1

On simple characteristics (Nov. 8, 1968)

Let S be a scheme let $\theta : \mathcal{E} \longrightarrow \mathcal{F}$ be a morphism of locally free sheaves of ranks e and f, respectively. Let $G_e(\mathcal{E} \oplus \mathcal{F})$ be the Grassmannian of e-planes in $\mathcal{E} \oplus \mathcal{F}$. Then

$$\begin{tikzcd} G_e(\mathcal{E}\oplus\mathcal{F}) \arrow[r, phantom, "\overset{\text{closed}}{\subset}"] & \mathbb{P}\left(\bigoplus_{i=0}^{\infty} \mathcal{H}om(\Lambda^i\mathcal{E}, \Lambda^i\mathcal{F})\right) \\ \mathcal{H}om(\mathcal{E},\mathcal{F}) \arrow[u, hook, "\text{dense open}"] & \end{tikzcd}$$

$$\alpha \longmapsto \sum_{i=0}^{\infty} \Lambda^i \alpha$$

where $\mathbb{P} = G_1$.

<u>Hörmander blow-up of θ:</u> Let $S[T] = S \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{Z}[T]$. Then have following situation

$$\begin{tikzcd} \operatorname{Spec} S[T,T^{-1}] \arrow[r, "\text{graph } T^{-1}\theta"] \arrow[dr, hook, bend right] & G_e(\mathcal{E}\oplus\mathcal{F})[T] \arrow[d] \arrow[r, phantom, "\subset"] & \mathbb{P}\left(\bigoplus_i \mathcal{H}om(\Lambda^i\mathcal{E},\Lambda^i\mathcal{F})\right)[\\ & S[T] & \end{tikzcd}$$

so we can construct the closure, which by a familiar calculation is

<u>Proj</u> $\bigoplus_{n=0}^{\infty} \mathcal{J}^n$ where $\mathcal{J}$ is the ideal in $\mathcal{O}_S[T]$ generated by ~~the element~~ the components locally of the ~~[illegible]~~ $T^{e-i}\Lambda^i\theta$

<u>Lemma:</u> Let $\theta : \mathcal{E} \to \mathcal{F}$ be a morphism of loc. free sheaves of ranks $e + f$ resp over S. Let $\mathcal{I}$ be the ideal generated by the elements of any local matrix representations of θ. Then $\mathcal{I} \simeq \operatorname{Ann} \Lambda^f(\operatorname{Coker}\theta)$

$$\simeq \operatorname{Ann} \Lambda^e(\operatorname{Ker}\theta^t)$$

Proof: May assume $S = \operatorname{Spec} R$ with $\mathcal{E}, \mathcal{F}$ free over R

$$R^e \xrightarrow{\theta} R^f \longrightarrow M \longrightarrow 0$$

$$\theta \sum_{i=1}^{e} r_i v_i = \sum_{j=1}^{f} r_i \theta_{ij} w_j$$

Have
$$R^e \otimes \Lambda^{f-1} R^f \xrightarrow{\beta} \Lambda^f R^f \longrightarrow \Lambda^f M \longrightarrow 0$$

$$\beta: \quad v_i \otimes (w_1 \wedge \dots \wedge \hat{w}_j \wedge \dots \wedge w_f) \longmapsto \pm \theta_{ij} (w_1 \wedge \dots \wedge w_n)$$

Thus $\operatorname{ann} \Lambda^f M = \underline{\sum_{ij} R\theta_{ij}}$. QED.

$$\therefore \mathcal{J} = \underline{\sum_{i=0}^{e} T^{e-i} \operatorname{ann} \Lambda^{\binom{f}{i}} \{\operatorname{Coker} \Lambda^i \theta\}}. \qquad \text{not much help.}$$

Recall that Hörmander's (test) estimates are obtained by taking a curve $t^{-n}(\xi_0 + t\xi_1 + t^2\xi_2 + \dots + t^{n-1}\xi_{n-1})$ heading to ∞. Then the graph $A^*(\xi(t))$ converges and we get a test estimate defined on

$$\frac{\{u(0) \in E^\delta \mid \exists\, u(t) \text{ with } A^\delta(\xi(t))u(t) \equiv 0 \ (t^{n-1})\}}{\{w_n \in E^\delta \mid \exists\, u(t) \text{ in } E^{\delta-1} \text{ with } u(0) = 0 \text{ and } A^{\delta-1}(\xi(t))u(t) \equiv w_n t^n \ (t^{n+1})\}}$$

and with induced transformations.

$$S_\xi(z)\, u_0 = \frac{A^\delta(\xi(t) + zt^n)\, u(t)}{t^n}$$

Claim $S_\xi(\xi_0) \equiv 0$? In effect $S_\xi(z) = S_{\xi + t^n z}$ and

~~$\varphi(t) + t^n \varphi_0 = (1+t^n)\varphi_0 + t\varphi_1 + \cdots + t^{n-1}\varphi_{n-1}$~~

$$t^{-n}\left(\varphi(t) + \varphi_0\right) = t^{-n}\left\{(1+t^n)\varphi_0 + t\varphi_1 + \cdots + t^{n-1}\varphi_{n-1}\right\}$$

Set $(t')^{-n} = t^{-n}(1+t^n) = \left(1 + \frac{1}{t^n}\right)$. Then

$$t^{-n}\left(\varphi(t) + \varphi_0\right) = (t')^{-n}\varphi_0 + t^{-n+1}\varphi_1 + \cdots + t^{-1}\varphi_{n-1}$$

$$\cong (t')^{-n}\varphi_0 + (t')^{-n+1}\varphi_1' + \cdots + (t')^{-1}\varphi_{n-1}' + c$$

where c is a linear combination of $\varphi_1, \dots, \varphi_{n-1}$?

Lemma. If graph $A(\varphi_n) \longrightarrow \Gamma$

then graph $A(\varphi_n) + B$ also converge to Γ' say and

$$S_{\Gamma'} = S_\Gamma + \overline{B}$$

$S_{\Gamma'}$ is map from $pr_1(\Gamma') \subset E$ to $\Gamma'/\Gamma' \cap F$

and
$$\begin{aligned} \mathrm{Hom}(E, F) &\longrightarrow \mathrm{Hom}(pr_1\Gamma', \Gamma'/\Gamma'\cap F) \\ B &\longmapsto \overline{B}. \end{aligned}$$

$$R^f \xrightarrow{\theta} R^e \longrightarrow M \longrightarrow 0$$

I_j = ideal in R generated by $j \times j$ minors of θ

Then
$$\boxed{\mathrm{Zero}(I_j) = \mathrm{Supp}\ \Lambda^{e-j+1} M}$$

and
$$\boxed{I_1 = \mathrm{ann}\ \Lambda^e M}$$

but if $M = R/fR \oplus R/fR$ we have that

$$(f^2R) = I_e \neq \mathrm{ann}\ M = (fR)$$

Simple characteristics (Nov. 10, 1968)

Ideas:

ⓐ Let M be an R-module of finite type, R noetherian and let $J_j = \operatorname{Ann} \Lambda^j M$. What is the result of blowing up the ideal generated by $T^j J_j$ in $R[T]$ and then setting $T = 0$?

ⓑ Is the following of simple characteristics type? $R = \mathbb{C}[X_{ij}]$ $1 \le i \le f$ $1 \le j \le e$ with $R^f \xrightarrow{\theta} R^e \longrightarrow M \longrightarrow 0$ given by the X_{ij}.

Let M be a modules of finite type over a regular local ring R which is a $\mathbb{C}$-algebra where $\mathbb{C} =$ res. field of R, and let $P_\bullet$ be a free finite type resolution of M. Let $A_n = \mathbb{C}[t]/(t^{n+1})$ and let $\varphi : R \longrightarrow A_n$ be a $\mathbb{C}$-morphism. The problem is to describe intrinsicly the Hörmander test complex associated to the map φ and to show that it depends only on M and not on the choice of $P_\bullet$. As R/k smooth ($k = \mathbb{C}$) have

$$0 \longrightarrow D^0(A_n/R, k) \longrightarrow \underset{\text{}}{\overset{\substack{k\cdot\text{derivation } f \mapsto f'(0) \\ \|}}{D^0(A_n/k, kt^{n+1})}} \longrightarrow \overset{\substack{(m/m^2)^\vee = T \\ \|}}{D^0(R/k, kt^{n+1})} \longrightarrow$$

$$\longrightarrow D^1(A_n/R, k) \longrightarrow \underset{\substack{\| \\ k\cdot\text{class of } A_{n+1}}}{D^1(A_n/k, kt^{n+1})} \longrightarrow 0$$

Two cases:

① tangent vector $z'(0)$ to curve is $\neq 0$, then $D^1(A_n/R, kt^{n+1})$ a torsor for $T/z'(0)$

② $z'(0) = 0$, then $D^1(A_n/R, kt^{n+1})$ torsor for T.

