Fall 68

## §1. Fundamental groupoids, singular cx, local coeff systems



If $C$ is a small category we define ~~a simplicial comp set~~ its singular complex $\operatorname{Sing} C$ to be the simplicial set whose $n$-simplices are diagrams

$$x_0 \to x_1 \to x_2 \cdots\cdots \to x_n$$

of maps in $C$ and with evident simplicial operations. Denoting by (Grpd) and (Cat) the categories of all small groupoids and small categories we have fully faithful functors

$$(\text{Grpd}) \xhookrightarrow[i]{} (\text{Cat}) \xrightarrow[\text{Sing}]{} (\text{s Sets}).$$

Both of these functors admit left adjoints which may be described as follows. The left adjoint to $i$ takes a category $C$ into the localization $S^{-1}C$, where $S$ is the family of all morphisms in $C$. We denote $S^{-1}C$ by $\pi(C)$ and call it the <u>fundamental groupoid</u> of $C$. The left adjoint of Sing takes a simplicial set $X$ into the category whose set of objects is $X_0$ and whose morphisms are the set of chains of 1-simplices modulo the equivalence relations on these chains coming from $X_2$. (Note: $i$ has a right adjoint but Sing does not).

The left adjoint to $\operatorname{Sing} \circ i$ will also be denoted by $\pi$, and if $X$ is a simplicial set $\pi(X)$ will be called the fundamental groupoid of $X$.

By a <u>local coefficient</u> system on a category $C$ (resp.

(with values in a category $\mathcal{A}$)

simplicial set $X$) we mean a covariant functor $F: C \longrightarrow \mathcal{A}$ such that for every morphism $u$ in $C$ we have that $F(u)$ is an isomorphism (resp. of local coefficient system on the category $\Delta/X$ of maps $\Delta[n] \longrightarrow X \quad n \geq 0$). It is easily seen that a local coefficient system on $C$ is the same as one on Sing $C$ and the same as a functor $\pi C \longrightarrow \mathcal{A}$.

~~Definition: If X is a simplicial set, then a covariant~~
~~A covariant system (resp. contravariant) system of coefficients with values in a category $\mathcal{A}$ on a simplicial set $X$ is a covariant (resp. contravariant) functor from $\Delta/X$ to $\mathcal{A}$.~~

~~Notation:~~

## §2. $D_{lc}(\mathcal{C})$ and the Whitehead theorem

Defn: $\mathcal{C}$ category (small)

$D(\mathcal{C})$ = derived category of $\mathrm{Hom}(\mathcal{C}, Ab)$

$D_{lc}(\mathcal{C})$ = full subcategory of $D(\mathcal{C})$ consisting of complexes whose homology groups are locally constant.

~~$D_{lc}(\mathcal{C})$ [illegible]~~

Given $\mathcal{C}_1 \xrightarrow{f} \mathcal{C}_2$

have

$$\begin{tikzcd} \mathrm{Hom}(\mathcal{C}_1, Ab) \arrow[r, shift left=2, "f_!"] \arrow[r, shift right=2, "f_*"'] & \mathrm{Hom}(\mathcal{C}_2, Ab) \arrow[l, "f^*"'] \end{tikzcd}$$

where $f_*$ and $f_!$ are Kan adjoints.

Moreover $\underline{\underline{L}} f_!$ exists on $D^-$

$\underline{\underline{R}} f_*$ exists on $D^+$

Clearly with $f^*$ $D_{lc}(\mathcal{C})$ is a contravariant functor of $\mathcal{C}$.

<u>Whitehead theorem</u>: Let $f : \mathcal{C}_1 \to \mathcal{C}_2$ be a ~~functor~~ morphism of small categories.

TFAE:

(i) $f^* : D^b_{lc}(\mathcal{C}_2) \longrightarrow D^b_{lc}(\mathcal{C}_1)$ equivalence

(ii) $\pi(f) : \pi(\mathcal{C}_1) \longrightarrow \pi(\mathcal{C}_2)$ equivalence + $\forall\ L \in LC(\mathcal{C}_2)$

$$R^q \varprojlim L \xrightarrow{\sim} R^q \varprojlim f^* L$$

(iii) $\left\{ \begin{array}{l} \forall \text{ set } S \text{ regarded as a discrete category} \quad \mathrm{Hom}(\mathcal{C}_2, S) \xrightarrow{\sim} \mathrm{Hom}(\mathcal{C}_1, S) \\ \forall \text{ gp. } G \text{ regarded as a category} \quad \mathrm{Hom}(\mathcal{C}_2, G) \xrightarrow{\sim} \mathrm{Hom}(\mathcal{C}_1, G) \\ \forall\ L \in Lc(\mathcal{C}_2) \quad R^q \varprojlim L \xrightarrow{\sim} R^q \varprojlim f^* L \end{array} \right.$

still needs proof.

Proof: (i) $\Rightarrow$ (ii). For any category $\mathcal{C}$ the category $\mathrm{Hom}(\pi\mathcal{C}, \mathcal{A}b)$ of local coefficient systems on $\mathcal{C}$ is embedded in $D^b_{lc}(\mathcal{C})$ as the complexes with at most one non-vanishing homology group in dimension zero. It follows from (i) then that $f^*: \mathrm{Hom}(\pi\mathcal{C}_2, \mathcal{A}b) \to \mathrm{Hom}(\pi\mathcal{C}_1, \mathcal{A}b)$ is an equivalence of categories. Now for any category $\mathcal{C}$, $\mathcal{C}$ may be recovered from $\mathrm{Hom}(\mathcal{C}, \mathcal{A}b)$ as the category of exact lim comp. tensor functors from $\mathrm{Hom}(\mathcal{C}, \mathcal{A}b)$ to $\mathcal{A}b$ (i.e. functors $\varphi$ together with isomorphisms $\varphi(F \otimes G) \cong \varphi(F) \otimes \varphi(G)$ which are coherent.)

But this implies that $\pi\mathcal{C}_1 \longrightarrow \pi\mathcal{C}_2$ is an equivalence. In effect by consideration of constant functors one sees that $\pi_0\mathcal{C}_2 \to \pi_0\mathcal{C}_1$ is bijective; by separate consideration of each component of $\mathcal{C}_2$ one reduces to the case where $f: \mathcal{C}_1 \to \mathcal{C}_2$ is a morphism of groups $f: G \to H$. One thus has to show that if ($H$ modules) $\longrightarrow$ ($G$ modules) are equivalence then $f$ is an isomorphism: $f$ surjective because $\mathrm{rank}_{\mathbb{Z}}(\mathbb{Z}[G/H])^G = 1$ and $\mathrm{rank}_{\mathbb{Z}}(\mathbb{Z}[G/H])^H \geq 2$ if $fH < G$. $f$ injective because $\mathrm{Ker}\, f$ acts trivially on any $G$ module. ($f^*$ fully faithful $\Longleftrightarrow$ $f$ surjective; $f^*$ equiv $\Leftrightarrow$ $f$ bijective).

Next note that
$$R^q \varprojlim_{\mathcal{C}} L = \mathrm{Ext}^0_{D^b_{lc}(\mathcal{C})}(\mathbb{Z}, L[q])$$
so that the isomorphism $R^q \varprojlim_{\mathcal{C}_2} L \simeq R^q \varprojlim_{\mathcal{C}_1} LF$ follows from fully faithfulness of $f^*$ on $D^b_{lc}$.

(ii) $\Rightarrow$ (i). First we show that $f^*$ is fully faithful. If $K, L \in D^b_{lc}(C)$ then there are spectral sequences

$$E_2^{pq} = \mathrm{Ext}^p(H^{-q}(K), L) \Rightarrow \mathrm{Ext}^{p+q}(K, L)$$

$$\mathrm{Ext}^p(K, H^q(L)) \Rightarrow \mathrm{Ext}^{p+q}(K, L)$$

where Ext denotes homomorphisms in the derived category. This reduces us to the case where $K, L$ are local coefficient systems. There is also a spectral sequence

$$E_2^{pq} = R^p\Gamma_C(\underline{\mathrm{Ext}}^q_C(K, L)) \Rightarrow \mathrm{Ext}_C^{p+q}(K, L)$$

where $\underline{\mathrm{Ext}}^*(K, \bullet)$ are the derived functors $\underline{\mathrm{Hom}}_C(K, \cdot)$.

CLAIM that as $K$ is a local coefficient system

$$\underline{\mathrm{Ext}}^q_C(K, L)(x) = \mathrm{Ext}^q_{ab}(K(x), L(x))$$

The point is that

$$\underline{\mathrm{Hom}}_C(K, L)(x) = \mathrm{Hom}_{C/x}(K, L) = \mathrm{Hom}_{ab}(K(x), L(x))$$

since $K$ is a local coefficient system
and $L$ inj $\Rightarrow$ $L(x)$ inj. since $(i_x)_!(A)(y) = \bigoplus_{x \to y} A$ exact

In particular if $K, L$ are both local coeff systems so is $\underline{\mathrm{Ext}}^q(K, L)$. Thus we are reduced to the case of proving $f^*$ induces an isomorphism on cohomology of local coefficient systems which is the second part of (ii).

Finally $f^*$ is essentially surjective since its image contains all the local coefficient systems on $C_2$ and these generate by successive triangles all of $D^b_{lc}$.

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<u>Corollary:</u> If $f: X \longrightarrow Y$ is a map of simplicial sets, then $f$ is a weak equivalence iff $f^*: D_{lc}(\Delta/X) \longleftarrow D_{lc}(\Delta/Y)$ is an equivalence of categories.

<u>Proof:</u> Use the criterion of Artin–Mazur for a map of simplicial sets to be a weak equivalence (HA, — ).

## §3. Review of work of André

If $X$ is a simplicial set and $F : (\Delta/X) \longrightarrow \mathcal{A}b$ then by $C^*(X, F)$ we mean the cosimplicial abelian group

$$C^q(X,F) = \prod_{x \in X_q} F(x)$$

if $\varphi : [p] \longrightarrow [q]$

$$\begin{tikzcd} C^p(X,F) \arrow[r, "\tilde{\varphi}_*"] \arrow[d, "\pi_{x\tilde{\varphi}}"] & C^q(X,F) \arrow[d, "\pi_x"] \\ F(x\tilde{\varphi}) \arrow[r, "F(\varphi)"] & F(x) \end{tikzcd} \qquad \begin{array}{l} x \in X_q \\ x\varphi \in X_p \end{array} \qquad \begin{tikzcd} \Delta(q) \arrow[dr, "x"'] & \Delta(p) \arrow[l, "\varphi"'] \arrow[d, "x\varphi"] \\ & X \end{tikzcd}$$

More simply $\quad C^*(X,F) : \Delta \longrightarrow \mathcal{A}b$

is $\quad f_*(F) \quad$ where $\quad f : \Delta/X \longrightarrow \Delta/\Delta(0) = \Delta$.

and where by our convention if $f$ is a functor then $f_!$ and $f_*$ are its Kan extensions.

<u>Prop 1:</u> $\quad R^q \varprojlim_{\Delta/X} F = \check{H}^q C^{\cdot}(X,F)$

<u>Proof:</u> $\quad \varprojlim_{\Delta/X} F = \check{H}^0 f_*$. $\quad f_*$ exact, and it carries injectives in $\underline{Hom}(\Delta/X, \mathcal{A}b)$ into injectives in $Hom(\Delta, \mathcal{A}b)$ since it has an exact left adjoint $f^*$. Thus $R^q \varprojlim = (R^q \check{H}^0) f_*$ by the spectral sequence of composite functors. Finally $R^q \check{H}^0 = \check{H}^q$ as one sees by either Dold-Puppe or effaceability (using that $\check{H}^+ C^{\cdot}(\Delta(n), A) = 0$.)

If $\mathcal{C}$ is a category then we may define a functor

$$\Delta/\mathrm{Sing}\,\mathcal{C} \xrightarrow{\ \gamma\ } \mathcal{C} \qquad (\text{resp. } \xrightarrow{\ \gamma^\circ\ } \mathcal{C}^\circ)$$

by sending a $n$-simplex $x_0 \to \cdots \to x_n$ into its last vertex $x_n$ (resp. first vertex $x_0$). In this way given $F : \mathcal{C} \to \mathcal{A}b$ (resp $F : \mathcal{C}^\circ \to \mathcal{A}b$) we can form the groups $H^q(\mathrm{Sing}\,\mathcal{C}, F\gamma)$ (resp $H^q(\mathrm{Sing}\,\mathcal{C}, F\gamma^\circ)$) which by prop. 1 are $R^q\varprojlim$'s

Prop. 2:

(i) If $F \in \mathrm{Hom}(\mathcal{C}, \mathcal{A}b)$ then $\gamma^* : R^q \varprojlim_{\mathcal{C}} F \xrightarrow{\ \sim\ } R^q \varprojlim_{\Delta/\mathrm{Sing}\,\mathcal{C}} F\gamma$

(ii) If $F \in \mathrm{Hom}(\mathcal{C}^\circ, \mathcal{A}b)$ then $\gamma^{\circ *} : R^q \varprojlim_{\mathcal{C}^\circ} F \xrightarrow{\ \sim\ } R^q \varprojlim_{\Delta/\mathrm{Sing}\,\mathcal{C}} F\gamma^\circ$

Proof: (i). Check that $\gamma^*$ is an isomorphism in dimension zero and that the right sides is effaceable as a functor of $F \in \mathrm{Hom}(\mathcal{C}, \mathcal{A}b)$. Enough to show that if $i_x : e \to \mathcal{C}$, then $R^q \varprojlim_{\Delta/\mathrm{Sing}\,\mathcal{C}} [(i_x)_* A]\gamma = 0$ for any abelian group $A$ + object $x$. ~~But by composite spectral sequence this is same as $R^q\varprojlim$~~ By proposition 1 have to calculate $\check{H}^+ C^\cdot(\mathrm{Sing}\,\mathcal{C}, (i_x)_* A^\gamma)$

$$C^\cdot(\mathrm{Sing}\,\mathcal{C}, (i_x)_* A^\gamma) = \prod_{x_0 \to \cdots \to x_q} \prod_{x_q \to x} A$$

and this gives $0$ by the cone construction.
(ii) similar.

$$\text{\underline{Cor 1}:}\quad \begin{tikzcd} D^b_{lc}(\Delta/\mathrm{Sing}\,\mathcal{C}) & D^b_{lc}(\mathcal{C}) \arrow[l, "\sim"'] \\ & D^b_{lc}(\mathcal{C}^\circ) \arrow[ul, "\wr"] \end{tikzcd}$$

$$\text{\underline{Cor 2}:}\qquad D^b_{lc}(\Delta/X) \simeq D^b_{lc}((\Delta/X)^\circ)$$

Remark: ~~[illegible]~~

A. Corollary 2 was the original problem which motivated this investigation. In effect ~~[illegible]~~ $D^b_{lc}(\Delta/X)$ was suggested by Deligne's descent theory for objects in a derived category as a good candidate for study. In effect $\mathrm{Hom}((\Delta/X), \mathcal{A}b)$ is the category of simplicial sheaves over $X$ regarded as a simplicial (discrete) topological space. On the other hand $\mathrm{Hom}((\Delta/X)^\circ, \mathcal{A}b)$ appears natural from André's work

B. Putting $\mathcal{C} = \Delta/X$ in corollary 1 we obtain an ~~[illegible]~~ equivalence $D^b_{lc}(\Delta/\mathrm{Sing}(\Delta/X)) \simeq D^b_{lc}(\Delta/X)$ which ~~[illegible]~~ suggests that the simplicial sets $\mathrm{Sing}\,\Delta/X$ and $\Delta/X$ have the same (weak) homotopy types. We shall prove this in the following section.

## §4. Every simplicial set is homotopy equivalent to the singular complex of a category.

Theorem: If $X$ is a simplicial set, then $X$ is isomorphic in the homotopy category to $Sing(\Delta/X)$.

