarrow X \times_S (V - Y)$ $\ni$ $(g'')^{-1} B$ f''-proper for all B closed in V $\ni$ $B \cap Y = \phi$. Picture.



This proves surjectivity; injectivity similar but more messy.

Above needs a lot of rewriting: Perhaps useful to introduce $\overline{W}$ from the beginning.

<u>Case 3</u>: Let $X \xrightarrow{i} U \xrightarrow{p} S$ be a factorization of $Y \longrightarrow S$ where i is a closed embedding and p is smooth. So now we expect an exact sequence

$$\cdots \operatorname{Hom}^q_S((U,X),(Y)) \longrightarrow \operatorname{Hom}^q_S((U),(Y)) \longrightarrow \operatorname{Hom}^q_S((X),(Y)) \xrightarrow{\delta} \cdots$$

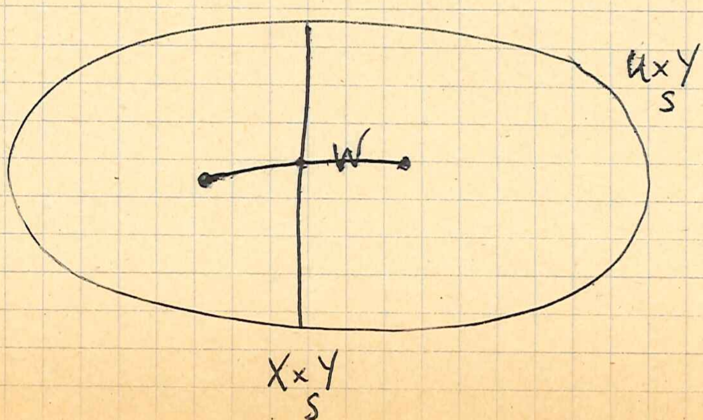
So set

$\operatorname{Hom}^q_S((X),(Y))$ = bordism classes of maps $(f,g): W \longrightarrow U \times_S Y$ where f is proper and oriented of dim q and where ∂W lies over $U - X$.

Compatibility with case 1: Assume X, Y are transversal. Then you have a map

$$\operatorname{Hom}^q_S((X),(Y))^{③} \longrightarrow \operatorname{Hom}^q_S((X),(Y))^{①}$$

$$\begin{tikzcd}
 & W \arrow[d] & X \times_U W \arrow[l] \arrow[d] \\
Y & U \times_S Y \arrow[d] & X \times_S Y \arrow[l] \arrow[d] \\
S & U & X \arrow[l]
\end{tikzcd}$$



Map in opposite direction given by sending $Z \longrightarrow X \times_S Y$ into $N|Z$ where N is a normal tube of $X \times_S Y$ in $U \times_S Y$.

Case 4: Let $X \xrightarrow{i} U \xrightarrow{p} S$, $Y \xrightarrow{j} V \xrightarrow{g} S$ be as before.

$\mathrm{Hom}^q_S((X),(Y))$ = bordism classes of: (Z, W, φ):

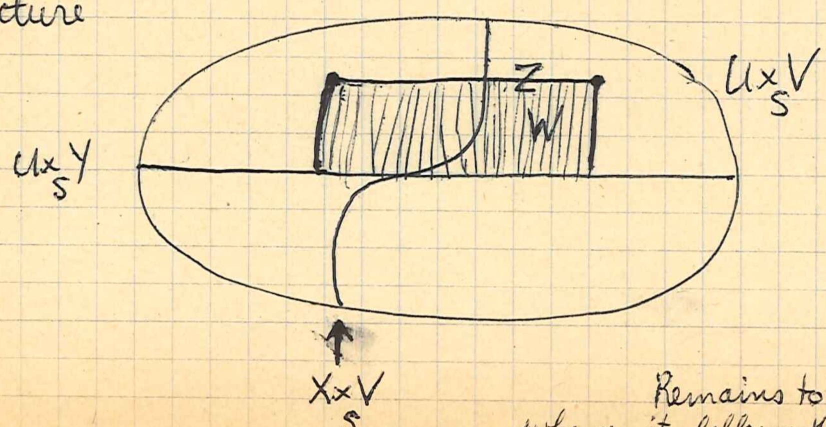
$(f,g): Z \longrightarrow U \times_S V$ — f proper-oriented of dim q; ∂Z lies over $U - X$

$(f',g'): W \longrightarrow U \times_S (V-Y)$ — f' proper-oriented of dim $q-1$; $\overline{W}$ proper over U, i.e. if B is closed in V, $B \cap Y = \phi$, then $g'^{-1}B$ is f' proper.