Proposition: Let R be a (local) noetherian ring and let M be an R-module of finite type. TFAE

(i) $M \simeq \bigoplus_{i=1}^{q} R/f_i R$ where f_i are regular elements, $f_{i+1} \mid f_i$

(ii) $\operatorname{Ann} \Lambda^i M$ is an invertible ideal of R for each i

Proof: (i) $\Rightarrow$ (ii). Suppose $M = \bigoplus_{i=1}^{q} R/\mathfrak{a}_i$ where

$$\mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \cdots \subset \mathfrak{a}_q$$

Then

$$\Lambda^i M = \bigoplus_{1 \le j_1 < \dots < j_i \le q} R/\mathfrak{a}_{j_1} + \dots + \mathfrak{a}_{j_i} = \bigoplus_{1 \le j_1 < \dots < j_i \le q} R/\mathfrak{a}_{j_i}.$$

$$\therefore \operatorname{Ann} \Lambda^i M = \mathfrak{a}_i$$

(ii) $\Rightarrow$ (i). Induction on $q = \dim M \otimes_R k$. ~~By assumption~~

Let

$$R^{\ell} \xrightarrow{\theta} R^{q} \longrightarrow M \longrightarrow 0$$

be a minimal presentation of M so that $\theta_{ij} \in \mathfrak{m}$ $1 \le i \le q$, $1 \le j \le \ell$. Claim

$$\operatorname{Ann} \Lambda^q M = \sum_{\substack{1 \le i \le q \\ 1 \le j \le \ell}} R\theta_{ij}$$

in effect, $\exists$ exact seq

$$R^{\ell} \otimes \Lambda^{q-1} R^{q} \longrightarrow \Lambda^{q} R^{q} \longrightarrow \Lambda^{q} M \longrightarrow 0$$

$$v_j \otimes (e_1 \wedge \cdots \hat{e}_i \cdots \wedge e_q) \longmapsto \theta_{ij}(e_1 \wedge \cdots \wedge e_n).$$

By assumption, $\operatorname{Ann} \Lambda^q M$ is principal, say $= Rf$, f non-zero divisor of R. Thus may form the matrix $\theta'_{ij} = f^{-1}\theta_{ij}$ and define N by exact sequence $R^{\ell} \xrightarrow{\theta'} R^{g} \longrightarrow N \longrightarrow 0$. ~~By assumption~~ As $\sum_{ij} R\theta'_{ij} = R$, not all element of θ' lie in $\mathfrak{m}$ and so this presentation of N is not minimal. Consequently after multiplying θ' on the left and right by invertible matrices we may assume that θ' and θ are of the form

$$\theta' = \left[\begin{array}{c|c} \eta & \\ \hline & s\{\ \boxed{id} \end{array}\right] \qquad \theta = \left[\begin{array}{c|c} f\eta & \\ \hline & \boxed{f\,id} \end{array}\right]$$

where $\eta_{ij} \in \mathfrak{m}$. In other words we have $M = M' \oplus (R/fR)^s$ with $s > 0$ and $M'/fM' \simeq (R/fR)^{g-s}$. We shall now apply our induction hypothesis to M'. Noting that

$$\Lambda^i M \simeq \bigoplus \Lambda^j (R/fR)^s \otimes \Lambda^{i-j} M'$$
$$= \Lambda^i M' \oplus \underbrace{\bigoplus_{j>0} \Lambda^j (R/fR)^s \otimes_{R/fR} \Lambda^{i-j} (R/fR)^{g-s}}_{\text{free over } R/fR}$$

~~we see that~~ ~~and the~~ and $\operatorname{Ann} \Lambda^i M' \subset \operatorname{Ann} \Lambda^i (R/fR)^{g-s} = fR$ for $0 \le i \le g-s$, we see that

$$\operatorname{Ann} \Lambda^i M = \operatorname{Ann} \Lambda^i M' \qquad \text{for} \quad 0 \le i \le g-s$$

is principal by hypothesis. Thus induction can be applied

QED.

Nice argument: If $\Lambda^g M \simeq R/fR \neq 0$ and f is regular, then $M \simeq (R/fR)^s \oplus M'$ with $s > 0$.

or you note that if M has elementary divisors so ~~does~~ do all the $\Lambda^q M$.

M / local ring R rank q

$$\boxed{\Lambda^q M \simeq R}$$

$\Longleftrightarrow$ M free rank q

M rank q
$\downarrow$
R

$$\Lambda^q M \simeq R/\underbrace{\mathrm{ann}\,\Lambda^q M}_{\parallel\ \mathfrak{a}_q}$$

blowup

locally $M = M' \oplus R/fR$

where $M'/fM' \simeq (R/fR)^{q-1}$

and $\Lambda^{q-1} M = \underbrace{\Lambda^{q-1} M'}_{\text{rank } 1} \oplus (R/fR)^{q-1}$

So locally upstairs

$$\Lambda^{q-1} M \simeq A/_{\mathrm{ann}\,\Lambda^{q-1}M} \oplus \Big[A/\underbrace{\mathrm{ann}\,\Lambda^q M}_{\text{principal}}\Big]^{q-1}$$

so blowing up $\mathrm{ann}\,\Lambda^{q-1} M$

we may it principal

blow up $\mathrm{ann}\,\Lambda^q M$ first

then $\mathrm{ann}\,\Lambda^{q-1} M$

etc. Each time one blows up the isoms. don't change

$$0 \to K \to M \xrightarrow{t_1} I \to 0$$

$$0 \to I \xrightarrow{\deg 1} M \to M/t_1M \to 0$$

$$H_1(M)_k \to \underline{H_1(I)_k} \xrightarrow{\partial} \overline{H_0(K)_k \to H_0(M)_k \to H_0(I)_{k+1}}$$

$$H_1(M)_k \to H_1(M/t_1M)_k \to H_0(I)_k \to H_1(M)_k$$

$$\left.\begin{array}{l} H_0(M)_+ = 0 \\ H_1(M)_+ = 0 \end{array}\right\} \Longrightarrow \begin{array}{l} H_0(M/t_1M)_+ = 0 \\ H_1(M/t_1M)_+ = 0 \end{array}$$

also get $\quad H_q(t_2, .., t_n; M/t_1M)_+ = 0 \quad q = 0, 1$

so by induction M/t_1M involutive and all $t_2, .., t_n$ s.q.r.

$$H_q(t_2, .., t_n; M/t_1M)_+ = 0$$

$$\Longrightarrow \quad M/t_1M \text{ involutive} \qquad t_2, .., t_n \text{ s.q.r.}$$

Element t

$$0 \to M_1/(t_1 \cdots t_{q-1})M_0 \xrightarrow{t_q} M_2/(t_1, .., t_{q-1})M_1$$

$$\subset (t_1 \cdots t_q)M_1 \subset \cdots \qquad \subset (t_1, .., t_n)M_1 = M_2$$

Thus the last conditions may be phrased,

$$\dim M_2 = \sum_{q=0}^{n-1} \dim M_1/(t_1, .., t_q)M_0 \qquad \text{always} \leq .$$

or you note that if M has elementary divisors so ~~does~~ do all the $\Lambda^{j} M$.

~~$\Lambda^q M$~~ $\quad M/$local ring $R \quad$ rank q

$$\boxed{\Lambda^q M \simeq R}$$

$\iff M$ free rank q

M rank q, $\quad M / R$

$$\Lambda^q M \simeq R/\underbrace{\mathrm{ann}\, \Lambda^q M}_{= \sigma_q}$$

blowup.

locally $\quad M = M' \oplus R/fR$

where $\quad M'/fM' \simeq (R/fR)^{q-1}$

and $\quad \Lambda^{q-1} M = \underbrace{\Lambda^{q-1} M'}_{\text{rank } 1} \oplus (R/fR)^{q-1}$

So locally upstairs

$$\underbrace{\Lambda^{q-1} M \simeq A/_{\mathrm{ann}\, \Lambda^{q-1} M} \oplus \Big[A/\underbrace{\mathrm{ann}\, \Lambda^{q} M}_{\text{principal}}\Big]^{q-1}}$$

so blowing up $\quad \mathrm{Ann}\, \Lambda^{q-1} M$

we may it principal

blow up $\quad \mathrm{Ann}\, \Lambda^{q} M$ first

then $\quad \mathrm{Ann}\, \Lambda^{q-1} M$

etc. $\qquad$ Each time one blows up the isoms. don't change

Summary of research on simple characteristics (Nov. 18, 1968)

1. If M is a finite type module over a reg. local ring R with residue field k, then a necessary condition that M have "simple characteristic" is that $gr^m M$ be involutive over $gr^m R = S(m/m^2)$. Question: From viewpoint of commutative algebra we want to require this for all localization M_p over R_p. Can one deduce this from the test complexes at the maximal ideal?