Proof: ~~Consider the categories $\Delta$ and $\Delta^d$ the discrete subcategory~~

Let $\Delta^d$ be the (discrete) category whose objects are $[n]$ $n \in \mathbb{Z}$ but where the morphisms are just identity maps, and let $i : \Delta^d \to \Delta$ be the inclusion functor. Then we get a standard resolution of any simplicial set from the (adjoint) functors

$$\mathrm{Hom}(\Delta^d, \text{sets}) \underset{i^*}{\overset{i_!}{\rightleftarrows}} \mathrm{Hom}(\Delta, \text{sets}).$$

Let

$$\cdots \; P_1 \rightrightarrows P_0 \to X$$

be the standard resolution so that

$$P_k = \coprod_{x_0 \to \cdots \to x_k \,\in\, Sing_k(\Delta/X)} \Delta[\dim x_0]$$

Thus $\pi_0^v(P.) = Sing(\Delta/X)$. We are going to show that the maps

$$\begin{tikzcd} & \Delta P \arrow[dl] \arrow[dr] & \\ \pi_0^v(P) & & \pi_0^h(P) = X \end{tikzcd}$$

are weak homotopy equivalences. We shall use the Whitehead theorem: to show isom $H^0( \,, S)$ any set $S$, $H^1(X, G)$ any gp $G$,

$H^*(X, L)$ any local coefficient system $L$.

So now if $S$ is a set then

$$Y = \mathrm{Hom}(P, S)$$

is a bi-cosimplicial set for which one always has

$$H^0(\Delta Y) = H^0_v H^0_h(Y)$$

i.e.
$$H^0(\Delta P, S) = H^0(X, S)$$

If $G'$ is a group then

$$G = \mathrm{Hom}(P, G')$$

is a bi-cosimplicial group. Recall that if $G$ is a cosimplicial gp we define $H^1(G) = Z^1(G)$ module equivalence where

$$Z^1(G) = \{\alpha \mid G_1 \ni (\partial_0\alpha)(\partial_1\alpha)^{-1}(\partial_2\alpha) = 1\}$$

& $\quad \alpha' \sim \alpha \iff \exists \beta \in G_0$ with $\alpha' = (\partial_1\beta)\alpha(\partial_0\beta)^{-1}$

Lemma: If $G$ is a bi-cosimplicial group, then there is an exact sequence of pointed sets

$$0 \longrightarrow H^1_v H^0_h G \xrightarrow{\psi} H^1(\Delta G) \xrightarrow{\varphi} H^0_v H^1_h(G)$$

Proof: Given $\alpha \in G_{11}$ $\ni$

(*)
$$\partial^h_1 \partial^v_1 \alpha = (\partial^h_2 \partial^v_2 \alpha)(\partial^h_0 \partial^v_0 \alpha)$$

applying $(\sigma^v_0)^*$ to both sides we get

(2)
$$\partial^h_1 \alpha = \partial^h_2(\partial^v_1 \sigma^v_0 \alpha) \cdot \partial^h_0 \alpha$$

Apply $\sigma^v_0$ again we find that $\sigma^v_0 \alpha \in Z^1_h(G)'$. Clearly

$\partial_0^v(\sigma_0^v\alpha) = \partial_1^v(\sigma_0^v\alpha)$ so $\sigma_0^v\alpha$ defines an element of $H_v^0 H_h^1(G)$, which one easily checks depends only on the equivalence class $cl\alpha$ of $\alpha$; This is how $\varphi$ is defined. If $\varphi(cl\alpha) = *$, then $\exists \beta \in G_{00}$ with

$$\sigma_0^v\alpha = \partial_1^h\beta(\partial_0^h\beta)^{-1}$$

Replacing $\alpha$ by the equivalent cycle

$$\alpha' = (\partial_1^h\partial_1^v\beta)^{-1}\alpha(\partial_0^h\partial_0^v\beta)$$

we may assume that

(3) $$\sigma_0^v\alpha' = 1.$$

~~This shows~~ Plugging this in (2) + applying $\sigma_1^h$ we get

$$\alpha = \partial_0^h\sigma_0^h\alpha$$

where just as we saw for $\sigma_0^v\alpha$, $\sigma_0^h\alpha \in$ ~~[illegible]~~ $Z_v^1(G)'$ in fact $\sigma_0^h\alpha \in Z_v^1 H_h^0(G)$. Conversely given $\beta \in Z_v^1 H_h^0(G)$ we may set $\alpha = \partial_0^h\beta$ and get an element of $H^1(\Delta G)$. This defines the map $\psi$ + we have shown that $\mathrm{Im}\,\psi = \mathrm{Ker}\,\varphi$.

To see that $\psi$ is well-defined ~~injective suppose $\beta', \beta \in Z_v^1 H^0$~~ note that if $\beta' = (\partial_1^v\gamma)(\beta)(\partial_0^v\gamma)^{-1}$ $\partial_0^h\gamma = \partial_1^h\gamma$ then

$$\partial_0^h\beta' = (\partial_1^h\partial_1^v\gamma)(\beta)(\partial_0^h\partial_0^v\gamma)^{-1}.$$

To see that $\psi$ is injective suppose that

$$\partial_0^h\beta' = (\partial_1^h\partial_1^v\gamma)\,\partial_0^h\beta\,(\partial_0^h\partial_0^v\gamma)^{-1}$$

and apply $\sigma_0^h$ to get $\beta' = \partial_1^v\gamma\cdot\beta\cdot(\partial_0^v\gamma)^{-1}$, and applying $\sigma_0^v$

to get (may always assume $\beta, \beta' \in Z^1_v H^0_h(G)$ normalized)

$$1 = (\partial_1^h \gamma) \cdot (\partial_0^h \gamma)^{-1}$$

so that $\beta$ + $\beta'$ are cohomologous. QED.

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Applying this lemma to $G = \mathrm{Hom}(P^{\#}, G')$ we have that

$$H^1_h(G) = 0 \qquad H^0_h(G) = \mathrm{Hom}(X, G')$$

since $P$ horizontally is a resolution of $X$. Thus the lemma shows that

$$H^1(X, G') \simeq H^1(\Delta P, G')$$

implying that $\Delta P \longrightarrow X$ induces an isomorphism on fundamental groupoids.

Finally if $L$ is a local coefficient system on $X$, we find that for the bi-cosimplicial abelian gp $A = \mathbf{C}(P, L)$ that

$$H^0_h(A) = \mathbf{C}(X, L) \qquad H^+_h(A) = 0$$

since $P$ resolves $X$. Thus the spectral sequence of a bicosimplicial abelian group shows that

$$H^*(X, L) \xrightarrow{\sim} H^*(\Delta P, L)$$

Thus by the Artin-Mazur Whitehead theorem we find that

$$\Delta P \longrightarrow X$$

is a weak homotopy equivalence. By similar arguments

$$\Delta P \longrightarrow \mathrm{Sing}(\Delta / X)$$

is a weak homotopy equivalence.

QED

Let $X$ be a bisimplicial set. ~~To define its homotopy type~~

~~the total~~

Problem. Let $\nabla X$ be the diagonal. Then $(\Delta\times\Delta/X)$ and $\Delta/\nabla X$ have the same homotopy types.

There is a functor $f : (\Delta/\nabla X) \longrightarrow (\Delta\times\Delta/X)$ to which we want to apply the Artin–Mazur criterion. So let $F : (\Delta\times\Delta/X)^{*} \longrightarrow ab$ be given. Then if $p : (\Delta\times\Delta/X) \longrightarrow (\Delta\times\Delta)/pt.$ is the obvious map we have that $p_*$ is exact. (In effect

$$p_*(F)(u) = \varprojlim_{u \to pV} F(V)$$

where the limit is taken over

$$\begin{array}{ccc} V & \longrightarrow & X \\ \uparrow & & \downarrow \\ u & \longrightarrow & pt. \end{array}$$

Thus

$$p_*(F)(u) = \prod_{u \to X} F(u) \quad \text{which is exact})$$

Thus

$$R^q \varprojlim_{\Delta\times\Delta/X} F = R^q \varprojlim_{\Delta\times\Delta} (p_* F).$$

Now the functors $R^q \varprojlim_{\Delta\times\Delta}$ are easy to determine since $\underline{\mathrm{Hom}}((\Delta\times\Delta)^{\circ}, ab)$ is the category of bi-cosimplicial abelian groups, which by Dold-Puppe is equivalent to the category of double cochain complexes. The functor $\varprojlim_{\Delta\times\Delta}$ is the total $H^0_{tot}$. One sees then that the derived functors are $H^q_{tot}$ which by EZC is $H^q \circ \nabla$. Thus in fact we have shown that

$$R^q \varprojlim_{\Delta\times\Delta/X} F \xrightarrow{\sim} R^q \varprojlim_{\Delta/\nabla X} f^* F$$

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<u>Proposition</u>: Let $\pi$ and $A$ be abelian groups. Then

$$\mathrm{Hom}(\pi, A) = H^2(\pi, 2; A) \simeq \cdots$$
$$\mathrm{Ext}^1(\pi, A) = H^3(\pi, 2; A) \simeq H^4(\pi, 3; A) \simeq \cdots$$
$$\mathrm{Hom}(\Gamma_2 \pi, A) = H^4(\pi, 2; A)$$
$$0 \longrightarrow H^5(\pi, 3; A) \longrightarrow \mathrm{Hom}(\Gamma_2\pi, A) \longrightarrow \mathrm{Hom}(S_2\pi, A) \quad \text{exact.}$$

<u>Proof</u>: Let $K$ be a free simplicial abelian group of type $(\pi, 2)$. Then

$$H^q(\pi, 2; A) = \check{H}^q \, \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}K, A)$$

where $\mathbb{Z}K$ is the free abelian group generated by $K$ as a simplicial set. Now $\mathbb{Z}K = \mathbb{Z}[K] \oplus \mathbb{Z}$ where $\mathbb{Z}[K]$ is the group ring of the simplicial abelian group $K$, thus

$$H^q(\pi, 2; A) = \check{H}^q \, \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[K], A) \qquad q > 0.$$

~~As $K$ is $\mathbb{Z}$-free the augmentation ideal $I$ in $\mathbb{Z}K$ is regular so $\pi_q(I^{r+1}) = 0$ for $q \le r$ and so the filtration~~

$$\cancel{F_r = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}K/I^{r+1}, A)}$$

~~gives rise to a convergent spectral sequence.~~ ~~[illegible]~~ ~~Also~~

$$\cancel{I^r/I^{r+1} = S_r K \simeq \Sigma^r \Lambda_r \Omega K \simeq \Sigma^{2r} \Gamma_r \Omega^2 K}$$

~~showing that $\pi_q(I^r/I^{r+1}) = 0$ for $q < 2r$.~~ ~~[illegible]~~

~~$F_2 \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}K/I^3, A)$ induces isomorphisms on [illegible] $\check{H}^q$ for $q \le 5$~~

~~As $I^r/I^{r+1}$ is $\mathbb{Z}$ free we have exact sequence~~

~~$0 \to F_r \to F_{r+1} \to \mathrm{Hom}_{\mathbb{Z}}(I^r/I^{r+1}, A)$~~

Let $I$ be the augmentation ideal of $\mathbb{Z}[K]$; ~~so that~~ then $\mathbb{Z}[K] = \mathbb{Z} \oplus I$ and

$$H^q(\pi, 2; A) = \check{H}^q\{\mathrm{Hom}_{\mathbb{Z}}(I, A)\} \qquad q > 0$$

Set $Q = \mathrm{Hom}_{\mathbb{Z}}(I, A)$ and $F_r Q = \mathrm{Hom}_{\mathbb{Z}}(I/I^{r+1}, A)$. Now as $K$ is $\mathbb{Z}$-free, $I$ is regular in $\mathbb{Z}[K]$, hence

$$I^r/I^{r+1} = S_r K$$

and

$$\pi_q(I^{r+1}) = 0 \qquad \text{for } q \leq r,$$

by my convergence theorem. Thus from the exact sequence

$$0 \longrightarrow F_r Q \longrightarrow Q \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(I^{r+1}, A) \longrightarrow 0$$

and the fact that $I^{r+1}$ is $\mathbb{Z}$-free we see that $F_r Q \longrightarrow Q$ induces isos on $\check{H}^j$ for $j \ll r$. But

$$I^r/I^{r+1} = S_r K \sim \Sigma^r \Lambda^r \Omega K \sim \Sigma^{2r} \Gamma^r \Omega^2 K$$

and the exact sequence

$$0 \longrightarrow F_{r-1} Q \longrightarrow F_r Q \longrightarrow \mathrm{Hom}(I^r/I^{r+1}, A) \longrightarrow 0$$

show that $\check{H}^j F_{r-1} Q \xrightarrow{\sim} \check{H}^j Q$ for $j < 2r$. So in fact we get

$$H^2(\pi, 2; A) = \check{H}^2(F_1 Q) = \check{H}^2\,\mathrm{Hom}_{\mathbb{Z}}(K, A) = \mathrm{Hom}(\pi, A)$$

$$H^3(\pi, 2; A) = \check{H}^3\,\mathrm{Hom}_{\mathbb{Z}}(K, A) = \mathrm{Ext}^1(\pi, A)$$

$$0 \to \overset{0}{\check{H}^4\,\mathrm{Hom}(K, A)} \longrightarrow \check{H}^4 Q \longrightarrow \check{H}^4\,\mathrm{Hom}(\Sigma^4 \Gamma^2 \Omega^2 K, A) \to$$
$$\to \overset{0}{\check{H}^5\,\mathrm{Hom}(K, A)}$$

so

$$H^4(\pi, 2; A) = \mathrm{Hom}(\Gamma^2 \pi, A)$$

To calculate $H^*(\pi, 3; A)$ replace $K$ by a $K(\pi, 3)$. This time will find $\pi_j\, I^r/I^{r+1} = 0$ $\quad j \leq 2r$ so will find that $\check{H}^j F_r Q \xrightarrow{\approx} \check{H}^j Q$ for $j \leq 2r$. Thus for $r = 2$ we have

$$\check{H}^5 F_2 Q = \check{H}^5 Q$$

$$\underset{\substack{\| \\ \check{H}^5 \operatorname{Hom}(K,A) = 0}}{\check{H}^5 F_1 Q} \longrightarrow \check{H}^5 F_2 Q \longrightarrow \check{H}^5 \operatorname{Hom}(S_2 K, A) \longrightarrow \check{H}^6 \operatorname{Hom}(K, A) = 0$$

$$S_2 K \cong \Sigma^4 \Gamma^2 \Omega^2 K$$

$$H^5(\pi, 3; A) = \operatorname{Hom}(\pi_1 \Gamma^2 \Omega^2 K, A)$$

But if $V = \Omega^2 K = K(\pi, 1)$ we have

$$0 \longrightarrow \Gamma^2 V \longrightarrow V \otimes V \longrightarrow \Lambda^2 V \longrightarrow 0$$

$$\Lambda^2 V = \Sigma^2 \Gamma^2 \Omega V$$

so that

$$\pi \otimes \pi \longrightarrow \Gamma^2 \pi \longrightarrow \pi_1 \Gamma^2 V \longrightarrow 0$$

is exact. Thus

$$0 \longrightarrow H^5(\pi, 3; A) \longrightarrow \operatorname{Hom}(\Gamma^2 \pi, A) \longrightarrow \operatorname{Hom}(S^2 \pi, A)$$

is exact as asserted.

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Remarks: 1. It seems reasonable that the map $H^4(\pi, 2; A) \longrightarrow \operatorname{Hom}(S^2\pi, A)$ is given by associating to a space $X$ with homotopy groups $\pi = \pi_2 X$, $A = \pi_3 A$ the Whitehead product $\pi \times \pi \longrightarrow A$.

Additional calculations

$$\begin{tikzcd}
 & & s(x,y) \arrow[r, mapsto] & s(x,y)-s(y,x) & \\
0 \arrow[r] & \mathrm{Ext}^1(\pi, A) \arrow[r] & H^2(\pi, 1; A) \arrow[r] & \mathrm{Hom}(\Lambda^2\pi, A) \arrow[r] \arrow[d, "\text{commutator or Whitehead product}"] & 0 \\
 & & & \mathrm{Hom}(\pi\otimes\pi, A) &
\end{tikzcd}$$

abelian ext. all ext. bilinear cocycles

shows that sequences split canonically if $2 : A \xrightarrow{\sim} A$.

---

$$0 \to H^3(S_2K; A) \longrightarrow H^3(\pi, 1; A) \longrightarrow \mathrm{Hom}(\Lambda^3\pi, A) \to H^4(\mathrm{Hom}(S_2K, A))$$

where $K = K(\pi, 1)$

$$H^3(S_2K; A) = H^1(\mathrm{Hom}(\Lambda_2 L, A)) \qquad H^4(\mathrm{Hom}(S_2K, A)) \cong H^2(\mathrm{Hom}(\Lambda_2 L, A))$$

where $L = K(\pi, 0)$ a free resolution of $\pi$

---

Problem: What is the map $H^4(\pi, 2; A) = \mathrm{Hom}(\Gamma_2\pi, A) \longrightarrow H^3(\pi, 1; A)$ ?