$\varphi: \underbrace{Z \hookrightarrow \partial W}_{|V-Y|}$ — embedding over $U \times_S (V-Y)$ compatible with orientations.

the limiting values of $\partial W - \mathrm{Int}\, Z$ in $U \times_S Y$ is proper over U and situated over $U - X$.

Picture



Remains to check compatible with Cases 3+4 whence it follows that the definitions given there are independent of choice of U, V.

<u>Remark</u>: We think of (X) as the homology of X relative to S, however, ~~we~~ we have just defined the dual of the category with objects $f_* \mathcal{O}_X$. In effect we expected to have

① $\operatorname{Hom}^q_S(f_*\mathcal{O}_X, g_*\mathcal{O}_Y) =$ bordism classes $Z \xrightarrow{(f,g)} X \times_S Y$
g proper ~~~~ oriented of dim q.
if X, Y transversal over S.

② $Y \xrightarrow{j} V \xrightarrow{q} S$ fact. of g; q transversal to f

have

$$0 \longrightarrow \bar{j}_! \mathcal{O}_{V-Y} \longrightarrow \mathcal{O}_V \longrightarrow j_* \mathcal{O}_Y \longrightarrow 0$$

where $\bar{j}: V - Y \hookrightarrow V$ is the inclusion. Thus get long exact sequence

$$\cdots \operatorname{Hom}^q(f_*\mathcal{O}_X, g_*\mathcal{O}_Y) \longrightarrow \operatorname{Hom}^{q+1}(f_*\mathcal{O}_X, q_*\bar{j}_!\mathcal{O}_{V-Y}) \longrightarrow \operatorname{Hom}^{q+1}(f_*\mathcal{O}_X, q_*\mathcal{O}_V) \cdots$$

so get bordism classes of $W \longrightarrow X \times_S V$ proper oriented of dimension $(-q)$ over V and ∂W situated over $V - Y$.

③ $X \xrightarrow{i} U \xrightarrow{p} S$ fact. of p, p trans. to g and $\bar{\imath}: U - X \hookrightarrow U$ inclusion

$$0 \longrightarrow \bar{\imath}_! \mathcal{O}_{U-X} \longrightarrow \mathcal{O}_U \longrightarrow i_* \mathcal{O}_X \longrightarrow 0$$

$$\operatorname{Hom}^q(f_*\mathcal{O}_X, g_*\mathcal{O}_Y) \longrightarrow \operatorname{Hom}^q(p_*\mathcal{O}_U, g_*\mathcal{O}_Y) \longrightarrow \operatorname{Hom}^q(p_*\bar{\imath}_!\mathcal{O}_{U-X}, g_*\mathcal{O}_Y)$$

$Z \longrightarrow U \times_S Y$ prop orient dim $-q$ $/Y$.

$W \longrightarrow (U-X) \times_S Y$ transforms closed subsets B of U not meeting X into sets proper over Y.

Base change theorem in transversal case:

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

transversal cartesian $\Longrightarrow$ $g^* f_* = f'_* g'^*$ for cohomology

Proof: Assume g of dimension d, so that

FALSE see page 10

$$g^! = \omega_{Y'/Y} \otimes g^*$$
$$g'^! = \omega_{X'/X} \otimes g'^* \qquad \omega_{X'/X} = f'^* \omega_{Y'/Y}$$

where $\omega_{Y'/Y}$ is the orientation sheaf of Y'/Y located in dimension $+d$.
Then