2. $gr^m M$ involutive $\Longrightarrow$ Cartan expressions for M as $\oplus ST_i \otimes P_i$ remain valid?

3. Let $g = \bigoplus_{n=0}^{\infty} g_n$ be a graded module over ST which is involutive. Classify $\widehat{ST}$ modules M ~~such that~~ endowed with an isomorphism $g \simeq gr^m M$. Let C_n be the isomorphism classes of ST modules M_n endowed with isom $gr^m M_n \cong \bigoplus_{k=0}^{n} g_k$. Then given $M_{n-1} \in C_{n-1}$, the fiber of the map $C_n \to C_{n-1}$ over M_{n-1} is the subset of $Ext^1_{ST}(M_{n-1}, g_n) = Hom_k(Tor_1^{ST}(k, M_{n-1}), g_n)$ consisting of the extensions

$$0 \to g_n \longrightarrow Q \longrightarrow M_{n-1} \to 0$$

such that the map $S_nT \otimes g_0 \to g_n$ induced by this extension is the one given by the module structure of g. But as g is involutive one has

Lemma: $\exists$ exact sequence

$$0 \longrightarrow g_n \longrightarrow Tor_1^{ST}(k, M_{n-1}) \longrightarrow H_1(g) \longrightarrow 0$$

where $H_1(g)$ denotes Kosyul homology of g.

Proof: Let $\alpha \in H_1(g) = \mathrm{Ker}\{T\otimes g_0 \to g_1\}$ and represent α by $\alpha_0 \in T\otimes M_{n-1}$. Assume i largest $\ni$ $d\alpha_0 \in m^i M_{n-1}$. Then $i \geq 2$ and as $T\otimes g_{i-1} \to g_i$ is onto by involutiveness, can find $\beta \in T\otimes m^{i-1}M$ so that $d(\alpha_0-\beta) \in m^{i+1}M$. This shows that $i=\infty$ and that $\mathrm{Tor}_1^{ST}(k, M_{n-1}) \longrightarrow H_1(g)$ is onto. Next if $\alpha \in T\otimes M_{n-1}$, $d\alpha=0$ we move α by a boundary until $\alpha \in T\otimes m^i M_{n-1}$ with i largest. If $i<n-1$, then $\bar\alpha \in \mathrm{Ker}\{T\otimes g_i \to g_{i+1}\}$ so can make i larger. Thus can assume $\alpha \in T\otimes g_{n-1} \subset T\otimes m^{n-1}M_{n-1}$. But by same reasoning as before $B_1\{T\otimes M_{n-1}\} \cap T\otimes g_{n-1} = \mathrm{Im}\{\Lambda^2 T\otimes g_{n-2} \to T\otimes g_n\}$ so the image of α in g_n determines coh. class.

Thus get

$$0 \longrightarrow \mathrm{Hom}_k(H_1(g), g_n) \longrightarrow \mathrm{Hom}_k(\mathrm{Tor}_1^{ST}(k, M_{n-1}), g_n) \xrightarrow{\pi} \mathrm{Hom}(g_n, g_n) \longrightarrow 0$$

where I will assume that $\pi^{-1}\{id\}$ is the extensions I am looking for.

Conclusion: $C_n \longrightarrow C_{n-1}$ is a torsor for the group

$$\mathrm{Hom}_k(H_1(g), g_n).$$

This tends to suggest that the moduli space C is an infinite dimensional affine space with tangent space $\mathrm{Hom}_k(H_1(g), M)$ at M.

~~Suppose that M is a blown up!~~

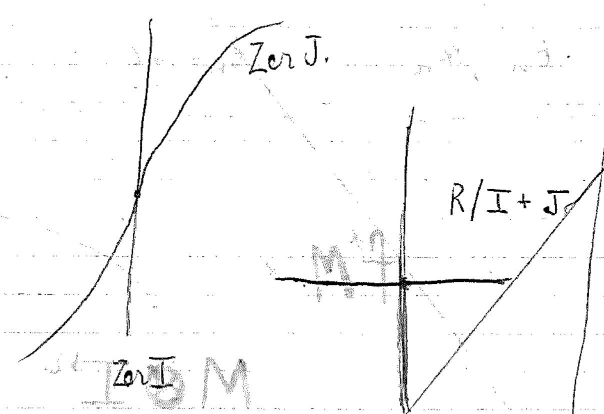
Given M over R, suppose $M \cong \bigoplus R/f_i R$. with

$f_2 | f_1$, $f_3 | f_2$, ...

Question: Suppose R/I non-sing. R/J non-sing.

Whether blowing up R/I + R/J is non-sing. ?

~~Next question is what to do about lines~~



$R/I \oplus R/J$

$R/I+J$

$R/I \oplus R/J$ has element divisors $\iff$ I+J princip

I+J = fR

$f^{-1}I + f^{-1}J = 1$

∴ either I = fR

or J = fR.

Suppose $V = V_1 \oplus V_2$

~~so that $V/?$ ~ direct sum of $V_1 + V_2$.~~

Blowup equals plane containing V_1 + point

Zero $I = W$ is a linear subspace of V of dim k say

$B_I = (A, v)$ $v \in V$ $W < A$ dim k+1 $v \in A$

~~Consider now~~ Assume now $\underset{Z}{Z(J)}$ meets W properly ?

ie. dim l = dim $V - k$. and the ∩ is 0-diml. Consider pairs now where (A, v) but $v \in Z$ $W < A \ni v$ dim k+1.

Is this non-singular?

~~On differential equations with simple character~~

Suppose Θ is a 2×2 matrix.

$$Tf_1, \dots, Tf_n, g_1, \dots, g_m$$

$$(t,x) \longmapsto \underline{tf_1(x), \dots, tf_n(x)}, g_1(x), \dots, g_m(x)$$

and setting $t=0$.

want all limits t_n, x_n $\underline{t_n \to 0}$

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example

R M
$\underline{m}$

$f^*M \times$

$M \otimes I^n$

$A/J \otimes I^n = I^n / I^{n+1}$

$$\begin{array}{l} Y = \operatorname{Proj} R \oplus m \oplus m^2 \oplus \dots \\ \downarrow f \\ \operatorname{Spec} R \end{array}$$

$$\underline{f^*M} \otimes_R k = \widetilde{\operatorname{gr} M} \text{ on } \operatorname{Proj} \operatorname{gr} R. \quad \text{clear!!!}$$

---

may be the condition is that

$$\begin{tikzcd} f^*M \quad \mathrm{Blow}(M) \arrow[d, "f^*"] & \arrow[l] \arrow[d] \\ M \quad \operatorname{Spec} R & \operatorname{Spec} k \arrow[l] \end{tikzcd}$$

Question: Let M be a graded involutive ST module such that $\tilde{M}$ is a line bundle on a non-singular subvariety of $\mathbb{P}(T^\vee)$. ~~Then~~ Let Char be the characteristic cone of M ~~[illegible]~~ and let L be the line bundle on Char-0 given by ~~[illegible]~~

$$L(\lambda) = \mathrm{Hom}_{ST}(M, k(\lambda)) \qquad \lambda \in \mathrm{Char}-$$

Define

$$\varphi : L \longrightarrow \mathrm{Hom}_{ST}(M, H_\infty)$$

by

$$(\ell, \lambda) \longmapsto (m \longmapsto \ell(m) e^\lambda).$$

Does φ extend to a morphism of a line bundle over the ~~[illegible]~~ canonical desingularization of Char?

Subquestion: Does L extend to the canonical desingularization of Char?

~~[illegible] to follow~~ Yes

$$L = \{ (e, \lambda) \mid e \in E^\circ, \lambda \in T^\vee - 0 \text{ and } A(\lambda) e = 0 \}.$$

$$\widetilde{\mathrm{Char}} = \{ (\ell, \lambda) \mid \ell \in \mathbb{P}(T^\vee), \lambda \in T^\vee \quad \lambda \in \ell \}.$$

$$\tilde{L} = \{ (\ell, e, \lambda) \mid \ell \in \mathbb{P}(T^\vee), \lambda \in \ell, A(\ell) e = 0 \}.$$

$L \subset \tilde{L} \xrightarrow{pr_{13}} \widetilde{\mathrm{Char}}$ line bundle since fiber is $\mathrm{Ker}\, A(\ell)$.

$\mathrm{Char}-0 \subset \widetilde{\mathrm{Char}} \longrightarrow \mathrm{Char}$ birational.

$\varphi : L \to \{\text{solutions}\}$ $\qquad \varphi(e, \lambda) = e \cdot e^{\lambda(z)}$

$\tilde{\varphi}(l, e, \lambda) = e \cdot e^{\lambda(z)}$ certainly defined if $\lambda \neq 0$. Above question is ?
(but not correct question maybe?)