$\mathbb{Z}[\#(P)] \longrightarrow \mathbb{Z}[\#(A)]$ ~~completely clear~~!

~~So take an A~~

$$\begin{tikzcd}
\overline{W}(P) \arrow[r, "\text{w.eq. in general}"] \arrow[d] & \overline{W}(A) \arrow[d] \\
\overline{W}(\# P) \arrow[r] & \overline{W}(\# A)
\end{tikzcd}$$

weak equivalence since P's are free!

$$\boxed{E_2^{pq} = \operatorname{Ext}^p_{\mathbb{Z}}(\mathbb{L}\Lambda_q(A), G) \Longrightarrow H^{p+q}(A, G)}$$

$$\begin{array}{l} 0 \to E_2^{10} \longrightarrow H^1 \longrightarrow E_2^{01} \longrightarrow E_2^{20} \longrightarrow H^2 \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad \| \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad 0 \end{array}$$

---

*There should be a systematic procedure for handling these non-linear situations*

---

$\cancel{\operatorname{Ext}^1(V, \mathbb{Z}/2\mathbb{Z})}$

$$\begin{array}{ccccccc} 0 \longrightarrow & \operatorname{Ext}^1(V, \mathbb{F}_2) & \longrightarrow & H^2(V, \mathbb{F}_2) & \longrightarrow & \operatorname{Hom}(\Lambda_2 V, \mathbb{F}_2) & \\ & \| & & \| & & & \\ 0 \longrightarrow & V' & \longrightarrow & S^2 V' & \Longrightarrow & \Lambda^2 V' & \longrightarrow 0 \end{array}$$

doesn't split naturally

$$\boxed{E_2^{pq} = \mathrm{Ext}^p(\mathbb{L}\Lambda_q(A), G) \Longrightarrow H^{p+q}(A, G)}$$

A abelian group $\square$

$$\mathbb{Z}[A] \longrightarrow A$$

$$\sum n_i[a_i] \longmapsto \sum n_i a_i$$

Questions ~~Is~~ $\mathbb{Z}[A]$ a $\lambda$-ring

$\boxed{\text{Yes}}$ $R(\hat{A})$ $\hat{A} = \mathrm{Hom}(A, S^1)$ Pontryagin dual

---

To construct a category $C^{(ab)}$ ~~of the Lawvere type~~ endowed with a functor

$$C(A) \longrightarrow \mathrm{Lawv}(\ )$$

such that

(i) $\mathrm{Lawv}(A)$

$$\mathrm{Sing}\, C(A)$$

---

Point can't get a functor of form $\mathbb{Z} S$ where $S$ is a functor on $(ab)$ to s.sets $\to \mathbb{Z}S$ resolve $A$, otherwise get functorial

December 1, 1968

More motives:

Let $\mathcal{V}$ be category of compact $C^\infty$ unoriented manifolds $F : \mathcal{V} \longrightarrow \mathcal{M}$ the motive category. $\mathcal{M}$ is (a graded) ~~an~~ additive category where the sum of $f_* g^*$, $f'_* g'^*$ is

$$\begin{tikzcd} & U \arrow[dl, "g"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd} \qquad \begin{tikzcd} & V \arrow[dl, "g'"'] \arrow[dr, "f'"] & \\ X & & Y \end{tikzcd}$$

$$\begin{tikzcd} & U \amalg V \arrow[dl, "g+g'"'] \arrow[dr, "f+g'"] & \\ X & & Y \end{tikzcd}.$$

Proposition: (i) $F(X \amalg Y) \cong FX \oplus FY$

(ii) $F$ is universal for ~~[illegible]~~ cohomology functors $G : \mathcal{V} \longrightarrow \mathcal{A}$ into an additive category such that

$$G(X \amalg Y) \cong GX \oplus GY$$

---

Consider the universal category associated to ~~[illegible]~~ $\mathcal{V}^n$ where if $\underline{f} = (f_1, \ldots, f_n)$, then $\underline{f}_* = (f_{1*}, \ldots, f_{n*})$, $\underline{f}^* = (f_1^*, \ldots, f_n^*)$. The universal category is ~~[illegible]~~ $F^n : \mathcal{V}^n \longrightarrow \mathcal{M}^n$. Note $\mathcal{M}^n$ not additive as we can't form $(X_1, \ldots, X_n) \amalg (Y_1, \ldots, Y_n)$. The universal additive (cohomology theory) ~~[illegible]~~ is $X_1, \ldots, X_n \longmapsto FX_1 \otimes \cdots \otimes FX_n$ from $\mathcal{V}^n$ to $\mathcal{M}^{\otimes n}$ where if $\mathcal{A}$ and $\mathcal{B}$ are additive cats then $\mathcal{A} \otimes \mathcal{B}$ has $\mathcal{A} \times \mathcal{B}$ for objects and

$$\mathrm{Hom}_{\mathcal{A} \otimes \mathcal{B}}(A \otimes B, A' \otimes B') = \mathrm{Hom}_{\mathcal{A}}(A, A') \otimes \mathrm{Hom}_{\mathcal{B}}(B, B')$$

In particular have

$$\begin{tikzcd}
(X_1,\dots,X_n) & \mathcal{V}^n \arrow[r] \arrow[d] & \mathcal{V} \arrow[d, "F"] & X_1\times\dots\times X_n \\
FX_1\otimes\dots\otimes FX_n \arrow[rrr, bend right] & \mathcal{M}^{\otimes n} \arrow[r] & \mathcal{M} & F(X_1\times\dots\times X_n)
\end{tikzcd}$$

so I get a $\otimes$ functor on $\mathcal{M}$ by

$$FX\otimes FY = F(X\times Y).$$

---

<u>Consequences of the Künneth theorem:</u> $\quad R = \eta_*(pt.) = \mathrm{Hom}_{\mathcal{M}}(pt, pt.)$ comm. ring

<u>Proposition:</u> Suppose $X\in\mathcal{V}$ and $\Delta : X \longrightarrow X\times X$ belongs to the image of $\eta_*(X)\otimes_R \eta_*(X) \longrightarrow \eta_*(X\times X)$. Then

$$\eta_*(X) \simeq \mathrm{Hom}_{\mathcal{M}}(pt, X) \simeq \mathrm{Hom}_{\mathcal{M}}(X, pt.)$$

is a finitely generated projective $R$-module and

$$\mathrm{Hom}_{\mathcal{M}}(pt, X) \otimes_R \mathrm{Hom}_{\mathcal{M}}(X, pt) \longrightarrow R$$

is a perfect pairing. Moreover for any $Y\in\mathcal{V}$ (not. nec. compact)

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{M}}(X, pt.)\otimes \mathrm{Hom}_{\mathcal{M}}(pt, Y) \arrow[r, "\sim"] & \mathrm{Hom}_{\mathcal{M}}(X, Y) \arrow[d, "S"] \\
\cancel{\eta(X)\otimes\eta(Y) \to \eta(X\times X)} & \mathrm{Hom}_R(\eta_* X, \eta_* Y)
\end{tikzcd}$$

**Proof:** Recall isomorphism

$$\mathfrak{N}_*(X\times X) \xrightarrow{\#} \mathrm{Hom}_{\mathfrak{M}}(X,X)$$

$$\{U \xrightarrow{(f,g)} X\times X\} \longmapsto g_* f^*$$

given $\alpha : U \to X$ $\beta : V \to X$ then $\#(\alpha\times\beta) = \{\beta_* pr_{2*}\, pr_1^*\, \alpha^*$ $= \beta_* \pi_V^* \pi_{U*}^{\#} \alpha^* = \#\beta \circ \#\alpha$ where $\#\alpha \in \mathrm{Hom}_{\mathfrak{M}}(X, pt.)$ and $\#\beta \in \mathrm{Hom}(pt, X)$. Thus we get elements

$$\alpha_i, \beta_i \in \mathfrak{N}_*(X)$$

such that

$$x = \sum_{i=1}^{n} \beta_i \langle \alpha_i, x\rangle \qquad \text{all } ~~\text{for}~~ \; x \in \mathfrak{N}_*(X)$$

where $\langle \alpha_i, x\rangle \in R$ is the product ~~of~~ $\mathrm{Hom}_{\mathfrak{M}}(pt, X) \otimes \mathrm{Hom}_{\mathfrak{M}}(X, pt) \to R$. ~~Hence~~ This formulas show $\mathfrak{N}_*(X)$ is a retract of $R^n$ so is finitely generated and projective. The map

$$\begin{aligned} \mathfrak{N}_*(X) &\longrightarrow \mathrm{Hom}_R(\mathfrak{N}_*(X), R) \\ \alpha &\longmapsto \{x \longmapsto \langle \alpha, x\rangle\} \end{aligned}$$

is an ~~perfect pairing~~ isomorphism because: Injective: $x = \sum \beta_i \langle \alpha_i, x\rangle$
Surjective: Given $\lambda : \mathfrak{N}_*(X) \to R$ have

$$\lambda(x) = \sum \lambda(\beta_i) \langle \alpha_i, x\rangle$$

so $\lambda$ is a ~~sum of~~ linear combination of $\alpha_i$. Finally the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathfrak{M}}(X,pt)\otimes \mathrm{Hom}_{\mathfrak{M}}(pt,X)\otimes \mathrm{Hom}_{\mathfrak{M}}(X,Y) \arrow[d] \arrow[dr] & \\
\mathrm{Hom}_{\mathfrak{M}}(X,pt)\otimes \mathrm{Hom}_{\mathfrak{M}}(pt,Y) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_{\mathfrak{M}}(X,Y) \arrow[d] \\
\mathrm{Hom}_R(\eta X, R)\otimes_R \mathrm{Hom}(R, \eta Y) \arrow[r, "\sim"] & \mathrm{Hom}_R(\eta X, \eta Y)
\end{tikzcd}$$

where surjectivity comes from the fact that $id_X$ comes from $\mathrm{Hom}_{\mathcal{M}}(X, pt) \otimes \mathrm{Hom}_{\mathcal{M}}(pt, X)$. $\therefore$ ~~Ha~~

$$\mathrm{Hom}_{\mathcal{M}}(X, Y) \cong \mathrm{Hom}_R(\eta X, \eta Y).$$

---

Corollary: If Künneth holds, then $\mathcal{M}$ is a full subcategory of the finitely generated projective quadratic $R$-modules ~~with non degenerate~~.

---

~~Motives: therefore results~~

Problem: Where does the motive groupoid of operations enter into the theory, e.g. $A = \pi_*(MO \wedge MO)$?

Remarks: 1 Tate motive appears only when you worry about orientation e.g. must first orient fundamental class of $\mathbb{P}^1$.

Problem: Morphisms of motive categories.

# Motives (December 8, 1968)

## I. Cohomology with compact support.

For $H_c^*(X)$ have

$$\begin{cases} \text{integration } f_!: H_c^k(X) \longrightarrow H_c^{k+\dim Y-\dim X}(Y) & \text{if } f \text{ oriented} \\ \text{pull back } f^!: H_c^k(X) \longrightarrow H_c^k(Y) & \text{if } f \text{ proper} \end{cases}$$

This leads one to consider category $\mathcal{M}_c$ with

$$\mathrm{Hom}^q_{\mathcal{M}_c}(X,Y) = \text{bordism classes } U \xrightarrow{(f,g)} X \times Y \qquad : g_! f^!$$

$f$ proper, $g$ oriented where

$q = \dim g = \dim Y - \dim X$.

which is solution of universal problem with $f_!$ and $f^!$ instead of $f_*$ and $f^*$.

$$\eta_c^q(X) = \mathrm{Hom}_{\mathcal{M}_c}(pt, X) = \text{bordism classes } U \xrightarrow{g} X$$

$U$ compact $g$ oriented, $\dim g = q$

Canonical map

$$\eta_c^q(X) \longrightarrow \eta^q(X) \qquad \text{since } g \text{ is in particular proper.}$$

Duality thm: $\boxed{\eta_c^q(X) \xrightarrow{\sim} \eta_{n-q}(X) \quad \text{if } X \text{ oriented.}}$

Proof: $\eta_c^q(X) = [U \xrightarrow{g} X]$, $U$ compact, $g$ oriented, $\dim g = q$

$\eta_{n-q}(X) = [U \xrightarrow{f} X]$, $U$ compact oriented, $\dim U = n-q$

Index conventions:

$$\eta_q(X) = \mathrm{Hom}_m^{-q}(pt, X) = \{S^q, M\wedge X\} = \pi_q(M\wedge X)$$

$$= \{\text{bordisms } U \to X,\ U \text{ oriented comp. dim. } q\}$$

$$= H_q(X, M).$$

$$\eta^q(X) = \mathrm{Hom}_m^{q}(X, pt)$$

$$= \{S^{-q}X, M\} = H^q(X, M).$$

$$= \left\{\text{bordisms } [U \xrightarrow{g} X] \quad \begin{array}{l} g \text{ proper oriented} \\ \dim g = q \end{array}\right\}$$

$$\eta^q(X;Y) = \mathrm{Hom}_m^{q}(X,Y) = \{S^{-q}X, M\wedge Y\}$$

$$= \left\{\text{bordisms } [U \xrightarrow{(f,g)} X\times Y] \;\middle|\; f \text{ proper oriented, } \dim f = q\right\}$$

$$\longrightarrow \bigoplus \mathrm{Hom}(H_k(X), H_{k-q}(Y)) \longrightarrow \bigoplus \mathrm{Hom}(H^k(Y), H^{k+q}(X)).$$

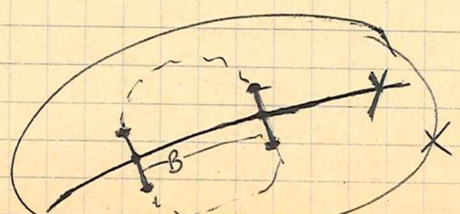
---

Advantages of compact support.

Lemma 1: Let $i: Y \to X$ be a closed submanifold, let $j: U \to X$ be the complement. Then there is a long exact sequence

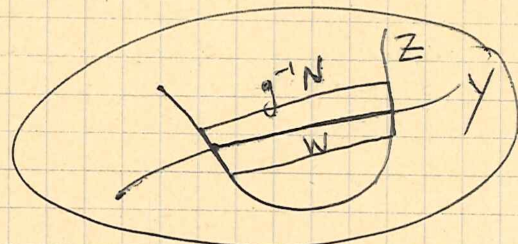
$$\cdots \to \eta_c^q(U) \xrightarrow{j!} \eta_c^q(X) \xrightarrow{i^!} \eta_c^q(Y) \xrightarrow{\delta} \eta_c^{q+1}(U) \to \cdots$$

Proof: defn. of $\delta$. Given $Z \xrightarrow{g} Y$ $Z$ compact, $g$ oriented. Let $B = \{(z,v) \mid v$ is a unit tangent vector to $g(z)$ perpendicular to $Y\}$ and map $B \to X$ by an exponential. Then $\delta[Z \to Y] = [\partial B \to X]$




If this is a boundary in $U$, then get a manifold $V$ in ~~X~~ meeting $Y$ transversally i.e. $i^! V = V \cap Y = Z$. Have to check orientation ✓

Exactness at $\eta_c^*(X)$. Given $Z \xrightarrow{f} X$ make transversal to $Y$. If $f^{-1}Y \longrightarrow Y$ is the boundary of $W \xrightarrow{g} Y$, consider the normal tubes of $i$ restricted to $W$, $g^{-1}N = \{(w,v) \mid w \in W,\ v$ normal to $Y$ ~~passing the~~ at $g(w)$




Then $g^{-1}N$ gives a surgery of $Z$ with a manifold $Z' \longrightarrow U$.

---

The above lemma is too geometric.

<u>Lemma 2</u>: (<u>Mayer - Vietoris</u>) $U, V$ open $\subset X$

$$\longrightarrow \eta_c^q(U \cap V) \longrightarrow \eta_c^q(U) \oplus \eta_c^q(V) \longrightarrow \eta_c^q(U \cup V) \xrightarrow{\delta} \eta_c^{q+1}(U \cap V)$$

<u>Proof</u>: Any manifold $W \subset U \cup V$ possibly with boundary may be split $W = W_1 \cup W_2$ where $W_1 \subset U$ and $W_2 \subset U$, by choosing a smooth function $0 \le f \le 1$ on $W$ with $f^{-1}[0,1] \subset V$, $f^{-1}[0,1) \subset V$ and taking a regular value near $\frac{1}{2}$.