$$\begin{aligned} \mathrm{Hom}_{Y'}(A, g^* f_* B) &= \mathrm{Hom}_{Y'}(A, \omega_{Y'/Y}^{-1} \otimes g^! f_* B) \\ &= \mathrm{Hom}_{Y'}(\omega_{Y'/Y} \otimes A, g^! f_* B) \\ &= \mathrm{Hom}_{Y'}(f^* g_! (\omega_{Y'/Y} \otimes A), B) \\ &= \mathrm{Hom}_{Y'}(g_! f'^* (\omega_{Y'/Y} \otimes A), B) \\ &= \mathrm{Hom}_{X'}(f'^* A, \omega_{X'/X}^{-1} \otimes g'^! B) \\ &= \mathrm{Hom}_{Y'}(A, f'_* g'^* B). \end{aligned}$$

---

may be true only for constructible sheaves?

Base change theorem in general for cohomology

$$\begin{tikzcd}
X' \arrow[r, "g'"] \arrow[d, "i'"] \arrow[dd, bend right, "f'"'] & X \arrow[d, "i"] \arrow[dd, bend left, "f"] \\
U' \arrow[r, "g''"] \arrow[d, "p'"] & U \arrow[d, "p"] \\
Y' \arrow[r, "g"'] & Y
\end{tikzcd}$$

i closed
p transversal to g

$$g^* f_* = g^* p_* i_*$$

$$= p'_* g''^* i_* \qquad \text{transversal base change}$$

$$= p'_* i'_* g'^* \qquad \text{proper base change , } i \text{ proper}$$

$$= f'_* g'^* .$$

<u>Remark:</u> Above argument should work in general for locally compact spaces under constructibility assumptions on g because then by Verdier we can expect a formula of form

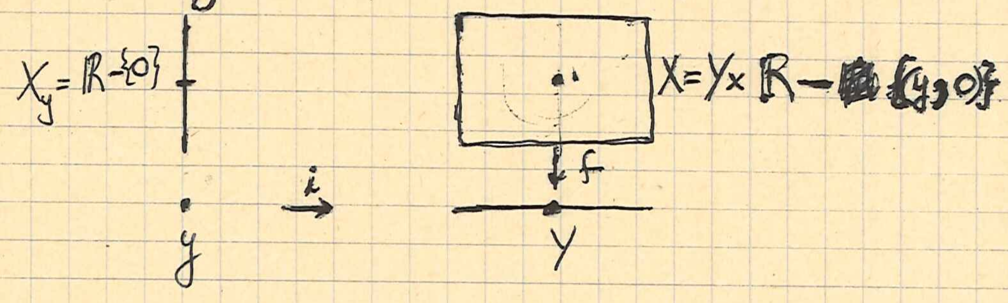
$$g^! A = g^! \mathbb{Z} \otimes g^* A$$

stable under base changes.

Preceding derivation false because the formula

$$i^! F = \omega \otimes i^* F$$

is not generally valid for a closed embedding i unless F is locally constant. <u>Example</u>: Remove a point from $Y \times \mathbb{R}$



Then

$$(f_* \mathbb{Z}_X)_y = \varinjlim_{U \ni y} H^0(f^{-1}U, \mathbb{Z}) = \mathbb{Z} \quad \text{since } f^{-1}U \text{ is connected}$$

but

$$H^0(f^{-1}y, \mathbb{Z}) = \mathbb{Z} \times \mathbb{Z} \quad \text{since there are two components.}$$

Thus the base change formula doesn't hold.
The problem with proof given on page 8 is that

$$\underline{R} i^! (\underline{R} f_* \mathbb{Z}) \neq \omega \otimes i^* (\underline{R} f_* \mathbb{Z}_X)$$

For example take $Y = \mathbb{R}$.

$$f_* \mathbb{Z}_X = \mathbb{Z}_Y$$

$$R^1 f_* \mathbb{Z}_X = i_* \mathbb{Z}_y$$

$$R^q f_* \mathbb{Z}_X = 0 \qquad q \geq 2$$

$$\therefore \quad H^q(i^* \underline{R} f_* \mathbb{Z}_X) = \begin{cases} \mathbb{Z} & q = 0 \\ \mathbb{Z} & q = 1 \\ 0 & q \geq 2 \end{cases}$$