??????
????

If $\lambda = 0$ then the map $(l, e) \longmapsto e$ non-trivial

~~Could have equally had $e \cdot f(\lambda(z))$~~

~~Since $P(D)\{e \cdot f(\lambda(z))\} = f'(\lambda(z)) P(\lambda) e = 0$~~

~~$\tilde{\varphi}(l, e, \lambda)$~~

~~$D_1^i f(\sum \lambda_i z_i) = D^{i-1} f'(\sum \lambda_i z_i) \lambda_1$~~

~~$= f^{(i)}(\sum \lambda_i z_i) \lambda_1^i$~~

Question 2: According to my theory the map $\tilde{\varphi}: \Sigma \longrightarrow (\mathrm{Soln})$ extends to a map $\mathbf{D_0}(\widetilde{\mathrm{Char}}, \Sigma) \longrightarrow (\mathrm{Solns})$

What does it mean for this map to be an isomorphism?

$$f(z) = \int e^{z\overline{w}} f(w)\, dG$$

$P(D) f$

$P(\overline{w})\, f(w)$

fundamental solution here? Then

$P(D)$: onto.

$$\begin{array}{ccc}
\operatorname{Proj} \bigoplus_{n\geq 0} \mathfrak{m}^n & \longleftarrow & \operatorname{Proj} \bigoplus_n \mathfrak{m}^n/\mathfrak{m}^{n+1} \\
\downarrow & & \downarrow \\
\operatorname{Spec} R & \longleftarrow & \operatorname{Spec} k
\end{array}$$

modules: $\bigoplus \mathfrak{m}^n \otimes_R M \twoheadrightarrow \bigoplus \mathfrak{m}^n M \to \boxed{\bigoplus \left(\mathfrak{m}^n/\mathfrak{m}^{n+1}\right) \otimes \left(M/\mathfrak{m}M\right)} \twoheadrightarrow \mathfrak{m}^n M/\mathfrak{m}^{n+1} M.$

$\tilde{M}$.

---

strict transform of a module.

If J is an ideal get total transform

$$\mathcal{O}/\mathcal{J} \simeq f^*\left(\widetilde{R/J}\right) = \widetilde{\bigoplus I^n/JI^n}$$

$$0 \to \bigoplus JI^n \to \bigoplus I^n \to \bigoplus I^n/JI^n \to 0$$

---

total transform: $\widetilde{\bigoplus_{n\geq 0} I^n J}$ is an ideal in $\mathcal{O}$

strict transform: $\widetilde{\bigoplus_{n\geq 0} I^n \cap J}$ —— $\mathcal{O}$.

<u>Check</u>: if $J \supset I^n$ some n, then strict transform is empty.

~~Statistical mechanics~~ (after Mackey).

X symplectic manifold, fundamental 2-form Ω.

H proper function on X, real-valued

To H is associated a flow X_H.

In statistical mechanics one seeks a probability distribution $\rho\Omega^n$, representing the most ~~[illegible]~~ random distribution having a given energy. Thus want to maximize

$$\int \rho \log \rho \; \Omega^n \quad =: \text{information about the system}$$

subject to conditions

$$\int \rho\, \Omega^n = 1 \qquad \int \rho H\, \Omega^n = E.$$

Not quite one assumes a given energy E ~~and ~~ assume $dH \neq 0$ on $\{H=E\} = Z_E$ and calculate the minimal distribution.

Hence have volume element $\Omega^{n-1} \wedge dH$ on Z_E or

$$i(X_H)\,\Omega^n = n\, dH \wedge \Omega^{n-1}$$

so want ~~[illegible]~~ $\int \rho \, dH \wedge \Omega^{n-1} = 1$

max. $\int \rho \log \rho \, dV$ $\quad \int \rho\, dV = 1$

sequence of numbers

$$\sum \lambda_i = E$$

$$\sum \lambda_i \log \lambda_i \quad \text{minimum.}$$

Example: dim $T=2$.

$\mathcal{F}$ torsion coherent sheaf on $\mathbb{P}(T^\vee)$

Assume $\infty \notin \operatorname{supp} \mathcal{F}$ and write

$\mathcal{F} = \tilde{M}$ $\quad M$ $\overset{(\text{torsion}}{\text{module}}$ over $\mathbb{C}[D]$.

Then M ~~[illegible]~~ is a direct sum of cyclic modules, so

$$M \simeq \bigoplus_i \mathbb{C}[D]/(D-\alpha_i)^{n_i}$$

and simple characteristics occurs if all $n_i = 1$. In terms of elementary divisors this means what?

$$M = R/p_1^{a_1} \oplus \cdots \oplus R/p_n^{a_n}$$

$$\Lambda^i M = \bigoplus_{1 \le j_1 < \cdots < j_i \le n} R/(p_{j_1}^{a_{j_1}}, \ldots, p_{j_i}^{a_{j_i}}) \qquad ?$$

So min. poly has no multiple factors. This means

$$R/f_1R \oplus R/f_2R \oplus \cdots \oplus R/f_nR$$

where $f_2 | f_1$, $f_3 | f_1$, ...

so min poly is $\operatorname{Ann} M$. ~~[illegible]~~.

---

dim $R=2$. If M is a finite type $\overset{\text{torsion}}{\text{module}}$ over a reg. loc. ring of dim 2, then by Iwasawa M is equivalent ~~[illegible]~~ mod finite length modules to $\overset{(\text{finite})}{\text{a}}$ direct sum of cyclic modules.

$$0 \to \underbrace{H^0_{\mathfrak{m}}(M)}_{\text{finite length}} \to M \to H^0(\operatorname{Spec} R - \{\mathfrak{m}\}, \tilde{M}) \longrightarrow \underbrace{H^1_{\mathfrak{m}}(M)} \to 0$$

$$\operatorname{Hom}_R\left(H^q_{\mathfrak{m}}(M), H^2_{\mathfrak{m}}(R)\right) \simeq \operatorname{Ext}_R^{2-q}(M, R)$$

Problem: $\boxed{\text{What are dist. killed by } P(\xi)?}$

~~$P(D)$~~ dist. on $\check{T}$ killed by $P(\xi)$

$P(D)$

$\xi \in \check{T} \quad e \in E^0$

$\varphi = \delta_\xi \cdot e$ = dist. given by

$$\boxed{\varphi(u) = u(\xi)e}$$

distributions on

given a map $f: Z \longrightarrow V$ smooth

where V is a complete loc. convex topological vector space it extends uniquely to a map

$$\begin{tikzcd} \mathrm{Dist}(Z) \arrow[r] & V. \\ Z \arrow[u] \arrow[ur, dashed] & \end{tikzcd}$$

$$C_1(Z) \xrightarrow{\partial} C_0(Z)$$

currents: Given a map of degree q from Z to V ie an element $C^q(Z, V)$

$$C^q(Z, V) = \mathrm{Hom}(C_q(Z), V)$$

where $C_q(Z)$ = q-currents

So now the representation problem appears as follows. To define a map of $Z \longrightarrow$ solutions so that $C_0(Z) \xrightarrow{\sim}$ solutions.

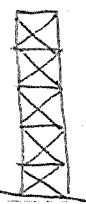
$$\sum_{i=1}^{n} \dim M_1/(t_1, \dots, t_{i-1})M_0 = \dim M_2$$

$$T \otimes M_0 \longrightarrow M_1 \longrightarrow 0$$

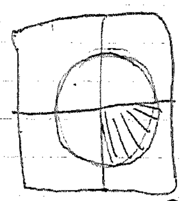
The equations of the Hilbert scheme (Chow varieties) ???

$$\sum_{i=1}^{n} \dim M_1/(t_1, \dots, t_{i-1})M_0 = \dim M_2$$

$$M_1/(t_1,\dots,t_{i-1})M_0 \xrightarrow{t_i} \frac{(t_1,\dots,t_i)M_1}{(t_1,\dots,t_{i-1})M_1} \longrightarrow 0$$



Description: Given M_0, T. To consider ~~quotients~~ flags F in T and quotients



$$\cancel{M_0 \otimes T \longrightarrow M_1 \longrightarrow}$$

$$M_0 \otimes T \twoheadrightarrow M_1 \twoheadrightarrow Q_1 \twoheadrightarrow Q_2 \twoheadrightarrow \cdots \twoheadrightarrow Q_n = 0.$$

$$\cancel{M_0 \otimes S^2T \longrightarrow M_2 \longrightarrow M_1/TM_0} \qquad M_0 \otimes S^2T \longrightarrow M_2$$

such that (i) $M_0 \otimes T_i \longrightarrow Q_i$ is zero

(ii)

$$\cancel{M_0 \otimes S^2T \longrightarrow M_2}$$

$$\cancel{M_1 \otimes T \longrightarrow M_0 \otimes A}$$

$$\cancel{M_0 \otimes T \otimes T \longrightarrow M_1}$$

$$\downarrow$$

$$\cancel{M_0 \otimes S^2T \longrightarrow M_2}$$

(F, Q, M_2)

closed subvariety of Flag $\times$

$(F, Q, M_2) \longrightarrow M_2$ OKAY map. Consider fibre $=$

$K_1 = \operatorname{Ker} M_0 \otimes T \to M$

then

$$K_1 \otimes T \longrightarrow M_0 \otimes T \otimes T$$
$$\downarrow$$
$$M_0 \otimes S^2T$$
$$\downarrow$$
M_2 is C

Elementary divisors:

Prop: R local noeth ring, M f.t. R module. TFAE:

(i) $M \simeq \bigoplus_{i=1}^{q} R/f_i R$ f_i non-zero divisor $f_{i+1} | f_i$

(ii) ~~[illegible]~~ $\operatorname{Ann} \Lambda^i M =$ ~~[illegible]~~ $f_i R$ where f_i is a non-zero divisor.