<u>Remark</u>: Above lemmas hold for homology $\eta_q$ except that $i$ must be oriented in lemma 1.

---

<u>Suspension</u>: $$\eta_c^q(X) \xrightarrow{\simeq} \eta_c^{q+1}(X \times \mathbb{R})$$

<u>Thom isomorphism</u>: If $E \longrightarrow X$ is a vector bundle oriented dim $d$

$$0_! : \eta_c^q(X) \xrightarrow{\cong} \eta_c^{q+d}(E)$$

<u>Proof of Thom isomorphism:</u> a) Take limits on both sides then use Mayer-Vietoris + Weil to cover $X$ by convex balls.

b). Observe that $f \sim g \Rightarrow f_! = g_!$ because enough to check for

$$X \underset{i_1}{\overset{i_0}{\rightrightarrows}} X \times \mathbb{R}$$

which comes from ~~suspension~~ cylinder construction. Thus have

$\pi_! : \eta_c^{q+d}(E) \longrightarrow \eta_c^q(X)$ + $\pi_! 0_! = id$ $0_! \pi_! = id$.

---

<u>Künneth theorem:</u> If $\eta_c^*(Y)$ is flat as $R$-module, then

$$\eta_c^*(X) \otimes_R \eta_c^*(Y) \xrightarrow{\sim} \eta_c^*(X \times Y)$$

Same assertions for $\eta_*$.

☆ Make into spectral sequence.

---

<u>Problems:</u>

1. Find a spectral formula for $\eta_c^q(X)$, using $H_c^*(X) = \tilde{H}^*(X \cup \{\infty\})$

2. Can you find an axiomatic description of the following category $\mathcal{C}$ obtained by **piecing** $\mathfrak{M}_c$ and $\mathfrak{M}$ together?

Objects $\mathcal{C}$ = $\mathrm{Ob}\,\mathfrak{M}_c \amalg \mathrm{Ob}\,\mathfrak{M}$ denoted $X_c$, $X$ resp.

$$\begin{cases} \mathrm{Hom}_{\mathcal{C}}(X_c, Y_c) = \mathrm{Hom}_{\mathfrak{M}_c}(X, Y) \\ \mathrm{Hom}_{\mathcal{C}}(X, Y) = \mathrm{Hom}_{\mathfrak{M}}(Y, X) \\ \mathrm{Hom}_{\mathcal{C}}(X_c, Y) = \text{bordism classes of } (f,g) : U \longrightarrow X \times Y \\ \qquad (f,g) \text{ proper} \quad g \text{ oriented} \\ \mathrm{Hom}_{\mathcal{C}}(X, Y_c) = \text{bordism classes of } (f,g) : U \longrightarrow X \times Y \\ \qquad U \text{ compact}, \ g \text{ oriented} \end{cases}$$

Are you forced to enlarge type of functors?

3. bordism theory with singularities
   change of orientation morphism

4. Characteristic classes values in $\mathfrak{N}^*$, operations in $\mathfrak{N}^*$ via
   Thom isomorphism $\mathfrak{N}^*(MO) \simeq \mathfrak{N}^*(BO)$.

~~Generalization of 1-point compactification~~

[illegible] Let $c: X \to Y$ be a map of locally compact spaces; $Z = X \cup Y$ with topology given by

Neighborhoods: of a point $x$ same

of a point $y$ are things cont. $[(X-K) \overset{\cup Y}{\cap} [c^{-1}U \cup U]$

$U$ open of $Y$ containg $y$.

$K$ proper over $Y$.

finite ~~intersections~~ ✓

Hausdorff: to separate

$x \quad x'$ ✓

$x \quad y$ let $K$ be a compact nbd of $x$ in $X$

then $(\text{Int } K) \cap [(X-K) \cup Y] = \phi$

$y \quad y'$ Let $U, V$ be dis. open nbds of $y, y'$ in $Y$

$[c^{-1}U \cup U] \cap [c^{-1}V \cup V] = \phi.$

✓

$\bar{c} = c + id: Z \longrightarrow Y$ proper:

Let $B$ be compact in $Y$; claim $c^{-1}B \cup B$ compact.

Let $\mathfrak{U}$ be a covering by nbd as described. $B$ compact have

$i = 1, \dots, n \quad [(X-K_i) \cup Y] \cap (c^{-1}U_i \cap U_i) \qquad B \subset \bigcup_{i=1}^{n} U_i$

let $K = \bigcup K_i$. Then we have covered all but

$$(c^{-1}B \cup B) - [(X-K) \cup Y] \cap \bigcup_i (c^{-1}U_i \cap U_i).$$

$$= c^{-1}B \cap K \quad \text{which is compact in } X.$$

as $K$ is proper over $Y$.

$\therefore$ $Z$ loc. compact. any compact nbd. of $x$ in $X$ is a nbd of $x$ in $Z$

$c^{-1}B \cup B$ compact nbd of $y$ in $Z$ [illegible] $B$ comp. nbd of $y$ in $Y$.

<u>Universal property</u>: Given

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "c"'] & W \arrow[d, "g"] \\ & Y \end{tikzcd}$$

Hausdorff

with $\underbrace{\text{i open } \cancel{\text{dense}}}$, there is a unique map $W \xrightarrow{\varphi} Z$ compatible with $i, \bar{c}$.

<u>Proof</u>: ~~As $i$ dense given $w \in W$ have $w = \lim x_\nu$ and set $\varphi(w) = \lim x_\nu$ in $Z$.~~

$$\begin{array}{llll} \text{If} & w = i(x) & \text{set} & \varphi(w) = x \\ & w \notin i(X) & \text{set} & \varphi(w) = g(w). \end{array}$$

Claim $\varphi$ continuous. cont at $x \in X$: Inverse image of $V \ni x$, $V \subset X$ under $\varphi$ is $i(V)$, open in $W$. cont at $w$: Given a nbd $(X-K) \cup Y \cap \{c^{-1}U \cup U\}$ of $w$ its inverse image under $\varphi$ is $\{g^{-1}U\} \cap \{W - i(K)\}$. But $i(K)$ closed in $W$ because ~~g is proper~~ K proper over Y.

<u>Uniqueness of $\varphi$</u>: $\varphi, \bar{\varphi}$ coincide on $i(X)$ which is dense.

---

<u>Proposition</u>: $H^q_{pr/Y}(X) = \operatorname{Ker}\{H^q(Z) \xrightarrow{i^*} H^q(Y)\}$

Pseudo-proof: Take a nice resolution of $\mathbb{Z}_Z$ call it $A(Z)$. Then $A(Z) \longrightarrow A(N)$ restriction is onto where $N$ is a regular nbd. of $Y$ in $X$ and kernel is cocycles vanishing on $N$. But $Z$-Int $N$ is proper over $Y$. I.E. we assume $A_\Phi(X) \longrightarrow \operatorname{Ker}\{A(Z) \to A(Y)\}$ is a weak equivalence, where $\Phi$ = closed subsets of $Z$ not meeting $Y$ i.e. subsets of $X$ proper over $Y$.

---

December 10, 1968:

Construction of the category of motives over a point Pt.

$$\boxed{\mathrm{Hom}_m^q(X, Y) \xleftarrow{\sim} \{S^{-q} \wedge X, M \wedge Y\} : \Phi}$$

Definition of $\Phi$: Given $\alpha \in \{S^{-q} \wedge X, M \wedge Y\}$ choose $N$ and a repres.

$$u: S^{N-q} \wedge X \longrightarrow M(N) \wedge Y$$

where $u$ is smooth on the complement of $u^{-1}\{\infty\}$ which we shall denote $V$.

$$\begin{tikzcd} & B(N) \times Y \arrow[d, hook, "i"] \\ V \arrow[r, "u"] & E(N) \times Y \end{tikzcd}$$

By Thom there is an automorphism of a nbd of $B(N) \times Y$ which produces a homotopy of $u$ not moving $u^{-1}\{\infty\}$. Thus may assume $u$ is transversal to $i$ and form intersection

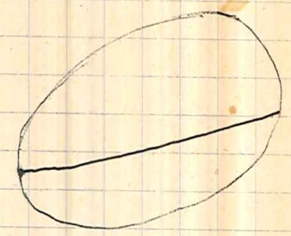
$$\begin{tikzcd} Z^{r-q} \arrow[r, "u'"] \arrow[d, hook, "i'"] & B(N) \times Y \arrow[d, hook, "i"] \\ V \arrow[r, "u"] \arrow[d, hook] & E(N) \times Y \\ R^{N-q} \times X^r & \end{tikzcd}$$

Since $Z$ doesn't meet a neighborhood of $\infty$ in $S^{N-q} \wedge X$ and is closed it is proper over $X$. Also oriented. Verify $\Phi$ independent of choices, clear by transversality approximation.

$$\boxed{\{S^{-q}\wedge X^+, M\wedge Y^+\} \xrightarrow{\sim} \mathrm{Hom}^q_{m_c}(Y,X).}$$

$$\begin{tikzcd}
S^{N-q}\wedge X^+ \arrow[rr, "u"] & & M(N)\wedge Y^+ \\
\mathbb{R}^{N-q}\times X \arrow[u, phantom, "\cup\ \text{open}"] & & E(N)\times Y \\
V \arrow[u, hook, "\text{open}"] \arrow[urr, "\text{proper}"] & & \\
Z \arrow[u, hook, "\text{closed}"] \arrow[rr, "\text{proper}"] & & B(N)\times Y \arrow[uu, hook]
\end{tikzcd}$$

since $X^+$ compact.



$$\begin{tikzcd}
S^{N-q}\wedge X^+ \arrow[d, phantom, "\shortparallel"] & & M(N)\wedge Y^+ \arrow[d, phantom, "\shortparallel"] \\
(\mathbb{R}^{N-q}\times X)^+ \arrow[rr, "u", "\text{proper}"'] & & (E(N)\times Y)^+ \\
U \arrow[u, phantom, "\cup\ \text{open}"] \arrow[rr, "u"] & & E(N)\times Y \arrow[u, phantom, "\cup\ \text{open}"] \\
Z^{n-q} \arrow[u, hook, "cl"] \arrow[rr, "\text{proper}"] & & B(N)\times Y \arrow[u, hook, "cl"]
\end{tikzcd}$$

~~$\infty$~~ since $X^+$ compact

Thus $Z^{n-q}$ proper over $Y$, oriented over $X^n$ defined element of $\mathrm{Hom}^q_{m_c}(Y,X)$.

---

$$\boxed{\{S^{-q}\wedge X^+, M\wedge Y\} \xrightarrow{\sim} \text{bordisms } (f,g): Z^{n-q}\to X^n\times Y \quad Z \text{ compact, } f \text{ oriented.}}$$

because in this case $B(N)\times Y$ doesn't meet $\infty$ so $Z$ doesn't meet $\infty$ in ~~[illegible]~~ $(\mathbb{R}^{N-q}\times X)^+$ and so $Z$ is compact.

$$\boxed{\{S^{-q}\wedge X,\ M\wedge Y^+\} \quad = \quad \text{classes of } Z \text{ proper over } X\times Y \text{ oriented over } X}$$

$$\begin{tikzcd}
S^{N-q}\wedge X \arrow[r] & M(N)\wedge Y^+ \\
\mathbb{R}^{N-q}\times X \arrow[u, phantom, "\cup"] \arrow[r] & (E(N)\times Y)^+ \arrow[u, phantom, "S\cup \text{ open}"] \\
U \arrow[u, phantom, "\cup \text{ open}"] \arrow[r] & E(N)\times Y \arrow[u, phantom, "\cup \text{ open}"] \\
Z \arrow[u, hook] \arrow[r] & B(N)\times Y \arrow[u, hook, "\text{cl.}"']
\end{tikzcd}$$

Claim $U$ proper over $X\times E(N)\times Y$. In effect if $u_n$ has images converging in $X\times E(N)\times Y$, then in $S^{N-q}\wedge X = [\mathbb{R}^{N-q}\times X] \cup \infty\times X$ it stay out of a nbd of $\infty\times X$ i.e. is in $K\times X$, $K$ compact. As it converges over $X$ done.

$\therefore$ $Z$ ~~compact over~~ proper over $X\times Y$
oriented over $X$.

---

Conversely

$$\begin{tikzcd}
Z \arrow[dd, hook] \arrow[dr] \arrow[rr] & & B(N)\times Y \\
& V \arrow[dl] \arrow[r] & E(N)\times Y \arrow[dd] \\
\mathbb{R}^{N-q}\times X \arrow[d] & & \\
S^{N-q}\wedge X \arrow[r] & V\cup X \arrow[r] & M(N)\wedge Y^+
\end{tikzcd}$$

$V$ proper / $X\times E(N)\times Y$ ✓

---

<u>Lemma</u>: If $V$ proper over $X\times Y$, get map $V\cup X \longrightarrow Y^+$.

Check: $\begin{cases} M \wedge X & \text{universal homology of } X \\ M \wedge X^{+} & \text{universal } \infty\text{-homology.} \end{cases}$

~~[illegible]~~

Check $\{S^{-q} \wedge X^{+}, M\} = H^{q}_{c}(X)$

$\{S^{-q} \wedge X, M\} = H^{q}(X)$

---

NEXT $f : \partial X \to Y$ $\qquad H^{*}_{p/Y}(X)$.

$H^{*}(X) \leftarrow H^{*}_{c}(Y)$

full bdry.

$\partial X \xrightarrow{f} Y$ to form $\boxed{X \cup_{f} Y}$

~~Next suppose $f$ com~~

$H_{p/Y}(X) = \operatorname{Ker}\{H^{*}(X \cup_{f} Y) \longrightarrow H^{*}(Y)\}$

$\operatorname{Hom}(?, pt) = \operatorname{Ker} \operatorname{Hom}(X \cup_{f} Y, pt)$

$$\begin{array}{ccccc} \partial X & \longrightarrow & X & \longrightarrow & X^{+} \\ \downarrow f & & \downarrow & & \\ Y & \longrightarrow & X \cup_{f} Y & \longrightarrow & X^{+} \end{array}$$

---

$H^{*}_{p/Y}(X) \longrightarrow H^{*}(X \cup Y) \rightleftarrows H^{*}(Y)$

$$\begin{tikzcd} Z \arrow[r, "\text{or}"] \arrow[dr, bend right, "\text{proper}"'] & X \arrow[d, "f"] \\ & Y \end{tikzcd} \qquad \begin{tikzcd} Z \arrow[d, "q"] \\ X \times \mathbb{R}^N \arrow[d] \\ Y \end{tikzcd}$$

---

submanifolds.

$$\begin{tikzcd} Z^{n-q} \arrow[rrr] \arrow[rr] \arrow[dr, hook, "\text{or}"] \arrow[ddr, bend right, "\text{proper}"'] & & V \arrow[r] & B(q) \arrow[d] \\ & X^n \arrow[ur, hook] \arrow[d] & & E(q) \arrow[d] \\ & Y & & M(q) \end{tikzcd}$$

$V$ proper over $Y \times E(q)$ so by lemma page 3 get

$$V \underset{}{\times} Y \longrightarrow M(q)$$

hence $\quad X \times Y \longrightarrow M(q)$.

---

$$\left\{ \cancel{S^q \wedge (X \underset{f}{\times} Y)}, M \right\} = \text{bordisms oriented over } X \text{ proper over } Y.$$

$$\lim_{\overrightarrow{N}} \left[ \frac{\mathbb{R}^{N-q} \times (X \underset{f}{\times} Y)}{\mathbb{R}^{N-q} \times Y}, M(N) \right].$$

What is the homotopy type of the pair $(X \underset{f}{\times} Y, Y)$? Nothing recognizable since it depends strongly on the ~~homotopy type of~~ topological type of $X$. Maybe a proper homotopy invariant of $(X, Y, f)$?

~~[scribble]~~ This constructs cohomology of $X$ proper over $Y$.

---

simplicial abelian groups, i.e. functors $(\mathcal{F}in)^\circ \longrightarrow \mathcal{A}b$.

$$\Delta \longrightarrow \square$$

$$\mathcal{A}b^{\Delta^\circ} \rightleftarrows \mathcal{A}b^{\square^\circ}$$

**Let** A be a s. abelian group.

$$\mathcal{C} \longrightarrow \mathrm{Hom}(\square, \mathcal{A}b) \text{ is a } \otimes \text{ cat.}$$

$$\mathcal{C} \xrightarrow{H_0} \mathcal{A}b$$

---

$$\cdot\cdot R \rightrightarrows A \equiv A \otimes_R A \longrightarrow \cdots$$

is the de Rham complex.