$R^q(i^! f_*) \mathbb{Z}_X$ = coh. of X with supports $f^{-1} y$.

$$0 \to R^0(i^! f_*) \mathbb{Z}_X \to \underset{= \mathbb{Z}}{H^0(X)} \to \underset{\mathbb{Z} + \mathbb{Z}}{H^0(X - f^{-1}y)} \to$$

$$\to R^1(i^! f_*) \mathbb{Z}_X \to \underset{= \mathbb{Z}}{H^1(X)} \to \underset{0}{H^1(X - f^{-1}y)} \to \cdots$$

$$\therefore \quad R^q(i^! f_*) \mathbb{Z}_X = \begin{cases} \mathbb{Z} & q = 0 \\ \mathbb{Z} & q = 1 \\ 0 & q \geq 2 \end{cases}$$

& these don't coincide after the shift of dimension ± 1.

December 68

Let X, Y be C^∞ manifolds. Let Φ be a family of supports on X, let Ψ be a family of supports on Y. (family of supports = family of closed subsets containing ϕ, hereditary, and closed under finite unions). Let $X \cup_\Phi \{\infty\}$ be the space which is the union of X and a point ∞ and whose open sets are those of X and those of the form $(X-F)\cup\{\infty\}$ where $F \in \Phi$.

Definition of the abelian group $D^q(X,\Phi;Y,\Psi)$: Consider diagrams of the form

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$

where f, g are maps of C^∞ manifolds such that

(i) f is oriented of dimension q

(ii) $\forall F \in \Psi$, $\exists F' \in \Phi$ such that $f: g^{-1}F \to F'$ is proper. ~~The inverse of such a diagram is defined by reversing the orientation of f, and sum of two diagrams is defined by disjoint union. Finally two diagrams are called equivalent if they are bordant~~

Two diagrams $(f',g'): Z' \to X \times Y$ and $(f'',g''): Z'' \to X \times Y$ are called bordant ~~equivalent~~ iff $\exists$ a manifold with boundary Z and maps $(f,g): Z \to X \times Y$

? f oriented of dim $q-1$, $g^{-1}\Psi$ proper for Φ, whose boundary with induced orientations ~~is Z' and Z''~~ is $Z'' - Z'$. Boldism is an equivalence relation and $D^q(X,\Phi;Y,\Psi)$ is the group of equivalence classes with + induced by disjoint union.

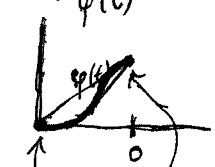
Problem is that if $Z_i \xrightarrow{f_i} X$ $\big(i = 1, 2\big)$ is smooth with common boundary $f_0 : Z_0 \to X$, then how to form $Z_1 \cup_{Z_0} Z_2 \xrightarrow{f_1 \cup_{f_0} f_2} X$ smooth? <u>Idea:</u> We know that near Z_0 have

$$\begin{tikzcd} Z_0 \times [0,1) \arrow[r, phantom, "\overset{\text{open}}{\subset}"] \arrow[d, "f_t"] & Z_1 \arrow[dl, "f_1"] \\ X & \end{tikzcd}$$

where f_t is a smooth homotopy starting with f_0

so slow down homotopy e.g.

$$t \longmapsto f_{\varphi(t)}$$

where 

∞ degree contact.

Doing this both for f_1 & f_2 one gets

$$\begin{tikzcd} Z_2 \supset Z_0 \times (-1,0] \arrow[dr, "f_t"'] & & Z_0 \times [0,1) \subset Z_1 \arrow[dl, "f_2"] \\ & X & \end{tikzcd}$$

OKAY

shows bordism is an equivalence relation.