Proof: (i) $\Longrightarrow$ (ii)

$$\Lambda^j M = \bigoplus_{1 \le i_1 < \cdots < i_j \le q} R/(f_{i_1}, \ldots, f_{i_j})R$$

$$= \bigoplus_{1 \le i_1 < \cdots < i_j \le q} R/f_{i_j} R$$

$$\therefore \operatorname{Ann} \Lambda^j M = f_j R.$$

Defn + Prop: If $\mathcal{F}$ is a ~~[illegible]~~ coherent sheaf on a noetherian schem X, then $\exists\ f : Y \to X$ universal $\ni f^*\mathcal{F}$ has elementary divisors. Y is called the result of blowing-up the ~~[illegible]~~ sheaf $\mathcal{F}$.

~~Proof: Construction of Y: [illegible] on X, [illegible]~~

~~[illegible] Next~~

For large q $\Lambda^q \mathcal{F} = 0$, so $\operatorname{Ann} \Lambda^q \mathcal{F}$ is ~~[illegible]~~ invertible. ~~[illegible]~~ Define a sequence of blow-ups

$$Y = Y_0 \quad \cdots\cdots \longrightarrow Y_{q-1} \to Y_q = X$$

by Y_{q} = result of blowing up the ideal $\operatorname{Ann} \Lambda^q \mathcal{F}_{q+1}$ where $\mathcal{F}_{q+1}$ = inverse image of $\mathcal{F}$ under $Y_{q+1} \to X$. By a local analysis one sees that $\operatorname{Ann} \Lambda^j \mathcal{F}_q$ are inv. for $j \ge q$, and hence stable under base change.

Check: If M has rank $\dim M \otimes_R k = r$ and $\operatorname{Ann} \Lambda^j M$ is ~~principal~~ invertible for $j \geq g$, then one knows that

$$M \simeq M' \oplus R/f_g R \oplus \cdots \oplus R/f_r R \quad \text{where}$$

$f_g, \ldots, f_r$ are invertible and

f_{j+1} / f_i.

and where $M'/f_g M'$ is free of rank $r-g-1$ over $R/f_g R$.

Proposition: Let R be a local noetherian ring, let M be an R module of finite type, and let $R^\ell \xrightarrow{\theta} R^k \longrightarrow M \longrightarrow 0$ be a finite presentation of R. TFAE

(i) $\operatorname{Ann} \Lambda^j M$ is invertible for $j > p$

(ii) $M' \simeq M' \oplus R/\mathfrak{a}_{p+1} \oplus \cdots$ where $\mathfrak{a}_j$ is invertible $\mathfrak{a}_j \subset \mathfrak{a}_{j+1} \subset \cdots$ and $M'/\mathfrak{a}_{p+1} M'$ is free of rank p over $R/\mathfrak{a}_{p+1}$.

(iii) The ideals I_j generated by the $j \times j$ minors of θ are invertible for ~~$j > p$~~ $k-j+1 > p$

Proof: Descending induction on p.

Initial step: Let $r = \operatorname{rank} M = \dim M \otimes_R k$. Then by autos. on R^ℓ and R^k can assume that θ is of the form

$$\begin{array}{c} r \\ k-r \end{array} \begin{bmatrix} \boxed{\theta'_1} & \\ & \boxed{\mathrm{id}} \end{bmatrix}$$

where $\theta'_{\cancel{j}} \equiv 0$ (mt).

$$I'_{\nu-k+r} = I_\nu .$$

Also $(\theta'_{ij}) = \operatorname{Ann} \Lambda^r M$. If $(\theta'_{ij}) = fR$ where f is a non-zero divisor, then the matrix $f^{-1}\theta'$ is no longer minimal so after an auto we get that θ is of the form

$$\begin{bmatrix} \boxed{f\theta''} & & \\ & \boxed{f} & \\ & & \boxed{\mathrm{id}} \end{bmatrix}$$

i.e. $M \simeq M' \oplus R/fR$ where M'/fM' is free of rank $r-1$ over R/fR.

Rest of prop. clear since once one knows (ii) one knows that θ is (similar to a matrix) of the form

$$\theta = \begin{bmatrix} \boxed{f_{p+1}\theta'} & & \\ & f_{p+1} & \\ & & f_k \end{bmatrix}$$

~~plane curve~~

$\cancel{f(x,y)=0}$ $\cancel{f \in k[[x,y]]}$

$\cancel{\dfrac{\partial f}{\partial x} \quad \dfrac{\partial f}{\partial y}}$

$$(t_1,\dots,t_{i-1})M \subset \underbrace{(t_1,\dots,t_i)}M$$

thus

<u>linear motives.</u> I want axioms to describe the linear category.

<u>Question</u>: To any polynomial P we consider ~~Fd~~ Quot^P.

Is there a dense Zariski open set of simple char equations?

<u>simple characteristics</u>:

A local noeth ring ~~$0 \to M' \to M \to M'' \to 0$~~

properties of simple characteristics.

a) M', M'' simple $\Longleftrightarrow$ $M' \oplus M''$ simple

b) $0 \to M' \to M \to M'' \to 0$

M, M'' simple $\Longrightarrow$ M' simple

c) A/I simple $\Longleftrightarrow$ I generated by part of a system of parameters.

non-linear homological algebra functor?

Thus simple should generalize { free modules, Semi-simple modules

In fact there should exist a homological functor L_n such that

$$M \text{ simple} \Longleftrightarrow L_+(M) = 0$$

NO

Then L_0 being right exact is of the form

$$L_0(M) = M \otimes K$$

$K = L_0(A)$

In general suppos

Let $\mathcal{A}$ be an abelian category and let $\mathcal{S}$ be a full additive subcategory with the following properties:

~~a) $M', M'' \in \mathcal{S} \Leftrightarrow M' \oplus M'' \in \mathcal{S}$~~

~~b) $0 \to M' \to M \to M'' \to 0$ exact, $M, M'' \in \mathcal{S} \Rightarrow$~~

a) $0 \to M' \to M \to M'' \to 0$ exact and $M'' \in \mathcal{S}$

then $M \in \mathcal{S} \Leftrightarrow M' \in \mathcal{S}$.

b) $\forall X \; \exists \; M \twoheadrightarrow X$ with $M \in \mathcal{S}$.

c) if X is a direct summand of $M \in \mathcal{S}$ then $X \in \mathcal{S}$

Then $\exists$ an abelian category $\mathcal{B}$ an additive right exact functor $F : \mathcal{A} \to \mathcal{B} \ni \mathcal{S} = F$ acyclic objects of $\mathcal{A}$.

Making relative homological algebra absolute!!

Want $\mathcal{S}$ to be the full subcategory of projective objects in $\mathcal{B}$.

Form $K(\mathcal{S})$ and apply Freyd and then cut down to complexes (concentrated in) of degree 0.

$$\mathcal{A} \hookrightarrow$$

$B \in \mathcal{B}$ $\qquad X \longmapsto \mathrm{Hom}(F(X), B)$ left ex.

Can expect to find $\mathcal{B} \subset \mathrm{Lex}(\mathcal{A}^{\circ}, \mathcal{Ab})$

and in fact expect ~~$B \in \mathcal{B}$ to be reasonable~~ for $X \in \mathcal{S}$ to be exact in $\mathcal{B}$.

Try $\quad \mathcal{B} = \left\{ F : \mathcal{A}^{\circ} \to \mathcal{Ab} \;\middle|\; \begin{array}{l} F \text{ left exact} \\ R^{g}F(\mathcal{S}) = 0 \quad g > 0 \end{array} \right.$ $\qquad$ No

The theory of Freyd:

$\mathcal{A}$ additive.