However

$$R \xrightarrow{\eta_R} A \underset{1\otimes id}{\overset{\Delta}{\rightrightarrows}} A \otimes_R A \equiv \cdots\cdots\cdots$$

exact so therefore should be able to determine

$R = \pi_*(M)$

$$\boxed{\begin{aligned} A &= \pi_*(M \wedge M) = \eta_*(M) \\ A \otimes_R A &= \eta_*(M \wedge M). \end{aligned}}$$

$$\Delta : A \longrightarrow A \otimes_R A$$
$$M \longrightarrow M \wedge M$$
$$x \longrightarrow 1 \otimes x$$

$\overset{x\ y}{M \wedge M}$

$\overset{x\ \ 1\ \ y}{M \wedge B \wedge M}$

$\overset{x\ \ 1}{(M \wedge M)} \underset{M}{\wedge} \overset{1\ \ y}{(M \wedge M)}$

$\overset{x\ y}{M \wedge M}$

$\overset{x\ y}{M \wedge M \wedge M}$

By Thm same as calculating

$$\eta_*(B) \rightrightarrows \eta_*(B \times B)$$

---

**Check:**

$$\begin{array}{cccccc}
x\otimes y & x\otimes y & \pi_*(M\wedge M) & \xrightarrow{\ \Delta\ } & & x\otimes y \mapsto (x\otimes 1)\otimes(1\otimes y) \\
\downarrow id\otimes 1 & \downarrow \Delta & \downarrow & & & \\
x\otimes y\otimes 1 & x\otimes 1\otimes y & \pi_*(M\wedge M\wedge M) & \xleftarrow{\ \cong\ } & \pi_*(M\wedge M)\otimes_R \pi_*(M\wedge M) & \\
 & & x\otimes(yz)\otimes w & \longleftarrow & (x\otimes y)\otimes(z\otimes w) &
\end{array}$$

Defn of $\Delta : A \longrightarrow A \otimes_R A$

$$\begin{tikzcd}
\pi_*(M \wedge M) \arrow[d, "x\otimes y \,\mapsto\, x\otimes 1\otimes y"'] \arrow[dr, dashed, "\Delta"] & \\
\pi_*(M\wedge M\wedge M) & \pi_*(M\wedge M) \otimes_{\pi_*(M)} \pi_*(M\wedge M) \arrow[l, "\cong"']
\end{tikzcd}$$

Claim that if we identify via the last ~~[illegible]~~ isom., then

$$\Delta : \dot{\eta}_*(M) \longrightarrow \dot{\eta}_*(M\wedge M)$$

is induced by $x \longmapsto 1 \otimes x$. Clear

---

Defn of $\mathrm{id} \otimes 1 : A \longrightarrow A \otimes_R A$

$$\begin{tikzcd}
\pi_*(M \wedge M) \arrow[d, "x\otimes y \,\mapsto\, x\otimes y\otimes 1"'] \arrow[dr, "\alpha \,\mapsto\, \alpha\otimes 1;\ x\otimes y \,\mapsto\, (x\otimes y)\otimes(1\otimes 1)"] & \\
\pi_*(M\wedge M\wedge M) & \pi_*(M\wedge M) \otimes_{\pi_*(M)} \pi_*(M\wedge M) \arrow[l, "\cong"']
\end{tikzcd}$$

By identification

$$\mathrm{id}\otimes 1 : \dot{\eta}_*(M) \longrightarrow \dot{\eta}_*(M\wedge M)$$

is ~~[illegible]~~ induced by $x \longmapsto y \otimes 1$.

---

**Conclude** that

Proposition: ~~If $A = \eta_*(M)$ is a flat $R = \eta_*(pt.)$ module (left or right, makes no difference by antipode), then~~

$$\dot{\eta}_*(\cancel{pt}) \longrightarrow \dot{\eta}_*(M) \overset{in_1}{\underset{in_2}{\rightrightarrows}} \dot{\eta}_*(M\wedge M)$$

is exact.

[illegible]

$$\sigma^1\{\delta^1\alpha = \delta^2\alpha\}$$
$$\alpha = \delta^1\sigma\alpha$$

$$\begin{tikzcd} \pi_*(M) \arrow[r, shift left] \arrow[r, shift right] & \pi_*(M\wedge M) \arrow[l, bend right] \arrow[r, shift left] \arrow[r] \arrow[r, shift right] & \pi_*(M\wedge M\wedge M) \arrow[l, bend right] \arrow[l, bend right=40] \end{tikzcd}$$

$$x \qquad\qquad 1$$

perfectly good cosimplicial ring, hence acyclic if we leave off $\delta_0$.

$$(x, y) \longmapsto (x, 1, y)$$
$$(x, y) \longmapsto (x, y, 1)$$
$$x \longmapsto (x, 1)$$

---

If I am correct, then this means that

$$\eta_*(pt) = \operatorname{Ker}\{\eta_*(B) \rightrightarrows \eta_*(B\times B)\}$$

~~In particular $\eta_*(pt.)$ is without torsion for MU NO~~

December 11, 1968

Thom transversality theorem: Let $Y$ be a closed submanifold of $X$, let $f: Z \to X$ be a map of manifolds, and let $U$ be an open set of $X$ containing $Y$. Let $F$ be a closed subset of $Y$ such at each $y \in F$, the map $f$ is transversal to $Y$ at $y$. Then there exists a diffeomorphisms $\varphi$ of $X$ such that $\varphi = id$ outside of $U$ and on $F$ and such that $\varphi(Y)$ is transversal to $f$. Furthermore $\varphi$ may be chosen arbitrarily close to the identity.

Proof: ~~Reduction to the case when X is a vector~~

1). Reduction to

Theorem 1': Let $X$ be a vector bundle over $Y$; let $f: Z \to X$ be a map and let $F$ be a closed subset of $Y$ such that at each $y \in F$. $f$ is transversal to $Y$ at $y$, where $Y$ is identified with the zero-section of $X$. Then $\{s \in \Gamma(X) \mid s = 0 \text{ on } F,\ f \text{ transversal to } s(Y)\}$ is a Baire set in $\Gamma(X)$.

Reduction: Take a tubular nbd. $N$ of $Y$ in $X$ and use exponential to identify $N$ with a vector bundle over $Y$. Then translation by a section of $N$ extends smoothly to a diffeo. of $X$ which is the identity outside of $N$.

Proof of 1':

Let $Z_n(K_n)$ be an exhaustion of $Z(Y-F)$ by compact manifolds with boundary. (ref.) Let $U_n = \{s \in \Gamma(X) \mid s = 0 \text{ on } F,\ s: K_n \longrightarrow X \text{ transversal to } f: Z_n \longrightarrow X\}$. Then as transversality is an open condition (ref.) it follows that $U_n$ is open in $W = \{s \in \Gamma(X) \mid s = 0 \text{ on } F\}$. ~~As $\Gamma(X)$ is a Baire space (ref.) it~~ We are reduced to proving that $U_n$ is dense in $W$ since then as $\Gamma(X)$ is a Baire space (ref.), $\bigcap U_n$ will be dense in $W$.

Let $s_0 \in W$ and let $V$ be a finite dimensional subspace of $W$ containing $s_0$ which spans the fiber $X(y)$ for each $y \in K_n$. $V$ exists since $K_n$ is compact. Then $K_n$ has a nbd $K_n'$ such that $V$ spans all fibers over $K_n'$. Consider

$$\begin{tikzcd}
\tilde{Z} \arrow[r] \arrow[d, "f'"] & Z \arrow[d, "f"] \\
K_n' \times V \arrow[r, "ev"] \arrow[d, "pr_2"] & X \\
V &
\end{tikzcd}$$

where $ev(y, s) = s(y)$ is a submersion (essentially a map of vector bundles covering open inclusion $K_n' \hookrightarrow X$), and where square is cartesian. Claim now that a regular point for $pr_2 f'$ is a section $s$ such that $s: K_n' \longrightarrow X$ is transversal to $f: Z \longrightarrow X$. Granted this $U_n \cap V$ is dense in $V$ by Sard hence $s_0 \in \overline{U}_n$ and so $U_n$ is dense in $W$.

Let $\tilde z = (y, s, z) \in \tilde Z$, that is $s(y) = f(z)$. Tangent space

~~$T_{\tilde Z}(\tilde z) \cong T_Y(y) \times_{T_X(sy)} T_Z(z)$ and map to~~

$$\begin{tikzcd} T_{\tilde Z}(\tilde z) \arrow[r, phantom, "\cong"] \arrow[d, "(pr_2 f')_*"] & \left(T_Y(y) \times T_V(s)\right) \times_{T_X(sy)} T_Z(z) \arrow[d, "pr_2"] \\ T_V(s) \arrow[r, dash, "\sim"] & T_V(s) \end{tikzcd}$$

where $T_Y(y) \times T_V(s) \longrightarrow T_X(sy)$ given by $\dot y + \dot s \longrightarrow s(\dot y) + \dot s(y)$.

~~Thus want to consider~~

$$\cancel{\{\dot s \mid \exists (\dot y, \dot s, \dot z) \quad s(\dot y) + \dot s(y) = f(\dot z) \in T_X(sy)\}}$$

~~Identifying $T_V(s)$ with $V$ one sees that this subspace contains all $\dot s$ with $\dot s(y) = 0$~~ Assuming that if $z$ is regular this means that for all $\dot s$ $\exists\, \dot y, \dot z$ with $s(\dot y) + \dot s(y) = f(\dot z)$, in particular ~~if $\dot s(y) = 0$~~ have $s_* T_Y(y) + f_* T_Z(z) = T_X(s(y))$. (Here identify $\dot s$ with ~~[illegible]~~ an element of $V$ and use that ~~[illegible]~~ $ev(y, V) = X(y)$.)
Thus equivalence of transversality of $s, f$ at $y, z$ with regularity of $\tilde z = (y, s, z)$.

QED.

<u>Remark:</u> By suitable growth conditions one may probably ignore $Y$ closed in $X$.

# Characteristic classes and motives

The bordism theories corresponding to $G = O$ and $G = U$ have the properties that the standard calculation of $H^*(BU)$ using the splitting principle works to calculate $\eta^*(BU)$. (Possibly also works for $Sp$), a phenomena associated to the fields $\mathbb{R}, \mathbb{C}, \mathbb{H}$). We now run through the details:

Thus let $R = \eta_*(pt)$ be the ground ring.

Lemma 1: $\eta^*(\mathbb{P}^n) \cong R[T]/(T^{n+1})$ where $T \in \eta^d(\mathbb{P}^n)$ is the class of the hyperplane $\mathbb{P}^{n-1} \longrightarrow \mathbb{P}^n$.

Proof:

$$\begin{tikzcd} 0 \arrow[r] & \eta^*(S^{dn}) \arrow[r] & \eta^*(\mathbb{P}^n) \arrow[r] & \eta^*(\mathbb{P}^{n-1}) \arrow[r, "\therefore"] & 0 \\ & & R[T] \arrow[u] \arrow[r] & R[T]/(T^n) \arrow[u] \arrow[r] & 0 \arrow[u, Rightarrow] \end{tikzcd}$$

But $\eta^*(S^{dn})$ is ~~generated as~~ free $R$ module with generator corresp. to a point which is $H^n$ corresponds to $T^n$. $\therefore$ Done by induction

---

Lemma 2: ~~If~~ Let $E$ is a vector bundle (of dim $n$) over $X$ (admitting a boundary) and let $L$ be the canonical line bundle $\mathcal{O}(-1)$ on $\mathbb{P}(E)$. Let $\xi_E \in \eta^2\{\mathbb{P}E\}$ be the first Chern class of $L^{-1} = \mathcal{O}(1)$ given by zeros of a generic section. Then $\eta^*(\mathbb{P}E)$ is a free module over $\eta^*(X)$ with basis $1, \ldots, \xi_E^{n-1}$

Proof: $\eta^*(U) \otimes_R \eta^*(\mathbb{P}^{n-1}) \longrightarrow \eta^*(\mathbb{P}(E|U))$ is shown to be an isomorphism by Mayer-Vietoris on $U$ since $X$ essentially compact.

---

Now using lemma 2 one sees splitting principle is valid and hence we can define Chern classes of vector bundles, e.g. elements

$$c_i \in \eta^{di}(B)$$

so that the usual formal properties hold, i.e.

$$c(E \oplus E') = c(E) \cdot c(E').$$

Lemma 3: $\eta^*(B(n)) = R[c_1, \dots, c_n]$

Proof: By induction on $n$.

$$B(n-1) \xrightarrow{\pi} B(n) \longrightarrow T(\pi) = M(n).$$

gives

$$\begin{tikzcd} 0 \arrow[r] & \eta^*(M(n)) \arrow[r] & \eta^*(B(n)) \arrow[r] & \eta^*(B(n-1)) \arrow[r] & 0 \\ & & R[c_1, \dots, c_n] \arrow[u] \arrow[r] & R[c_1, \dots, c_{n-1}] \arrow[u, "\wr"] \arrow[r] & 0 \arrow[u, Rightarrow] \end{tikzcd}$$

By Thom isomorphism $\eta^*(M(n))$ free over $\eta^*(B(n))$ with ~~the~~ generator $U_n$ = class of zero section $B(n) \longrightarrow M(n)$ which must be identified with $c_n$. By splitting principle we can assume $E = L_1 \oplus \cdots \oplus L_n$ whence its pretty clear. Thus done

---

Conclusion

$$\begin{cases} \eta^*(B) = \varprojlim R[c_1, \dots, c_n] \\ \Delta c = c \otimes c \qquad \varepsilon(c) = 1 \end{cases}$$

[redacted]

## Ideas on Motives

1. characteristic classes and operations
2. proper homotopy theory
3. parametrix (needed to define $f_*$) is it similar to a Steenrod diagonal approximation
4. realization of the triangulated motive category as M-spectra
5. twisted motive theories, Riemann-Roch [Conjecture
6. cobordism with singularities
7. linear motives in algebraic geometry
8. Variation of orientation group and descent.
9. Spectral Künneth theorem using nice decomposition of
10. Multiplicative structures $\otimes$, $\underline{Hom}(X, Y)$.

application: $\begin{cases} W & \text{power series in 1-variable} \\ W(t) = 1 + a_1 t + a_2 t^2 + \cdots & a_i \in \mathbb{Z}_2 \end{cases}$

when is $\Phi : H^*(X) \longrightarrow H^*(X)$ given by

$\Phi(f_* 1) = f_*\{W(\nu_f)\}$ $\qquad f : Z \to W$

well defined, e.g.

$$\boxed{\deg f = 1 \overset{?}{\Rightarrow} f_*(W(\nu_f)) = 1}$$

problem. In general $W = \varphi(w_1(\nu_f), \ldots, w_d(\nu_f))$ and $f_*\{w_i(\nu_f)\} = 0$.

But $f_* : H(Z) \to H(X)$ is onto adjoint to inclusion so kernel $f_*$ not an ideal

easier problem: Show that $f_* 1 = 0 \Rightarrow f_*\{W(\nu_f)\} = 0$. Here can write therefore

$$Z = Z_1 \times Z_2 \xrightarrow{pr_1} Z_1 \to X$$
$$f$$

$$Z_1 \times Z_2 \xrightarrow{pr_1} Z_1 \xrightarrow{g} X$$

to show $f_* 1 = 0 \Rightarrow f_! 1 = 0$. But $f_! 1 = g_!(pr_1)_! 1$ and $(pr_1)_* 1 = 0$ by reasons of dimension. Thus enough to show

that $(pr_1)_! 1 = 0$, and so by base change that $f_! 1 = 0$ ~~for all~~ whenever $f : Z \to pt.$ $\dim Z > 0$.

i.e. $\int_Z W(\nu_Z) = 0$ for all $Z$.

Therefore the problem is to decide exactly when char. no. always vanishes $\int_Z w(\nu_Z) = 0 \quad \forall Z \quad \dim Z > 0$.

Example: Let $Y \subset X$ be a submanifold. When can we blow up $Y$? Blow up $0$ in $V$. pairs $(\ell, x)$ $\ell \in \mathbb{P}(V)$ $x \in \ell$.