~~Now suppose~~

<u>Theorem:</u> Assume Φ, Ψ contains the compact subsets of X, Y resp. Then

$$\lim_{N\to\infty} \left[S^{N-q} \wedge (X \cup_{\Phi} \{\infty\}), M(N) \wedge (Y \cup_{\Psi} \{\infty\}) \right] \cong D^{q}(X, \Phi; Y, \Psi).$$

<u>Proof:</u> Start with left side. Given

$$\alpha \in [S^{N-q} \wedge \qquad].$$

represent α by a map f which is smooth off $f^{-1}\{\infty\}$ and transversal to $B(N) \times Y$. (To simplify matters about bigness of $B(N), M(N)$ ~~...~~ suppose in framed case so $B(N) = pt.$ $M(N) = S^N$) This is legitimate; in effect ~~consider~~ take a cont. map

$$f : S^{N-q} \wedge (X \cup \infty) \longrightarrow M(N) \times (Y \cup \infty)$$

let $V = f^{-1}(E(N) \times Y)$; it is open in $\mathbb{R}^{N-q} \times X$. By smoothing theory I may approximate $f|V$ by a smooth map homotopic to f in fact differing from f by any amount which may be assumed to go to zero as one goes toward the boundary(?).

$$\begin{array}{ccc} V & \longrightarrow & E(N) \times Y \\ \cap \text{ open} & & \\ \mathbb{R}^{N-q} \times X & & \end{array}$$

to complete here wrt sets proper over Φ.

to complete here wrt closed sets proper over Ψ

$\leftarrow$ to complete here wrt closed sets proper over Φ

Check if $U \subset X$, Φ on X, then $\exists$ canon. map ~~...~~ $X \cup_{\Phi} \{\infty\} \longrightarrow U \cup \{\infty\}$ $\Phi \cap$ closed sets of U.

<u>Lemma:</u> If U open in X, Φ family of supports on X, Φ_U = those members of Φ contained in U, then $\exists$ cont. map

$$X \cup_{\Phi} \{\infty\} \xrightarrow{\ \ j\ \ } U \cup_{\Phi_U} \{\infty\}.$$

$$x \longmapsto \begin{cases} x & \text{if } x \in U \\ \infty & \text{if } x \notin U. \end{cases}$$

<u>Proof:</u> If V open in U, then $j^{-1}V = V$ open in X, hence in $X \cup_{\Phi} \{\infty\}$.

If $V = (U-F) \cup \{\infty\}$, is a nbd of ∞ in $U \cup \{\infty\}$, then

$$j^{-1}(V) = X - F \cup \{\infty\}. \qquad \text{open since } F \in \Phi_U \subset \Phi.$$

~~[illegible]~~

f ~~must~~ have property that f^{-1} carries ~~sets~~ Φ-proper sets in $E(N) \times Y$ ~~into~~ Φ-proper sets of V, and I have to be able to approximate f by a smooth $\tilde{f}$ with same property.

<u>Example.</u> Suppose X, Y manifolds, Φ, Ψ closed submanifolds of X, Y resp. $f: X \longrightarrow Y$ continuous such that $f^{-1}\Psi \subset \Phi$. Can f always be approximated by a smooth $\tilde{f}$ having same property? I don't know.

Lemma: Let X, Y be manifolds, let Φ, Ψ be families of supports for X and Y respectively containing the compact sets. Then any continuous map ~~is homotopic to a map smooth on the complement~~ $f: X \longrightarrow Y$ ~~is~~ such that $f^{-1}\{\Psi\} \subset \Phi$ is approximable.

Lemma: Let Φ, Ψ be families of supports for manifolds X, Y respectively. Assume Φ, Ψ contain the compact subsets. ~~Then any continuous map $f: X \to Y$ Let $Map(X,$ let~~ Then

$$C^\infty(X_\Phi, Y_\Psi) \longrightarrow C^0(X_\Phi, Y_\Psi)$$

is a homotopy equivalence.

~~Proof / Density~~.

Smoothing in general

Linear theory.

~~X manifold, Φ as usual containing the compacts~~

~~f continuous $f: X \to V$ V vector space~~

~~f bounded on each member of Φ. (~~

X compact manifold

A closed subset

$f: X \longrightarrow V$ vector space

f continuous smooth on A (i.e. restriction of a smooth fn.)

then f uniform limit of smooth fns. coinciding with f on A.

May assume $f=0$ on A.

First suppose $A = \emptyset$. Then set

$$\tilde{f}(x) = \int k_\varepsilon(x,y)\, f(y)\, dy$$

where $k_\varepsilon(x,y)dy$ smooth on $X \times X$ and in limit approaches Δ.