(i) $\mathrm{Ob}\ \mathcal{D}(\mathcal{A}) = \mathrm{Fl}\ \mathcal{A}$

$$\mathrm{Hom}\left(\begin{smallmatrix}X\\ \downarrow f\\ Y\end{smallmatrix}, \begin{smallmatrix}X'\\ \downarrow f'\\ Y'\end{smallmatrix}\right) = \mathrm{Im}\left\{\mathrm{Hom}(f, f') \longrightarrow \mathrm{Hom}(X, Y')\right\}.$$

induced composition

$$\mathrm{can}: \mathcal{A} \longrightarrow \mathcal{D}(\mathcal{A})$$
$$X \longmapsto \mathrm{id}_X.$$

(ii) $\mathrm{Ob}\ \mathcal{C}(\mathcal{A}) = \mathrm{Fl}\ \mathcal{A}$

$$\mathrm{Hom}\left(\begin{smallmatrix}X\\ \downarrow f\\ Y\end{smallmatrix}, \begin{smallmatrix}X'\\ \downarrow f'\\ Y'\end{smallmatrix}\right) = \mathrm{Hom}(f, f') \Big/ \mathrm{Hom}(\overset{Y}{\cancel{X}}, X) +$$

induced composition.

$$\mathrm{can}: \mathcal{A} \longrightarrow \mathcal{C}(\mathcal{A})$$
$$X \longmapsto (0 \longrightarrow X)$$

(iii) $\mathrm{Ob}\ \mathcal{K}(\mathcal{A}) = \mathrm{Fl}(\mathcal{A})$

$$\mathrm{Hom}\left(\begin{smallmatrix}X\\ \downarrow f\\ Y\end{smallmatrix}, \begin{smallmatrix}X'\\ \downarrow f'\\ Y'\end{smallmatrix}\right) = \mathrm{Hom}(f, f') \Big/ \mathrm{Hom}(Y, X')$$ (same as $\mathcal{C}\mathcal{A}$

$$\mathrm{can}: \mathcal{A} \longrightarrow \mathcal{K}(\mathcal{A})$$
$$X \longmapsto (X \longrightarrow 0)$$

Note that $\mathcal{D}(\mathcal{A}), \mathcal{C}(\mathcal{A}), \mathcal{K}(\mathcal{A})$ are additive and that can is additive and fully faithful

~~[illegible]~~ Nov 29, 1968

Lemma: (Weyl). D elliptic diff op.
u distribution on Ω $\quad Du \in C^\infty(\Omega)$

$$\Longrightarrow \quad u \in C^\infty(\Omega).$$

Proof: Will prove u smooth near a fixed point of Ω which may assume is ~~[illegible]~~ the origin. Choose $\rho, \varphi, \psi \in C_0^\infty(\Omega)$ such that $\psi \equiv 1$ near 0, $\varphi \equiv 1$ near $\operatorname{Supp}\psi$, $\rho \equiv 1$ near $\operatorname{Supp}\varphi$. Claim can ~~solve a~~ find a pseudo-differential operator P of order $= -$ order D $\ni$

$$P\varphi D = \psi + K$$

where K is a smooth operator. In effect it suffices to solve

$$\hat{P}\hat{\varphi}\hat{D} = \hat{\psi}$$

in the algebra of symbols. But $\hat{D}^{-1}$ exists and $\hat{\psi}\hat{D}^{-1}$ has support $=$ that of ψ, hence if $\hat{P} = \hat{\psi}\hat{D}^{-1}$ we have $\hat{P}\hat{\varphi} = \hat{\psi}\hat{D}^{-1}\hat{\varphi} = \hat{\psi}\hat{D}^{-1}$. Now

$$\begin{array}{c} P\varphi D\rho u = \psi\rho u + \overset{\in C^\infty}{K\rho u} \\ \| \\ P\varphi D u \\ \cap \\ C^\infty \end{array}$$

$\therefore \ \psi\rho u \in C^\infty$ so $u \in C^\infty$ near 0. Q.E.D.

~~Remark: Observe that D could have been a pseudo-differential operator because $\varphi D\rho - \varphi D = \varphi D(\rho - 1)$ is smooth since $\operatorname{Supp}\varphi \cap \operatorname{Supp}(\rho - 1) = \phi$. Thus $P\varphi D\rho u = P\varphi D u + P\varphi D(\rho-1)$ [illegible] u~~

~~$\in C^\infty$~~

~~Choose $\rho' \in C_0^\infty(\Omega) \ni \rho' \equiv 1$ near $\operatorname{Supp}\rho$. Then $(\rho - 1) = (\rho - 1)\rho'$~~

~~where [illegible]~~

$$B = \{x \in \mathbb{R}^n \mid \|x\| \le 1\}$$

$$\boxed{\sum_{|\alpha|\le m} a_\alpha D^\alpha u = f}$$

$a_\alpha(x)$ function of $x \in B$

$u(x)$ ———— $x \in B$

$$(D^i u)(x) = \lim_{t\to 0} \frac{u(x+te_i) - u(x)}{t}$$

~~1. Set $b_\alpha(x) = a_\alpha(tx)$ a new function of $x \in B$ if $0 < t \le$~~

2. To calculate Restriction from B to tB and then homothecy.

$$B \hookleftarrow tB \xleftarrow{\;t\;} B$$

$$(x \mapsto f(x)) \longmapsto (x \mapsto f(x)) \longmapsto (x \mapsto f(tx))$$

$$\boxed{f \longmapsto \theta f \quad \text{transformation of functions on } B}$$

$$\theta \sum a_\alpha D^\alpha u = \theta f$$

$$\sum \theta(a_\alpha)\, \theta(D^\alpha u) = \theta f$$

what is $\theta(D^\alpha u)$ in terms of θu ?

ind.

$$[\theta(D^i v)](x) = \quad (D^i v)(tx) = \lim_{\varepsilon\to 0} \frac{v(tx+\varepsilon e_i) - v(tx)}{\varepsilon}$$

$$= \frac{1}{t} \lim_{\varepsilon \to 0} \frac{(\theta v)(x + \frac{\varepsilon}{t} e_i) - (\theta v)(x)}{\frac{\varepsilon}{t}} = \frac{1}{t}(D^i \theta v)(x)$$

$\therefore$

$$\boxed{\theta D^i v = \frac{1}{t} D^i \theta v} \qquad \boxed{\theta D^\alpha v = \frac{1}{t^{|\alpha|}} D^\alpha \theta v}$$

∴ new equation becomes

$$\boxed{\sum_{|\alpha|\leq m} (\theta a_\alpha) \frac{1}{t^{|\alpha|}} D^\alpha(\theta u) = \theta f}$$

would like to show soluble for t small. Interesting case is where ~~...~~ $m=1$.

$$\frac{1}{t}\sum_{|\alpha|=1} a_\alpha(0)\, D^\alpha(\theta u) + \sum_{|\alpha|=1} \frac{\theta a_\alpha - a_\alpha(0)}{t} D^\alpha(\theta u) + (\theta a_0)(\theta u) = \theta f$$

$$\sum_{|\alpha|=1} (\theta a_\alpha)\, D^\alpha(v) + t(\theta a_0) v = t(\theta f).$$

given f want to produce a t such that I can find a solution. Power series expansion at $t=0$ corresponds to formal power series expansion of solution.

e.g. $$f(tx) \doteq \sum_{i=0}^{\infty} t^i f_i(x) \qquad \text{Taylor series}$$

Then $$f(\varepsilon t x) \doteq \sum_{i=0}^{\infty} t^i f_i(\varepsilon x) \doteq \sum_{i=0}^{\infty} t^i \varepsilon^i f_i(x)$$

$$\therefore \boxed{f_i(\varepsilon x) = \varepsilon^i f_i(x)}$$

I know this (problem) is correctly posed and flat.

<u>Summary of theory</u>

1) Definition of Cartan scheme

2) Proof that Hilbert + Quot schemes are Cartan schemes
Cartan's criterion

~~3)~~

3) Fibration of open sets associated ~~with respect~~ to a ~~generic~~ flag (connectedness thm. of Hartshorne).