What is $\nu_f$ in this case?

recall that $\nu_f + \tau_Z = f^* \tau_X$.

$$\boxed{\nu_f = f^* \tau_X - \tau_Z}$$

$$W(\nu_f) = f^* W(\tau_X) \cdot W(\tau_Z)^{-1}$$

$$\therefore f_* W(\nu_f) = W(\tau_X) \cdot f_*(W(\tau_Z)^{-1}).$$

somehow the idea is that

$f_*(W(\nu_f))$ = characteristic classes of the center $Y$.

$Z$

$$\begin{array}{ccc} \tilde{Y} & \subset & Z \\ \downarrow & & \downarrow \\ Y & \subset & X \end{array}$$

$$0 \to f^*\Omega_X \to \Omega_Z \longrightarrow \Omega_f \to 0$$

seems reasonable then that $\boxed{\nu_f = i_*(\tau_{\tilde{Y}})}$

~~$\nu_f = i_*(\tau_{\tilde{Y}})$~~.

$Z$

$$H(BO) \ni W$$
$$\nu_Z \swarrow$$
$$H(Z) \xrightarrow{f_*} \mathbb{Z}_2 H(pt.)$$

$$H(BO) \otimes \eta \longrightarrow pt.$$

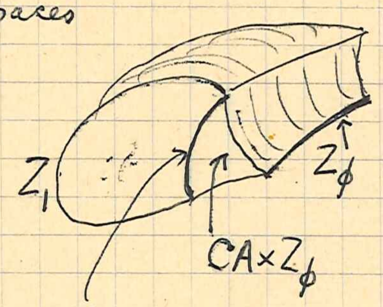
Thm: $\eta \hookrightarrow \mathrm{Hom}(H(BO), H(pt))$

Motives (December 16, 1968)

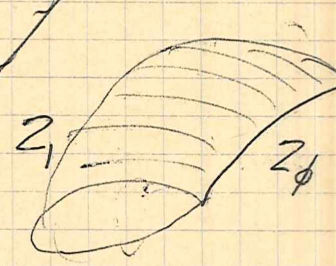
Sullivan's bordism theories using varieties with prescribed singularities:

Simplest case: unoriented varieties with only one singularity.

Let $A$ be a fixed closed manifold of dimension $a$. By a closed variety with singularity type $A$ (A-variety) of dimension $n$ we mean a triple $Z = (Z_1, Z_\phi, \varphi)$ where $Z_1$ is an $n$-manifold possibly with boundary, $Z_\phi$ is a $n-a-1$ closed manifold and $\varphi$ is an isomorphism of $A \times Z_\phi$ with $\partial Z_1$. We think of $Z$ ~~the space~~ as either of the spaces




$\partial Z_1 = A \times Z_\phi$




The idea is that $Z$ is a variety with singular locus $Z_\phi$ endowed with a trivialization $\cong CA \times Z_\phi$ of the normal tube around the singular locus.

By a map $f: Z \longrightarrow X$ where $X$ is a manifold we mean a pair of $C^\infty$ maps $f_1 : Z_1 \longrightarrow X$, $f_\phi : Z_\phi \longrightarrow X$ such that

$$\begin{array}{ccc} & Z_1 \xrightarrow{\;f_1\;} & X \\ \nearrow & & \uparrow f_\phi \\ \partial Z_1 \simeq A \times Z_\phi & \xrightarrow{pr_2} & Z_\phi \end{array}$$

commutes. By an A-cycle of dimension $n$ in $X$ we mean a map $f: Z \longrightarrow X$ where $Z$ is a closed variety with singular type $A$.

If $g: Y \longrightarrow X$ is a map of manifolds we say $g$ is transversal to the A-cycle $f: Z \longrightarrow X$ if $g$ is transversal to $f_i : Z_i \longrightarrow X$ $i = 1, \phi$. ~~In this case~~ (If $g$ is proper) let

$$Z_i' = Z_i \times_X Y \qquad \partial Z_1' = \partial Z_1 \times_X Y .$$

and let $\phi$ be the isomorphism composed of

$$A \times Z_0' = A \times Z_0 \times_X Y \underset{\varphi \times id}{\cong} \partial Z_1 \times_X Y = \partial Z_1' .$$

Then we obtain an A-cycle $Z' \xrightarrow{pr_2} Y$ called the <u>inverse image</u> of $f: Z \to X$, sometimes denoted $g^{-1}(f)$ or $f^{-1}(Y)$.

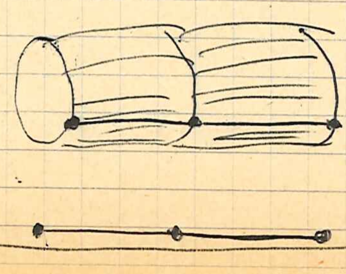
If $g: Y \longrightarrow X$ is not transversal to $f$ it can be made so by moving $f$ a little. In effect first move $f_\phi : Z_\phi \longrightarrow X$ until it is transversal to $g$. This homotopy of $f_\phi \, pr_2 \varphi^{-1} = f_1 | \partial Z_1$ may then be extended to $f_1$ (homotopy extension theorem valid for closed submanifold $A \hookrightarrow B$; in effect given $f$ on $A \times I$, $g$ on $B \times 0$ solve D.E. $\frac{d}{dt} F(x,t) = \frac{d}{dt} f(t)$, $F(x,0) = g(x)$ to get an extension over a nbd of $U \times I$, $U$ nbd of $I$. In this way can get a smooth extension over a nbd of $A \times I \cup X \times 0$ in $X \times I$ whence done.). Then $f_1$ transversal to $g$ over $\partial Z_1$ as already $(f_\phi)_*$ maps tangent space to $Z_\phi$ onto normal space of $g$ and because $pr_2$ is a <u>submersion</u>. Thus may move $f_1$ transversally to $g$ keeping it fixed on $\partial Z_1$, whence $g$ is transversal to $f$.

In order to insure that the inverse image be independent of the pushing one needs the notion of equivalent cycles. ~~Say that an A-variety Z is a boundary if there is an A-cycle $f: Z' \to B^1$ such that f is transversal to 0,~~

This seems to require the notion of pasting for which we now digress. Observe that we can speak of non-compact A-varieties and a proper map $f: V \longrightarrow X$ where $V$ is a not-necessarily compact A-variety and $X$ is a manifold. Let $\Delta(n)$ be the standard $n$-simplex considered as embedded in $\mathbb{R}^n$. Let $f: V \to \mathbb{R}^n$ be a proper A-variety over $\mathbb{R}^n$. Assume $f$ transversal to each face of $\Delta(n)$, so that over each face of $\Delta(n)$ we get an A-variety. By an A-variety over $\Delta(n)$ we mean something of the form $f^{-1}(\Delta(n)) \longrightarrow \Delta(n)$. Observe that the set of isomorphism classes of A-varieties over $\Delta(n)$ gives a simplicial set graded by relative dimension. Denote the isom. classes ~~the simplicial set~~ of A-varieties of (relative) dimension $n$ over $\Delta(q)$ by $I_A(n)_q$. The basic fact about pasting is:

**Lemma:** $I_A(n)$ is a Kan complex.

Partial Proof: Given A varieties $V_i$ over $\Delta(n-1)$ for $i \neq k$ fitting together, one can glue them together to get an "A-variety" over $V(n,k)$. Choosing a retraction of $\Delta(n) \longrightarrow V(n,k)$ and pulling back one obtains an A-variety over $\Delta(n)$. Example:



The above digression is too painful. The problem is that one must introduce A-varieties with boundary and paste two such along a common boundary, putting smooth ~~structures~~ on the union (rounding out creases). My feeling is that such considerations of boundaries are fundamentally irrelevant as they have no place in alg. geometry. The problem eventually is to eliminate them from the geometry by axiomatic considerations as we did in the smooth case.

We say that two A-cycles $f_1: Z_1 \longrightarrow X$, $f_2: Z_2 \longrightarrow X$ are equivalent or bordant if there exists an A-cycle $f: W \longrightarrow P \times X$ where $P$ is a connected manifold and points $p, q \in P$ such that $f^{-1}(p \times X) = f_1$, $f^{-1}(q \times X) = f_2$. Joining $p$ to $q$ by a path transversal to $f\, pr_1$ we may assume that $P = S^1$ or admitting boundaries $P = [0, 1]$, in which case what sits over $P$ is an A-variety with boundary:

Definition: An A-variety with boundary is a tuple $(Z_{01}, Z_0, Z_1, Z_\phi, \alpha; \beta, \gamma, \delta)$ where

$$\partial Z_\phi = \phi$$

$$\alpha : A \times Z_\phi \xrightarrow{\sim} \partial Z_1 \qquad \beta : Z_\phi \xrightarrow{\sim} \partial Z_0$$

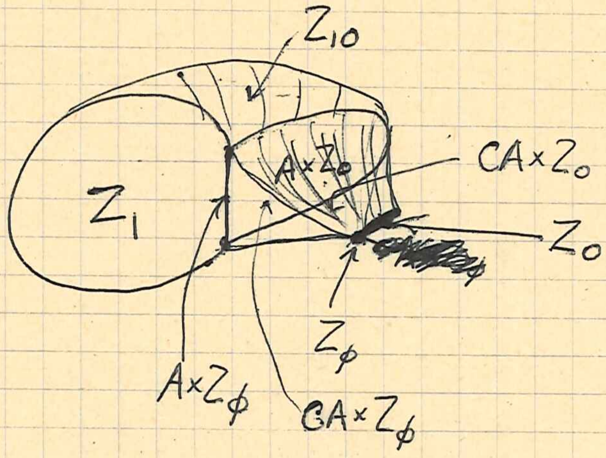
$$\gamma : Z_1 \hookrightarrow \partial Z_{01} \quad \text{embedding}$$

$$\delta : A \times Z_0 \hookrightarrow \partial Z_{01} \quad \text{"}$$

such that

$$\begin{tikzcd} & Z_1 \arrow[dr, "\gamma"] & \\ A \times Z_\phi \arrow[ur, "\alpha"] \arrow[dr, "id \times \beta"'] & & \partial Z_{01} \\ & A \times Z_0 \arrow[ur, "\delta"'] & \end{tikzcd}$$

is a smoothing of $Z_1 \underset{A \times Z_\phi}{\cup} A \times Z_0$. The boundary of this thing is $(Z_1, Z_\phi, \alpha)$. Picture:




---

Let $\mathcal{N}A_q(X)$ be the bordism classes of A-cycles in X of dimension q. Given $f: X \to Y$ have $f^*, f_*$ given by inverse and direct image and all the formulas holds (homotopy ✓, transversality ✓). ~~[illegible]~~ We now define products,

$$\mathcal{N}A_p(X) \otimes \mathcal{N}A_q(Y) \longrightarrow \mathcal{N}A_{p+q}(X \times Y).$$

induced by ordinary product of cycles.

Given $Z = (Z_1, Z_\phi, \varphi)$, $V = (V_1, V_\phi, \psi)$ set

$Z \times V = (Z_1 \times V_1, (Z_\phi \times V_1) \underset{A \times Z_\phi \times V_\phi}{\cup} (Z_1 \times V_\phi), \varphi * \psi)$ where $\varphi * \psi$ is the

$$\cancel{\partial(Z_1 \times V_1) = (A \times Z_\phi) \times V_1 \cup Z_1 \times (A \times V_\phi) = A \times (Z_\phi \times V_\phi)}$$

composition of the following isomorphisms

$$\partial(Z_1 \times V_1) \cong (\partial Z_1 \times V_1) \cup_{\partial Z_1 \times \partial V_1} (Z_1 \times \partial V_1) \qquad \text{(can.)}$$

$$\cong (A \times Z_\phi \times V_1) \cup_{A \times Z_\phi \times A \times V_\phi} (Z_1 \times A \times V_\phi) \qquad \text{(using } \varphi, \psi\text{)}$$

$$\cong A \times \left\{ (Z_\phi \times V_1) \cup_{A \times Z_\phi \times V_\phi} (Z_1 \times V_\phi) \right\}.$$

(necessitates the smoothing problem again.)

---

Thus we have constructed a ~~co~~homology functor $F : \mathcal{V} \longrightarrow \mathcal{Ab}$ endowed with products. Hence we obtain a category $\mathcal{MA}$ with

$$\mathrm{Hom}_{\mathcal{MA}}(X, Y) = F(X \times Y)$$

endowed with a tensor product $(X) \otimes (Y) = (X \times Y)$. Recall the sorites on ~~co~~homology theories with products.

~~Proposition: Equivalence between cohomology functors $F: \mathcal{V} \to \mathcal{Ab}$ with products and~~

Proposition: (1.) If $F : \mathcal{V} \longrightarrow \mathcal{Ab}$ is a ~~co~~homology theory with products, then one obtains an additive category $\mathcal{A}_F$ with tensor product and same objects as $\mathcal{V}$ with

$$\mathrm{Hom}_{\mathcal{A}_F}((X), (Y)) = F(X \times Y)$$

$$(X) \otimes (Y) = (X \times Y) \qquad 1 = (pt)$$

such that $F(X) = \mathrm{Hom}_{\mathcal{A}_F}(1, (X))$. Thus $\mathcal{V} \to \mathcal{A}_F$ $X \mapsto (X)$ is a ~~co~~homology theory with products such that Künneth holds.

(2.) Let $G: \mathcal{V} \longrightarrow \mathcal{A}$ be a homology theory with products such that Künneth holds. by which we mean that $G(X) \otimes G(Y) \xrightarrow{\sim} G(X \times Y)$ is an isomorphism for all $X, Y$ (it suffices to have for $X = Y$). Then

$$\mathrm{Hom}_{\mathcal{A}}(GX, GY) = \mathrm{Hom}_{\mathcal{A}}(1, G(X \times Y)).$$

~~and the category associated~~ so that if $G: \mathrm{Ob}\,\mathcal{V} \to \mathrm{Ob}\,\mathcal{A}$ is an isomorphism, then $\mathcal{A} \cong \mathcal{A}_F$ where $FX = \mathrm{Hom}_{\mathcal{A}}(1, GX)$.

(3). Therefore there is an equivalence of categories:

$$\left\{\begin{array}{l}\text{cat of hom.}\\ \text{theories } F: \mathcal{V} \to \mathcal{Ab}\\ \text{with products}\end{array}\right\} \cong \left\{\begin{array}{l}\text{cat. of hom.}\\ \text{theories } G: \mathcal{V} \longrightarrow \mathcal{A} \text{ with product}\\ \text{such that } \mathrm{Ob}\,G \text{ is an isom.}\\ \text{and Künneth holds}\end{array}\right\}$$

<u>Proof</u>: 1) To define composition:

$$F(X \times Y) \otimes F(Y \times Z) \xrightarrow{\boxtimes} F(X \times Y \times Y \times Z) \xrightarrow{(id_X \times \Delta \times id_Z)^*} F(X \times Y \times Z) \xrightarrow{(pr_{13})_*} F(X \times Z)$$

Define for $f: X \longrightarrow Y$, $\quad f_* = (\Gamma_f)_* 1_X \in F(X \times Y)$, $f^* = ({}^t\Gamma_f)^* 1_X \in F(Y \times X)$.

Finally set $(X) \otimes (Y) = (X \times Y)$ and check everything

2). Abstract Poincaré duality

3). One checks that if $u: F \longrightarrow F'$ is a morphism then get $\mathcal{A}_F \longrightarrow \mathcal{A}_{F'}$ compatible with $X \mapsto (X)$ and conversely given $\theta: G \longrightarrow G'$ one gets $F \longrightarrow F'$.

($\theta$ is a pair consisting of a functor $\theta_1: \mathcal{A} \to \mathcal{A}'$ and an isomorphism $\theta_0: \theta_1 G \longrightarrow G'$)

~~Understand coh. theories~~

**Determination of the motive category:**

If one can find $F: \mathcal{M} \longrightarrow Mod_R$ $\quad R = End_{\mathcal{M}}(1)$ such that Künneth holds $F(X)\otimes_R F(Y) \xrightarrow{\sim} F(X\times Y)$ and possibly something else, then $\mathcal{M}$ is the category of representations of a group $G$ over $R$. $G$ ~~auto group~~ $= Aut^{\otimes} F$.