~~compact smooth~~

Properties: $k_\varepsilon(x,y)\,dy \in$ form of degree n on $X \times X$ with support in a nbd. of Δ. Also want

$$\int k_\varepsilon(x,y)dy = 1 \qquad \text{all } x.$$

probably also want $k_\varepsilon(x,y)\,dy$ to be closed ?

Therefore if $k_\varepsilon(x,y)\,dy = U_\varepsilon$ is a Thom class within an ε nbds of Δ then for any distribution f have

$$\tilde{f}_\varepsilon = (pr_1)_* (U_\varepsilon \cdot pr_2^* f)$$

is smooth and $\lim_{\varepsilon \to 0} \tilde{f}_\varepsilon = f$

Nice to have U_ε closed as then

$$d\tilde{f}_\varepsilon = (\widetilde{df})_\varepsilon .$$

but not essential.

~~Now given a closed set A, want~~

General smoothing:

$$X \xrightarrow{f} Y \subset \mathbb{R}^n.$$

take tubular nbd. $N_\varepsilon(Y) \xrightarrow{\pi} Y$.

then
$$\pi f_\varepsilon : X \longrightarrow Y$$

is a smooth approximation to f, moreover if $f_\varepsilon = f$ on A then same for πf_ε!!!

Closed set $A \subset X$

then take smooth approximation of δ_Δ outside of A.

$k_\varepsilon(x,y)\, dy$

We have a fn. f $= 0$ on A.

thus take $\rho(x,x)\delta_\Delta$ where $\rho = 0$

Let X and Y be C^∞ manifolds let Φ, Ψ be families of supports on X and Y, respectively. Let X_Φ be the space $X \cup \{\infty\}$, where the open sets are those of X and those of the form $(X-F)\cup\{\infty\}$ where $F \in \Phi$. Similarly define Y_Ψ. To calculate

$$\{\bar{S}^q \wedge X_\Phi,\ M \wedge Y_\Psi\}.$$

Start with

$$\begin{tikzcd}
\alpha : S^{n-q} \wedge X_\Phi \arrow[r] & M(n) \wedge Y_\Psi \\
\cup & \cup \\
R^{n-q} \times X & E(n) \times Y \\
\cup & \\
V \arrow[uur, "\alpha'"] & B(n) \times Y \arrow[uu, hook, "\text{cl or dim } n"'] \\
Z^{x-q} \arrow[u, hook, "\text{cl or dim } n"] \arrow[ur, "\alpha''"] &
\end{tikzcd}$$

one-point completion of $R^{n-q} \times X$ wrt closed sets proper over a member of Φ

$\alpha^{-1}(E(n)\times Y)$

one-point completion of $E(n)\times Y$ wrt closed sets proper over a members of Ψ

Assume for the moment that α may be chosen smooth on $\alpha^{-1}(E(n)\times Y)$ and transversal to $B(n)\times Y$.* Then we get

$$\begin{tikzcd}
& Z \arrow[dl, "f"'] \arrow[dr, "g"] & \\
X & & Y
\end{tikzcd}$$

where f is oriented of dimension q, and where $F \in \Psi \Longrightarrow$ $g^{-1}F$ is proper over a member of Φ.

---

* found condition that $\exists$ nbd N of Δ_Y in $Y \times Y \ni$ $F \in \Psi$ $\Longrightarrow$ $N * F = \{y \mid \exists y' \in F,\ (y, y') \in N\} \in \Psi$.

Conversely given $Z \xrightarrow{(f,g)} X \times Y$ f oriented dim q, $g^{-1}\Phi$ proper over Φ
choose embedding $Z \longrightarrow \mathbb{R}^{n-q}$ and form diagram

$$\begin{array}{ccc}
Z & \xrightarrow{\quad \alpha \quad} & B(n) \times Y \\
\downarrow & & \downarrow \\
V & \xrightarrow{\quad \alpha' \quad} & E(n) \times Y \\
\downarrow & & \\
\mathbb{R}^{n-q} \times X & &
\end{array}$$

We know that α carries proper$/\Phi$ into proper $/\Phi$. Is this also true for α'? Must know that for some (closed) tube Q around Z within V that $Q \cap f'^{-1}F$ is proper over a member of Φ, where $f' = pr_2\alpha' : V \longrightarrow Y$. Example $q = n$. Then $Z \hookrightarrow X$ and we know that $g^{-1}\Phi \subset \Phi$. But we must also know that we must be able to find a closed tube N around Z so that $\pi^{-1}g^{-1}\Phi \subset \Phi$ where $\pi : N \longrightarrow X$ is the normal projection.