Possible improvements ~~(absurd~~

1.) absurd hope that under ~~for~~ the Plücker embedding of the <u>Grassmannian</u> things might simplify immensely.

suppose given

$$\begin{array}{ccc} R_{n-1} & \subset & R_n \\ \cap & & \cap \\ T_{n-1}\otimes E & \subset & E \end{array}$$

$$t_n \otimes E_n \subset R + T_{n-1}\otimes E$$

assume $E_n = E$ ie $s_n = 0$.

transversal

$$R/R_{n-1} \simeq (T/T_{n-1}) \otimes E_n$$

$\therefore$ ~~$T_{n-1}\otimes R_n$~~ ~~$R+T_{n-1}\otimes$~~

$$R + T_{n-1}\otimes E = T\otimes E_n + T_{n-1}\otimes E$$

$\boxed{T_{n-1}\otimes E \text{ and } R_n \text{ meet transversally.}}$

Again assume $s_n = 0$. Then T_{n-1} is ~~none~~ char iff $T_{n-1}\otimes E + R = T\otimes E$

Proof: $\lambda \in T^*$ char. $\iff \sigma_\lambda : E^* \longrightarrow R^*$ not injective

$\iff \sigma_\lambda^t : R \longrightarrow E$ not surj

$R \hookrightarrow T\otimes E \longrightarrow (T/T_\lambda)\otimes E$ not surj

$\iff R + T_\lambda \otimes E < T\otimes E$

$\mathrm{Cart}_{\underline{S}}$ Given $^{\mathcal{F}c}\{T_i\}$

get $(\mathrm{Cart}_{\underline{S}})^{\mathcal{F}}$ <u>open</u> in $\mathrm{Cart}_{\underline{S}}$

~~$\mathrm{Cart}_{\underline{S}}$~~ "

$\boxed{\text{Grassmanian}}$

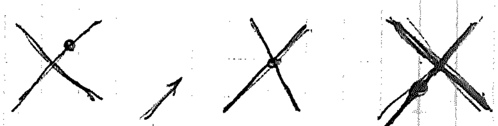
<u>induction</u>: is only possibility.

R involutive $\subset T \otimes E \Longrightarrow R \cap T_{n-1} \otimes E$ also involutive

<u>Program 1</u>:

<u>Problem</u> Consider case where $s_n = 0$. Then the sheaf $\mathcal{F}$ has nonchar points of $\mathbb{P}^{n-1}$ and such $T_{n-1} \subset T$ are part of generic flags (?). <u>Problem</u>: Describe $\{\mathcal{F} \mid \mathcal{F}_\lambda \neq 0\}$. Closed subset of Cart. <u>Any ideas?</u>

Intuitively means can solve Goursat problem there.

One of the first invariants is the associated graded ~~ring~~ module at λ. Seems unlikely that this should ~~be~~ ~~reasonable~~ vary flatly. e.g. 

However it should be a limited family. Thus one should ~~take~~ be able to take ~~the desired~~ parameter family blow up to point λ and then stratify until flat, finally

~~Problem~~:

Open set: $T_\lambda \subset T$ codim 1 $\qquad \mathbb{C}t_n \oplus T_\lambda = T$

$\{R \mid R \cap T_\lambda \otimes E$ has minimal dimension$\}$

$\lambda \in \mathbb{P}(T^*)$ want to blow up the point λ

$$\begin{tikzcd} & \tilde{X} = \{(\ell, x) \mid \ell \text{ line thru } v,\ x \in \ell\}. \arrow[dl] \\ \mathbb{P}(V) & \end{tikzcd}$$

---

dim $V = n+1$

$\mathbb{P}(V) =$ 1-diml subspaces of V.

$W \subset V$ subspace dim $r+1$

$$\tilde{X} = \{(E, \ell) \mid \overset{r+1}{W} \subset E^{r+2} \supset \ell\}.$$

~~$\tilde{X} = \{(E, \ell) \mid$~~

$$\tilde{X} \longrightarrow \mathbb{P}_1(V/W) \ni E$$

Therefore ~~what is the projective~~ $\tilde{X} = \mathbb{P}(\mathcal{E})$

where $\mathcal{E}$ is the bundle $\{E\}$ over $\mathbb{P}(V/W)$.

$\underline{r+1 + \mathcal{O}(-1)}$.

Given K_i want all R such that this be true

$$K_i = \{e \mid t_i \otimes e \in T_{i-1} \otimes M_0 + R\}.$$

Suppose true.

$$\mathcal{R}_g = \{R \subset T \otimes M_0 \mid K_i = \{e \mid t_i \otimes e \in T_{i-1} \otimes M_0 + R\}\}_{i=1,\dots,n}.$$

~~If true~~ If $R \in \mathcal{R}_g$ then

$$E/K_i \cong (T_i \otimes M_0) + R / T_{i-1} \otimes M_0 + R$$

so $\dim T \otimes M_0 / R = \sum_{i=1}^{n} \dim E/K_i$

$$= \sum_{i=1}^{n} \sum_{j \geq i} s_j = \sum j s_j \quad \checkmark$$

~~Next stage is to~~

$\mathcal{R} = \{R$

so I try $R \ni$

$$T \otimes K_i \subset T_{i-1} \otimes M_0 + R.$$

$$\{R \subset T \otimes M_0 \mid \operatorname{codim} R = \sum_{i=1}^{n} i s_i \text{ and } T \otimes K_i \subset T_{i-1} \otimes M_0 + R\}$$

Let's use Hilbert's scheme

$(\mathcal{F}, \mathcal{E}, \Phi)$

$\Phi_i \in \mathrm{Hom}(T_i \otimes E_i, T_{i-1} \otimes E)$

rest. to inc. on $T_{i-1} \otimes E_i$.

---

have

$$\begin{array}{ccc} \mathcal{H} & \longrightarrow & \mathrm{Grass}(T \otimes E) \\ \uparrow & & \cup \\ \mathcal{H}^+ & \longrightarrow & \mathrm{Cart} \end{array}$$

let $\mathcal{H}^+$ be inverse image

$$\mathcal{H} \longrightarrow Gr(T \otimes E) \times \mathrm{Flag}(T)$$

To describe $R \ni$ $\quad T_i \otimes E_i \subset T_{i-1} \otimes E + R$.

of the correct dimension.

$$\mathcal{H} = \{(\mathcal{F}, \mathcal{E}, \Phi)\}$$

$$\mathcal{H} \longrightarrow \underset{\displaystyle I}{\overset{}{\mathrm{Grass}}}(T \otimes E) \quad I \subset \mathrm{Grass}(T \otimes E)$$

$$\mathcal{H}^+ =$$

Claim we have that I is non-singular with

$$\mathcal{H} \longrightarrow \mathcal{F}l \times I$$

~~an open imm~~ an affine bundle over the open set of I for which some $\mathcal{F}$ works!

$(\mathcal{B}, \mathcal{E}, \Psi)$

t_i, t_j ~~ψ_{ij}~~ $\underline{\psi_{ij}}$ $\quad i > j.$

$i > j \quad \psi_{ij} \in \mathrm{Hom}(E_i, E).$

$\underline{i}$ $\qquad E_i \longrightarrow T_{i-1} \otimes E$

$$T_i \otimes E_i \subset T_{i-1} \otimes E + R$$

$$\Psi_i : T_i \otimes E_i \longrightarrow T_{i-1} \otimes E$$

~~$\Psi_i(t_j$~~ $\qquad \Psi_i \in \mathrm{Hom}(T_i \otimes E_i, T_{i-1} \otimes E)$

$\ni \Psi_i \big| T_{i-1} \otimes E_i$ is inclusion

$\mathcal{H} = \{ (\mathcal{F}, \mathcal{E}, \Psi) \mid \mathcal{F}$ flag in T
$\mathcal{E}$ s-flag in E
$\Psi = (\Psi_1, \dots, \Psi_n) \qquad \Psi_i \in \mathrm{Hom}(T_i \otimes E_i, T_{i-1} \otimes E),$
$\Psi_i \big| T_{i-1} \otimes E_i$ is inc. $\}$

---

this is an affine space for the group

$$\bigoplus_i \mathrm{Hom}(T_i / T_{i-1} \otimes E_i, T_{i-1} \otimes E).$$

$$\therefore \quad t_3 t_1 \otimes v_1 \equiv t_1 \big(t_2 \otimes \overbrace{\psi_{32}(v_1)}^{\in E_2} + t_1 \otimes \overbrace{\psi_{31}(v_1)}^{\in E_1} \big)$$

$$\cancel{t_3 t_1 \otimes v_1 \equiv}$$

$$0 \boxed{\equiv t_3 t_1 \otimes v_1 \equiv t_1^2 \psi_{21} \psi_{32}(v_1) + t_1^2 \psi_{31}(v_1)}$$

$$0 \equiv t_3 t_2 \otimes v_2 - t_3 t_1 \otimes \psi_{21}(v_2) \equiv t_2 \big(t_2 \otimes \psi_{32}(v_2) - t_1 \otimes \psi_{31}(v_2) \big)$$
$$- t_1 \big(t_2 \otimes \psi_{32} \psi_{21}(v_2) - t_1 \otimes \psi_{31} \psi_{21}(v_2) \big)$$

$$\begin{aligned} t_3(t_1 \otimes e_1) = {} & t_1 \big(t_2 \otimes e_2' - t_1 \otimes \psi_{21}(e_2') \big) + t_1 (t_1 \otimes e_1') \\ & + t_2 \big(t_2 \otimes e_2'' - t_1 \otimes \psi_{21}(e_2'') \big) \\ & + t_1 \big(t_3 \otimes e_3' - t_2 \otimes \psi_{32}(e_3') - t_1 \otimes \psi_{31}(e_3') \big) \\ & + t_2 \big(t_3 \otimes e_3'' - t_2 \otimes \psi_{32}(e_3'') - t_1 \otimes \psi_{31}(e_3'') \big) \\ & + t_3 \big(t_3 \otimes e_3''' - t_2 \otimes \psi_{32}(e_3''') - t_1 \otimes \psi_{31}(e_3''') \big) \end{aligned}$$

$$\begin{aligned} t_3(t_1 \otimes e_1) &= \cancel{t_1(t_2 \otimes} \; t_1 (t_3 \otimes e_1) \\ &= t_1 \big(t_3 \otimes e_1 - t_2 \otimes \psi_{23}(e_1) - t_1 \otimes \psi_{31}(e_1) \big) \\ &\quad + t_1 t_2 \otimes \psi_{23}(e_1) + t_1^2 \otimes \psi_{31}(e_1) \end{aligned}$$