$\boxed{\text{Relative version:}}$ Suppose can find two motive categories $\mathcal{M}_1 \xrightarrow{F} \mathcal{M}_2$ so that Künneth holds. Then $\mathcal{M}_1$ representations of a group in $\mathcal{M}_2$ ie a $G$ in $\mathcal{M}_2$

e.g. $\mathcal{M}_1$ = stable category
$\mathcal{M}_2$ = M-spectra.

problem is that

$$End_{\mathcal{M}_1}(1) \neq End_{\mathcal{M}_2}(1)$$

$$\{S,S\} \qquad\qquad \{M,M\}$$

however in any case one has descent

$$S \longrightarrow M \rightrightarrows M\wedge M \Rrightarrow M\wedge M\wedge M.$$

so that we have

$$\{S,S\} \longrightarrow \{S,M\} \rightrightarrows \{S, M\wedge M\} \Rrightarrow \{S, M\wedge M\wedge M\}.$$

Conclusion: In the topological situation it seems that Grothendieck's version very seldom occurs, i.e. with each cobordism theory is associated a ring spectrum so that the functors ~~all~~ always base change e.g. $F(X) = M' \otimes_M X$.

Thus: ~~If~~ $\mathcal{M}_1 \xrightarrow{F} \mathcal{M}_2$ is a change of motive categories it is given by $X \longmapsto M_2 \otimes_{M_1} X$ and

$$\begin{array}{ccc} \mathrm{End}_{\mathcal{M}_2}(M_2, M_2) & \longleftarrow & \mathrm{End}_{\mathcal{M}_1}(M_1, M_1), \\ \| & & \\ \pi_*(M_2) & \longleftarrow & \pi_*(M_1) \end{array}$$

is not an isomorphism.

Still, however, one can try ~~to~~ find a G action on $M_2$ whose invariants are $M_1$. Usual descent. G has to be replaced in general by a ~~[illegible]~~ category scheme with objects scheme $M_2$. For the $S \longrightarrow K$ map ~~[illegible]~~ ~~[illegible]~~ one gets a group since $\pi_*(K) = \mathbb{Z}/p\mathbb{Z}$ has no autos but this is theoretically rare.

---

However the situation ~~[illegible]~~ envisioned by Grothendieck offers a new idea: Thus if $A \longrightarrow B$ is Galois of group G we can't distinguish between (A-modules) and (B-modules with G action). The group G might be fairly uniform ($S_\infty$ ?)

<u>Summary:</u>

1. If $X$ is a pointed spaces, let $\bar{\mathbb{Z}}_2 X$ be the $\mathbb{Z}_2$ module generated by $X$ with basepoint $= 0$. Then (in the simplicial category at least)

$$\bar{\mathbb{Z}}_2 X \simeq K(\tilde{H}(X, \mathbb{Z}_2)).$$

e.g.

$$\bar{\mathbb{Z}}_2 S^n = K(\mathbb{Z}_2, n).$$

~~For example if n = 1, then points of~~ ~~$\bar{\mathbb{Z}}_2 S^1$ may be identified with finite subsets of~~ ~~$\mathbb{R}$ to which we uniquely associate~~

Question: Is $\bar{\mathbb{Z}}_2 S^1$ homeomorphic to $\mathbb{R}P^\infty$ as in complex case? What is the functor analogous to $\bar{\mathbb{Z}}_2$ in the real case, $\bar{\mathbb{Z}}$ in the complex case for the quaternions? (seems unlikely as $\bar{\mathbb{Z}}_2 S^1$ is a group

2. Key lemma about Steenrod algebra appears to be that $H^q(K(\mathbb{Z}_2, n), \mathbb{Z}_2) \longrightarrow H^q(K(\mathbb{Z}_2, 1)^n, \mathbb{Z}_2)$ is injective in $\underbrace{\text{stable range}}$ and image = symmetric part, $q < 2n$. In virtue of the diagram

$$\begin{array}{ccccc} MO(1) \wedge \cdots \wedge MO(1) & \longrightarrow & MO(n) & \longrightarrow & K(n) \\ & & & & \\ BO(1) \times \cdots \times BO(1) & \longrightarrow & BO(n) & & \end{array}$$

together with the fact that $MO(1) \cong \mathbb{R}P^\infty = K(1)$, this yields the fact that $U^*: H^*(K) \longrightarrow H^*(MO)$ is injective after which Hopf algebra theory yields additive structure of $H^*(MO)$.

Don't know whether this key fact permits one to define the squares.

<u>Proposition:</u> Let $\mathcal{V}$ be the category of compact smooth manifolds. Let $F$ be a cohomology functor on $\mathcal{V}$ with values in $\mathcal{Ab}$ with products, by which we mean endowed with $f^*, f_*$ as usual together with

$$\begin{cases} \alpha \otimes \beta \longmapsto \alpha \boxtimes \beta \\ F(X) \otimes F(Y) \longrightarrow F(X \times Y) \\ 1 \in F(pt.) \end{cases}$$

which is associative

$$\alpha \boxtimes (\beta \boxtimes \gamma) = (\alpha \boxtimes \beta) \boxtimes \gamma$$

$$\begin{matrix} 1 \boxtimes \alpha \simeq \alpha & \text{under isom} & X \times pt \simeq X \\ \alpha \boxtimes 1 \simeq \alpha & \longrightarrow & pt \times X \simeq X. \end{matrix}$$

and compatible with $f^*, f_*$ in the sense that

$$(f \times id)^* (\alpha \boxtimes \beta) = f^*\alpha \boxtimes \beta \qquad (f \times id)_* (\alpha \boxtimes \beta) = f_*\alpha \boxtimes \beta$$

$$(id \times f)^* (\alpha \boxtimes \beta) = \alpha \boxtimes f^*\beta \qquad (id \times f)_* (\alpha \boxtimes \beta) = \alpha \boxtimes f_*\beta.$$

(in other words the functor $F' : \mathcal{M} \longrightarrow \mathcal{Ab}$ is a morphism of additive categories with tensor product) Then one obtains a category $\mathcal{C}$ with $Ob\,\mathcal{C} = Ob\,\mathcal{V}$ and

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) = F(X \times Y)$$

where composition is

$$F(X \times Y) \otimes F(Y, Z) \xrightarrow{\boxtimes} F(X \times Y \times Y \times Z) \xrightarrow{(id \times \Delta \times id)^*} F(X \times Y \times Z) \xrightarrow{(pr_{13})_*} F(X, Z)$$

## Identity axiom.

Claim $(\Delta_X)_*\pi_X^*1 \in$ ~~[illegible]~~ $F(X\times X)$ is the identity in $\mathcal{C}$.

$$\begin{tikzcd}
(\Delta_X)_*\pi_X^*1_X \boxtimes f \\
(X\times X)\otimes(X\times Y) \arrow[r, "\boxtimes"] & (X\times X\times X\times Y) \arrow[r, "(id\times\Delta\times id)^*"] & (X\times X\times Y) \arrow[r, "(pr_{13})_*"] & (X\times Y) \\
(X)\otimes(X\times Y) \arrow[u, "(\Delta_X)_*\otimes id_*"] \arrow[r, "\boxtimes"] \arrow[ur, phantom, "\text{comp}"] & (X\times X\times Y) \arrow[u, "(\Delta\times id\times id)_*"] \arrow[r, "(\Delta\times id)^*"] \arrow[ur, phantom, "\text{cart.}"] & (X\times Y) \arrow[u, "(\Delta\times id)_*"] \arrow[ur, "id"'] \arrow[ur, phantom, "\text{funct}"] \\
(pt)\otimes(X\times Y) \arrow[u, "\pi_X^*\otimes id"] \arrow[r, "\boxtimes"] \arrow[ur, phantom, "\text{comp}"] & (pt\times X\times Y) \arrow[u, "(\pi_X\times id\times id)^*"] \arrow[d, "(pr_{12})^*"] \\
1 \otimes f \arrow[uuuu, bend left] \arrow[dr, mapsto] & (X\times Y) \arrow[uur, "id"'] \\
(X\times Y) & f
\end{tikzcd}$$

Therefore works because of axiom that

$$\begin{tikzcd}
F(pt)\otimes F(X) \arrow[r, "\boxtimes"] & F(pt\times X) \arrow[d, "(pr_2)_*"] \\
1 \otimes f \arrow[r, mapsto] & f \quad F(X)
\end{tikzcd}$$

(eg. $(pr_2)_*$ and $(\pi_X, id)^*$ are inverses)

these ~~[illegible]~~ isomorphism only one good $F(pt\times X)\to F(X)$ by lemma below.

---

**Lemma:** If $f: X\to Y$ is an isomorphism in $\mathcal{V}$ with inverse $g$, then $f_* = g^*$ (ie. $f_*f^* = 1 = f^*f_*$)

**Proof:**

$$\begin{tikzcd}
X \arrow[r, "id"] \arrow[d, "id"'] & X \arrow[d, "f"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

is cartesian $\Rightarrow$ $f^*f_* = id$

Similarly

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "f"'] & Y \arrow[d] \\
Y \arrow[r] & Y
\end{tikzcd}$$

shows $f_*f^* = id$.

---

4

$\boxed{\text{associativity of composition}}$

$$\begin{tikzcd}
(X\times Y)\otimes(Y\times Z)\otimes(Z\times W) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{assoc of } \boxtimes"] & (X\times Y\times Y\times Z)\otimes(Z\times W) \arrow[r] \arrow[d] \arrow[dr, phantom, "\substack{\text{compatibility of } \boxtimes \\ \text{with }^{*} \\ (f\times id)^{*}(x\boxtimes y)=f^{*}x\boxtimes y}"] & (X\times Y\times Z)\otimes(Z\times W) \arrow[r] \arrow[d] \arrow[dr, phantom, "(f_{*}\times id)(x\boxtimes y)=f_{*}x\boxtimes y"] & (X\times Z)\otimes(Z\times W) \arrow[d] \\
(X\times Y)\otimes(Y\times Z\times Z\times W) \arrow[r] \arrow[d] \arrow[dr, phantom, "\checkmark"] & (X\times Y\times Y\times Z\times Z\times W) \arrow[r, "\Delta_Y^{*}"] \arrow[d, "\Delta_Z^{*}"] \arrow[dr, phantom, "\text{funct. of }^{*}"] & (X\times Y\times Z\times Z\times W) \arrow[r, "(\ )_{*}"] \arrow[d, "(\ )^{*}"] \arrow[dr, phantom, "\text{cart. axiom}"] & (X\times Z\times Z\times W) \arrow[d, "(\ )^{*}"] \\
(X\times Y)\otimes(Y\times Z\times W) \arrow[r] \arrow[d] \arrow[dr, phantom, "\checkmark"] & (X\times Y\times Y\times Z\times W) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{cart. axiom}"] & (X\times Y\times Z\times W) \arrow[r, "(\ )_{*}"] \arrow[d] \arrow[dr, phantom, "\text{funct of } *"] & (X\times Z\times W) \arrow[d] \\
(X\times Y)\otimes(Y\times W) \arrow[r] & (X\times Y\times Y\times W) \arrow[r] & (X\times Y\times W) \arrow[r] & (X\times W)
\end{tikzcd}$$

$$\begin{tikzcd}
Y\times Z \arrow[r, "id_Y\times\Delta"] \arrow[d] & Y\times Z\times Z \arrow[d] \\
Z \arrow[r, "\Delta"] & Z\times Z
\end{tikzcd}$$

(a variant of Künneth $\Rightarrow$ Poincaré duality)

<u>Proposition:</u> Let $F: \mathcal{V} \longrightarrow \mathcal{A}$ be a cohomology functor with values in an additive category $\mathcal{A}$ with tensor product. Let $X \in \mathcal{V}$ and assume $1 \simeq F(1)$. Then

(i) $F(X) \otimes F(X) \xrightarrow{\sim} F(X \times X)$ ~~[illegible]~~.

$\Downarrow$

(ii) $\Phi: \mathrm{Hom}(1, F(X\times Y)) \xrightarrow{\sim} \mathrm{Hom}(FX, FY)$ for all $Y$ where $\Phi(\alpha)$ is the composition

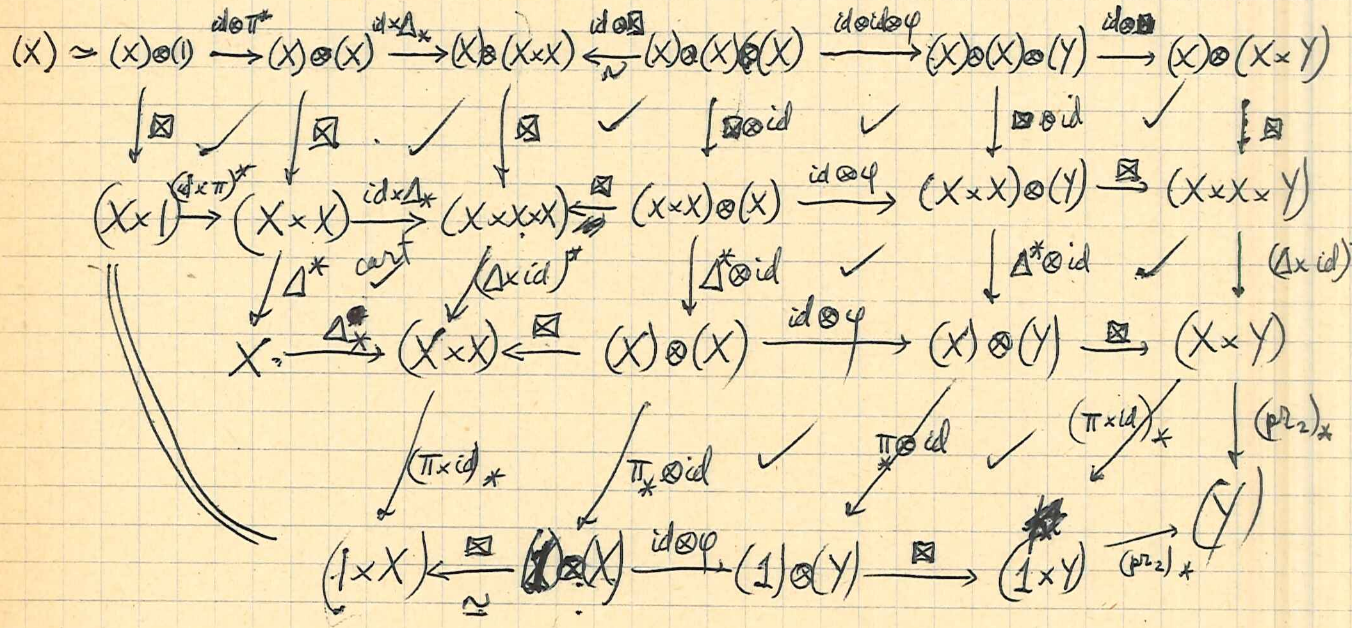
$$F(X) \cong F(X)\otimes 1 \xrightarrow{id\otimes\alpha} F(X)\otimes F(X\times Y) \xrightarrow{\boxtimes} F(X\times X\times Y) \xrightarrow{(\Delta\times id)^*} F(X\times Y) \xrightarrow{(pr_2)_*} F(Y).$$

<u>Proof:</u> Define $\Psi: \mathrm{Hom}(FX, FY) \longrightarrow \mathrm{Hom}(1, F(X\times Y))$ by $\Psi(\varphi) =$ composition

$$1 \to F1 \xrightarrow{\pi_X^*} FX \xrightarrow{(\Delta_X)_*} F(X\times X) \xleftarrow{\sim} F(X)\otimes F(X) \xrightarrow{id\otimes\varphi} FX\otimes FY \xrightarrow{\boxtimes} F(X\times Y)$$

<u>Will now show that $\Phi$ and $\Psi$ are inverses of each other.</u>

Given $\varphi: FX \to FY$ calc. $\Phi\Psi\varphi = \varphi$?



December 18, 1968.

Problem: To what extent is a homology theory with products on the category of smooth manifolds related to a $\underbrace{\text{cohomology}}^{\text{multiplicative}}$ theory a la Atiyah-Hirzebruch on the category of finite CW complexes?