<u>Conclusion:</u> The object $M \wedge X_{\Phi}$ where $X_{\Phi} = X \cup_{\Phi} \{pt\}$ is unreasonable except when Φ possesses good properties relative to smoothing.

Proposition:
$$\begin{cases} f: X \longrightarrow Y & \text{cont} \\ f^{-1}\Psi \subset \Phi \end{cases} \qquad \Phi, \Psi \text{ contain all compacts}$$

assume $\exists$ nbd. Q of Δ_Y in $Y \times Y$ such that

$$\left.\begin{array}{l} F \in \Phi \Longrightarrow Q * F = \{ y \mid \exists y' \quad y' \in F,\ (y, y') \in Q \} \\ \qquad \text{then } \overline{Q * F} \in \Psi \end{array}\right\}$$

Then f is a ~~uniform~~ limits of smooth functions f_n with $f_n^{-1}\Psi \subset \Phi$.

Proof: Endow Y with a complete Riemannian metric so that balls of radius ≤ 1 are convex + compact and so that $Q \subset \{(y, y') \mid \text{dist}(y, y') \leq 1\}$. Then set

$$f_\varepsilon(x) = \int k_\varepsilon(x, x') f(x')\, dx' \qquad \left(\begin{array}{l}\text{center of gravity} \\ \text{which exists by} \\ \text{convexity}\end{array}\right)$$

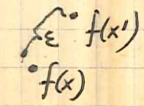
where $k_\varepsilon(x, x')\,dx'$ is a smooth form on $X \times X$ with support in $\{(x, x') \mid \text{dist}(fx, fx') \leq \varepsilon\}$ and $\ni$ $k_\varepsilon(x, x') \geq 0$ ~~and~~

$$\int k_\varepsilon(x, x')\, dx' = 1 \qquad \text{all } x$$

and $\lim\limits_{\varepsilon \to 0} k_\varepsilon(x, x')\, dx' = \delta(x, x')$.

Then f_ε is smooth and $\lim\limits_{\varepsilon \to 0} f_\varepsilon = f$

In fact $\text{dist}(f_\varepsilon(x), f(x)) \leq \varepsilon$



Have to show that

$$f_\varepsilon^{-1}\underline{\Phi} \subset \underline{\Phi}.$$

But

$$f_\varepsilon(x) \in F \in \underline{\Phi} \implies f(x) \in \overline{Q * F} \in \underline{\Phi}$$

$$\therefore \quad f_\varepsilon^{-1}(F) \subset f^{-1}\left(\overline{Q * F}\right) \in \underline{\Phi}$$

$$\therefore \quad f_\varepsilon^{-1}\underline{\Phi} \subset \underline{\Phi}.$$

QED.

Check that hypotheses on $\underline{\Phi}$ can't be improved:

Observe: if

$$f_\varepsilon(x) = \int k_\varepsilon(x, x')\, f(x')\, dx'$$

then all we can conclude is that

$$\left\{ \left(f_\varepsilon(x), f(x) \right) \,\middle|\, x \in X \right\}$$

is contained in some nbd^Q of Δ in $Y \times Y$. Thus given an estimate on how close f_ε is to f we must conclude that

$$f_\varepsilon^{-1} F \subset \underline{\Phi}$$

Thus from $\operatorname{dist}\{f_\varepsilon(x), f(x)\} \leq \varphi(x)$ must conclude $f_\varepsilon^{-1} F \subset \underline{\Phi}$.

i.e.

$$f_\varepsilon(x) \in F$$

$$\Downarrow$$

$$\rho\left(f(x), F\right) \leq \varphi(x) \implies x$$