Suppose $F_q$ is a homology theory for manifolds with products. If $K$ is a finite complex, then embed $K$ in $\mathbb{R}^n$ and take a regular neighborhood $N$ which is then a manifold of the homotopy type of $K$. Set $F_q(K) = F_q(N)$ and one extends $F$ from manifolds to finite complexes. One also has a $\overbrace{\text{map}}^{\text{product}}$

$$F_p(K) \otimes F_q(L) \longrightarrow F_{p+q}(K \times L)$$

To obtain cohomology one uses Alexander-Spanier duality. Thus ~~if~~ take $K$ embed it in $\mathbb{R}^n$ for $n$ large let $N_K$ be a regular neighborhood so that $N_K^+$ is the Alex.-Spanier dual of $X$. Then set

$$F^q(K) = \tilde{F}_{n-q}(N_K^+)$$

where for a pointed space $X$

$$\tilde{F}_*(X) = \operatorname{Ker}\{F_*(X) \longrightarrow F_*(pt)\}$$
$$= \operatorname{Cokernel}\{F_*(pt) \longrightarrow F_*(X)\}.$$

Given $K, L$ then get product embedding $K \times L \longrightarrow \mathbb{R}^n \times \mathbb{R}^{n'}$ with $N_{K\times L} = N_K \times N_L$ so $N_{K\times L}^+ = N_K^+ \wedge N_L^+$. Hence homology product gives map $\tilde{F}_p(N_K^+) \otimes \tilde{F}_q(N_L^+) \longrightarrow \tilde{F}_{p+q}(N_K^+ \wedge N_L^+)$

## Review of the Yoga of duality.

$$\begin{array}{ccc} X & D(X) & \\ \downarrow f & f_* \downarrow \downarrow f_! & \uparrow f^* \uparrow f^! \\ Y & D(Y) & \end{array}$$

$$\begin{cases} \mathrm{Hom}(f_! F, G) = \mathrm{Hom}(F, f^! G) \\ \mathrm{Hom}(f^* G, F) = \mathrm{Hom}(G, f_* F) \end{cases}$$

also have map $f_! F \longrightarrow f_* F$.

> Point is that $f_!$ has the good properties
> (i) base change
> (ii) triangle Mayer-Vietoris, excision.

~~[illegible]~~ Biduality

$$D_X = \underline{\mathrm{Hom}}( \;, \omega_X) \qquad \omega_X = \pi_X^! 1$$

$$\boxed{D_X f^* D_Y = f^!} \Longleftrightarrow \boxed{D_Y f_* D_X = f_!}$$

(using that $f_! - f^!$, $f^* - f_*$ are adjoints).
and that $D_X, D_Y$ are dualizing

$$\begin{cases} f \text{ proper} \Longrightarrow & f_! = f_* \\ f \text{ oriented} \Longrightarrow & f^* = f^! \end{cases}$$

$$\boxed{\begin{array}{l} f \text{ proper} \Longrightarrow \text{ dualization commutes with } f_* \\ f \text{ oriented} \Longrightarrow \text{ ——————————— } f^* \end{array}}$$

---

$f_! \downarrow \downarrow f_*$
pt.

$X \supset \underline{Y}$

$X$ is a partial comp. of $X - Y$

It is ultimately ~~necessary~~ to describe families of supports somehow —
I needed this because I want to define

$$\mathrm{Hom}\,(H^*(X), H^*(Y)) = H\,(X \times Y \text{ proper over } Y)$$

~~Hom(~~

$$\mathrm{Hom}((\pi_X)_* 1_X, (\pi_Y)_* 1_Y) = \mathrm{Hom}(\pi_Y^*(\pi_X)_* 1_X, 1_Y)$$

$$= \mathrm{Hom}((pr_2)_* \underbrace{(pr_1)^* 1_X}_{1_{X \times Y}}, 1_Y)$$

$$\begin{tikzcd} X \times Y \arrow[r, "pr_1"] \arrow[d, "pr_2"'] & X \arrow[d, "\pi_X"] \\ Y \arrow[r, "\pi_Y"'] & 1 \end{tikzcd}$$

$=$

~~a family of supports~~:

$X$ space $\Phi$ family of closed subsets of $X$, hereditary and stable under finite unions.

Example: Take ~~a point~~ a subset $A$ of $X$ and let $\Phi$ = closed sets not meeting $A$

$$\Gamma_{\Phi}(X,F) = \varinjlim_{Z \in \Phi} \mathrm{Hom}(\mathcal{O}_Z, F)$$

where $\mathcal{O}_Z = \mathrm{Coker}\{\mathcal{O}_{X-Z} \longrightarrow \mathcal{O}_X\}$. $\sqcap$

Problem: If $X$ loc. compact, compactify $X \xhookrightarrow{\text{open}} \bar{X}$ so that $Z \in \Phi \iff Z$ closed in $\bar{X}$ and $Z \subset X$.

Method: Let $\bar{X} = X \cup pt.$ where $X$ is an open set in $\bar{X}$ and where nbds. of $\infty$ are of form $(X-Z) \cup \infty$ where $Z \in \Phi$. (Assume $\Phi$ contains all compact subsets of $X$.) Then $\bar{X}$ Haus. i.e. $\forall x \in X$ choose $Z \ni \mathrm{Int}\, Z \ni x$ whence $(\mathrm{Int}\, Z) \cap (X - Z \cup \infty) = \phi$

equivalently $\Phi$ contains a nbd. of each pt.

$\bar{X}$ loc. compact. Need to show $\infty$ has a compact nbd. So assume $(X - \mathrm{Int}\, Z) \subset (\bigcup U_i) \cup (X - Z')$

i.e. $Z' - \mathrm{Int}\, Z \subset \bigcup U_i$. Therefore one needs $Z \ni Z' - \mathrm{Int}\, Z$ compact.

If $\Phi$ is a family of supports in $X$, then can form a space $X \underset{\Phi}{\cup} \{\infty\}$ where nbds. of $\{\infty\}$ are of form $(X-Z)\cup\{\infty\}$, $Z \in \Phi$, and where $\{\infty\}$ is closed. Special cases: $\Phi$ = all closed subsets of in which case $X \underset{\Phi}{\cup}\{\infty\} = X \sqcup \text{pt.}$; $\Phi$ = all compact subsets of $X$ in which case $X \underset{\Phi}{\cup}\{\infty\}$ is the 1-point compactification of $X$. $\square$

$$\{\infty\} \xhookrightarrow{i} \underset{\overset{\shortparallel}{X^+}}{X \underset{\Phi}{\cup}\{\infty\}} \xhookleftarrow{j} X$$

If $F$ is a sheaf on $X$, then $\quad i^* j_*(F) = \varinjlim_{Z\in\Phi} \Gamma(X-Z, F)$

$$i^* j_!(F) = \varinjlim_{Z\in\Phi} \Gamma((X-Z)\cup\{\infty\}, j_! F).$$

$j_! F$ = subsheaf of $j_* F$ consisting of germs vanishing near $\infty$.

$\therefore$

$$\boxed{\Gamma(X^+, j_! F) = \Gamma_\Phi(X, F)}\,.$$

$$0 \to j_! j^* G \longrightarrow G \longrightarrow i_* i^* G \longrightarrow 0$$

If $G^\circ$ is an injective resolution of $G$ over $X^+$, then $j^* G^\circ$ also injective res. of $j^* G$ (since $j^*$ has an exact left adjoint $j_!$ and since $j^*$ has a right adjoint $j_*$), hence get

$$\longrightarrow H^*_\Phi(X, G) \longrightarrow H^*(X, G) \longrightarrow H^*(\{\infty\}, G) \longrightarrow \cdots$$

$$\boxed{\begin{aligned} H^*_\Phi(X,G) = \tilde{H}^*(X^+,G) &= \mathrm{Ker}\{H^0(X,G) \longrightarrow G_\infty\} && *=0 \\ &= \mathrm{Coker}\{H^0(X,G) \longrightarrow G_\infty\} && *=1 \\ &= H^*(X,G) && * \geq 2 \end{aligned}}$$

$$\therefore \quad \boxed{H^*_\Phi(X) = \tilde{H}^*(X \cup_\Phi \{\infty\})}$$

---

Relation with bordism theory: Given

$$\begin{tikzcd} S^{N+8} \arrow[r, "f"] & M(N) \wedge (X \cup_\Phi \{\infty\}) \\ V^{N+8} \arrow[u, phantom, "\cup \text{ open}"] \arrow[r, "f'"] & E(N) \times X \arrow[u, phantom, "\cup"] \\ Z^8 \arrow[u, hook, "\text{closed codim } N"'] \arrow[r] & \underset{\text{compact.}}{B(N)} \times X \arrow[u, hook] \end{tikzcd}$$

$f'$ smooth transversal to $B(N) \times X$

Now given an $F \in \Phi$ it gives a closed subset of $X \cup_\Phi \{\infty\}$ not meeting infinity. Then $B(N) \times F$ is closed in $M(N) \wedge (X \cup_\Phi \{\infty\})$ & doesn't meet $\infty$ so if $u: Z \longrightarrow X$, then $u^{-1}F$ is compact. Conclude

$\{S, M \wedge (X \cup_\Phi \{\infty\})\} =$ bordism classes $u: Z \xrightarrow{\text{oriented}} X$ which are $\Phi$-proper, e.g. $u^{-1}F$ compact for all $F \in \Phi$.

<u>Examples:</u> $\Phi \ni X$. $\{S, M \wedge (X \amalg \text{pt.})\} = \{u: Z \longrightarrow X \quad Z \text{ compact}$

$\Phi =$ compacts $\{S, M \wedge X^+\} = \{u: Z \xrightarrow{\text{oriented}} X \text{ proper}\}$

$$\begin{array}{ccc}
S^{N+q} \wedge (X \cup_{\Phi} \{\infty\}) & \xrightarrow{\ f\ } & M(N) \wedge (Y \cup_{\Phi} \{\infty\}) \\
\cup & & \cup \\
\mathbb{R}^{N+q} \times X & & E(N) \times Y \\
\cup \text{ open} & \nearrow f & \updownarrow \\
V = f^{-1}(E(N)\times Y)) & & \\
\uparrow \text{closed codim } N \text{ orient} & & \\
Z & \xrightarrow[(\nu, u)]{} & B(N) \times Y
\end{array} \qquad \bigstar$$

**Definition:** Let $X, Y$ be pointed spaces $\underbrace{}_{\text{such that basepoints are closed}}$. Then $X \wedge Y$ is the space $X \times Y / X \vee Y$ with topology $\ni$ $(X - \{\infty\}) \times (Y - \{\infty\})$ is open and such that nbds of $X \vee Y$ are of form $(U \times Y) \cup (X \times V)$ where $U, V$ are nbds of $\infty$ in $X$ and $Y$ resp.

**Example:** If $X, Y$ compact, this topology is the quotient topology. Also if $X$ compact and $\{\infty\} \subset X$ is both open and closed, or if both $\{\infty\} \subset X$ + $\{\infty\} \subset Y$ are clopen.

With this definition we have

**Lemma:** $(X \cup_{\Phi} \{\infty\}) \wedge (Y \cup_{\Phi} \{\infty\}) = (X \times Y) \cup_{\Phi \times \Psi} \{\infty\}$.

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Return to above diagram $\bigstar$. One sees that $f^{-1}\{\text{Comp. of } E(N) \times F \in \Psi\}$ closed in $S^{N+q} \wedge (X \cup_{\Phi} \{\infty\})$ and doesn't meet $\infty$, hence contained in compact $\mathbb{R}^{N+q} \times$ element of $\Phi$.

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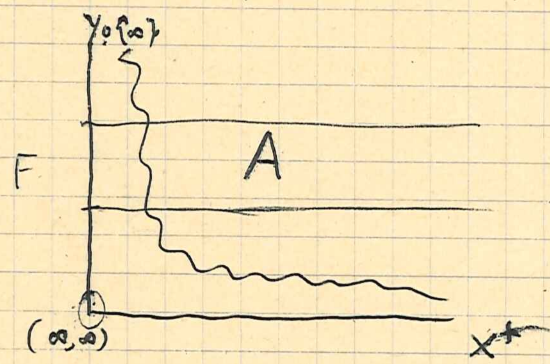
**Correction:** Just gave wrong definition of $X \wedge Y$.

Lemma: Let $X$ be a loc compact space, let $Y$ be a space, ~~with~~ ~~Let~~ $X^+$ be the 1-pt. compactification of $X$, and let $\Phi$ be a family of supports on $Y$. Then

$$X^+ \wedge (Y \cup_{\Phi} \{\infty\}) = (X \times Y) \cup_{\Psi} \{\infty\}$$

where $\Psi$ is the family of closed subsets $A$ of $X \times Y$ which are ~~$\Phi$ proper,~~ $\ni$ $pr_2 A \in \Phi$ and $\ni$ $pr_2 : A \longrightarrow pr_2 A$ is proper. (proper over a member of $\Phi$).
~~e.g. $A \cap (X \times F)$ compact for all $F \in \Phi$. proper over $F$ for all $F \in \Phi$.~~

Proof: As both spaces have same points and as $X \times Y$ is an open subspace of both, it suffices to compare neighborhoods of $\infty$. Take an open neighborhood of $\infty$ in the left space; it corresponds to an open nbd $N$ of $X^+ \vee \{Y \cup \{\infty\}\}$ in $X^+ \times (Y \cup \{\infty\})$




The complement of $N$ is a closed subset $A$ of $X \times Y$. Claim $A$ $\Psi$-proper: ~~Given~~ $F \in \Phi$ have to show $A \cap (X \times F)$ proper over $F$. So let $(x_\nu, y_\nu) \in A$, $y_\nu \in F$ $y_\nu \to y$; then $X^+$ compact $\Rightarrow$ $x_\nu$ ~~has~~ conv. subseq. $\therefore$ $(x_\nu, y_\nu) \longrightarrow (x,y) \in A$ so $x \neq \infty$. & limit belongs to $A \cap (X \times F)$. ✓ Conversely suppose $A \subset X \times Y$ is closed & $\Phi$-proper, yet $X^+ \vee \{Y \cup \infty\}$ ~~belongs to~~ meets the closure of $A$. Then have $(x_\nu, y_\nu) \in A$ with either $x_\nu \longrightarrow \infty$ or $y_\nu \longrightarrow \infty$. If

As $X^+$ is compact and $A$ doesn't meet $X^+ \times \{\infty\}$, $A$ doesn't meet $X^+ \times (Y - F \cup \{\infty\})$ e.g. $A \subset X \times F$. But $A$ is closed in $X^+ \times F$ hence proper over $F$, in particular $pr_2 A$ is closed hence $\in \Phi$. Conversely if $\exists\, F \in \Phi$ and $A$ is closed in $X \times Y$ and $A \subset X \times F$ is proper over $F$, then $A$ is ~~disjoint from~~ closed in $X^+ \times F$, hence closed in $X^+ \times (Y \cup_\Phi \{\infty\})$. ✓

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$$\begin{tikzcd}
S^{N+q} \wedge (X \cup_\Phi \{\infty\}) \arrow[r] & M(N) \wedge (Y \cup_\Phi \{\infty\}) \\
\mathbb{R}^{N+q} \times X \arrow[u, phantom, sloped, "\subset", "\text{comp. wrt subsets proper over a member of } \Phi"'] \arrow[r, phantom, "\text{cart.}"] & E(N) \times Y \arrow[u, phantom, sloped, "\subset", "\text{completion wrt subsets proper over a member of } \Phi"'] \\
V^{N+q+x} \arrow[u, phantom, sloped, "\subset"] \arrow[ur] & \\
Z^{q+x} \arrow[u, hook] \arrow[r, "{(\nu, u)}"'] \arrow[ur, phantom, "\text{cart.}"] & B(N) \times Y \arrow[uu, hook]
\end{tikzcd}$$

Therefore $u : Z \to Y$ has the property that every member of $u^{-1}\Phi$ is proper over a member of $\Phi$.

$$\begin{tikzcd}
& Z^{q+x} \arrow[dl, "\text{oriented}"'] \arrow[dr] & \\
X^x & & Y^y
\end{tikzcd}$$ .

should think intuitively of getting a map from

$$H^{k}_{\Phi}(X) \longrightarrow H^{k-q}_{\Phi}(X).$$

**Conclusion:**

$$\boxed{\begin{gathered}\left\{ S^{-q} \wedge \left(X \cup_{\Phi} \{\infty\}\right), {}^{Mn}\left(Y \cup_{\Phi} \{\infty\}\right) \right\} \\ \simeq \operatorname{Hom}^{q}_{(\text{motive})}\left( H^{*}_{\Phi}(Y),\ H^{*}_{\Phi}(X) \right).\end{gathered}}$$

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~~Objections~~:

Remaining work:

ⓐ Check this conclusion e.g. if $\Phi$ given by a single closed subset of $X$

ⓑ Sullivan spectra

ⓒ Construction of motive category as a triangulated category.

ⓓ Orientation

ⓔ Contravariant singularities.