$\boxed{e_3' = e_3}$

$$\psi_{23}(e_1) =$$

Thus for each i with $i > j$ we have

$$t_i \otimes v_j - \sum_{j \geq k \leq l} t_k \otimes a^{ij}_{kl}(v_j) \quad \in R$$

where $a^{ij}_{kl} \in \mathrm{Hom}(P_j, P_l)$ defined for $i > j$, $j \geq k \leq l$

~~These equations give us elements~~

By above dimension argument the elements

$$t_i \otimes v_{j\alpha} \qquad i \leq j \quad 1 \leq \alpha \leq s_j$$

form a basis of $T \otimes M_0$ modulo R.

hence the elements

$$t_i \otimes v_{j\alpha} - \sum_{\substack{k \leq j \\ l \geq k}} t_k \otimes a^{ij}_{kl}(v_{j\alpha}) \qquad \begin{array}{c} i > j \\ 1 \leq \alpha \leq s_j \end{array}$$

form a basis for R.

We saw that $\sum_{i \leq j} t_i \otimes P_j$ is a complement to R

Involutivity means that $\sum_{i \leq i' \leq j} t^i t^{i'} \otimes P_j$

is a complement to TR in $S^2T \otimes E$.

$R \subset T \otimes E$ involutive (type $s_0, \dots, s_n$) $\iff$

$\exists$ flag $\{T_i\}$ such that

if $E_i = \{ e \mid T_i \otimes e \subset T_{i-1} \otimes E + R \}$.

then $\dim E_i = s_0 + \cdots + s_{i-1}$

and $\dim M_2 = \sum \frac{i(i+1)}{2} s_i$

$$(\text{Flag } T) \times (\text{Flag } E) \longrightarrow \text{grass}(T \otimes E)$$

given T_i E_i can consider all R such that

$$T_i \otimes E_i \subset T_{i-1} \otimes E + R$$

~~Flag T × Flag E~~

Suppose that $\underline{T_i \otimes E_i \subset T_{i-1} \otimes E + R}$

and that $\dim \{T \otimes E/R\} = \sum i s_i$

Then consider R in $\dfrac{T_i \otimes E}{T_{i-1} \otimes E} \longleftarrow \dfrac{R \cap (T_i \otimes E)}{R \cap (T_{i-1} \otimes E)}$

$$0 \longleftarrow \frac{T_i M_0}{T_{i-1} M_0} \longleftarrow$$

$$\mathcal{H} = \left\{ (\mathcal{F}, \mathcal{E}, R) \;\middle|\; \begin{array}{l} \mathcal{F} \text{ Flag in } T \\ \mathcal{E} \text{ flag in } E \\ R \subset T \otimes E \end{array} \ni \; T_i \otimes E_i \subset T_{i-1} \otimes E + R \right\}$$

~~The~~ $\mathcal{H}$ clearly ~~projective variety~~ quasi - projective ~~computational~~

Hubert chooses t_i, then $\psi_{ij} : E_i \longrightarrow E$

$$\ni \quad t_i \otimes e_i - \sum_{j<i} t_j \otimes \psi_{ij}(e_i) \in R.$$

he gets a basis for R in this way.

altering t_i by $t_i + \sum_{j<i} \lambda_{ij} t_j = t_i'$

$$t_i^* \otimes e_i - \sum_{j<i} t_j \otimes \psi_{ij}(e_i) = t_i' \otimes e_i - \sum_{j<i} t_j \otimes (\psi_{ij}(e_i) + \lambda_{ij} e_i$$

Thus modifying t_i $\quad \psi_{ij}'(e_i) = \phi_{ij}(e_i) + \underbrace{\lambda_{ij}}\, e_i$

constant times inclusion of $E_i \longrightarrow E$

$$t_1 t_2 \otimes \left[e_2' - \psi_{21}(e_2'') - \psi_{32}(e_3') - \psi_{31}(e_3'') \right] = 0$$

$$t_1^2 \otimes \left[-\psi_{21}(e_2') - \psi_{31}(e_3') \right] = 0$$

$$t_1 t_3 \otimes \left[e_3' - \psi_{31}(e_3''') \right] = t_1 t_3 \otimes [e_1]$$

$$t_2^2 \otimes \left[e_2'' - \psi_{32}(e_3'') \right] = 0$$

$$t_2 t_3 \otimes \left[e_3'' - \psi_{32}(e_3''') \right] = 0$$

$$t_3^2 \otimes e_3''' = 0.$$

$$\boxed{e_3'' = 0 \quad e_3'' = 0 \quad e_2'' = 0}$$

$$\boxed{e_3' = e_1} \qquad \boxed{e_2' = \psi_{32}(e_1)}$$

no relations

$$t_2(t_1 \otimes e_1) = t_1(t_1 \otimes e_1') + t_1(t_2 \otimes e_2' - t_1 \otimes \psi_{21}(e_2'))$$
$$+ t_2(t_2 \otimes e_2'' - t_1 \otimes \psi_{21}(e_2''))$$

$$\begin{array}{ll} t_1^2 & \left\{ \begin{array}{l} e_1' = \psi_{21}(e_2') \\ e_1 = e_2' - \psi_{21}(e_2'') \\ 0 = e_2'' \end{array} \right. \\ t_1 t_2 & \\ t_2^2 & \end{array}$$

$$\boxed{\psi_{21}(e_1) = e_1'}$$

$e_2' = e_1$

$$\boxed{\psi_{21}(E_2) \subset E_1}$$

b

What is the standard desingularization of

$$Z = \{ R \mid R + T_\lambda \otimes E < T \otimes E \} \quad ?$$

Ans. $\tilde{Z} = \{ (R, H) \mid H$ hyperplane in $T \otimes E$ containing $T_\lambda \otimes E$ and $R \}$

Can this desing. be used to analyze the bad R?

Any chance that $\boxed{R \cap (T_{n-1} \otimes E) \text{ is involutive}}$.

meaning?

$$R / R \cap (T_{n-1} \otimes E) \hookrightarrow \frac{T \otimes E}{T_{n-1} \otimes E} = M$$

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & R_{n-1} \arrow[r] \arrow[d] & T_{n-1} \otimes E \arrow[r] \arrow[d] & T_{n-1} M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & R \arrow[r] \arrow[d] & T \otimes E \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & R/R_{n-1} \arrow[r] \arrow[d] & T/T_{n-1} \otimes E \arrow[r] \arrow[d] & M/T_{n-1} M \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

$R/R_{n-1} = t_n \otimes E_n$

Thus T_{n-1} char $\Longleftrightarrow$ $T_{n-1} M < M$.

however might be true that we can ~~wrote~~ split up

$T_{n-1} M =$

he has that

$$t_k\Big(t_i \otimes e_i - \sum_{j<i} t_j \otimes \psi_{ij}(e_i)\Big) = \sum_{p\le q} t_p\Big(t_q \otimes e_{pq} - \sum_{l\le q} t_l \otimes \psi_{ql}(e$$

recursively get the e_{pq} as functions of e_i

$$e_{pq} = P_{pq}(\psi)\, e_i$$

So only problem is to know that $e_{pq} \in E_q$.
Thus get involutivity conditions

$$P(\psi)\, E_i \subset E_q.$$

~~variety~~ Can you express all of this in terms of $\overset{\text{Grassmann}}{\text{of}}$ invariants? ~~of the~~

~~Yes~~

$$\boxed{\mathrm{Flag}(T) \times \mathrm{Flag}\,E \times \underline{\{\psi\}} \longrightarrow \mathrm{Grass}}$$

Point is we have an ~~affine~~ bundle over

$\mathrm{Flag}(T) \times \mathrm{Flag}'\,E$

consisting of

change

$$R \subset T \otimes E$$

$$T_i \otimes E_i \subset T_{i-1} \otimes E + R.$$

Problem: Analyze what happens when generically $T_\lambda M_0 = M_1$ but not for ~~...~~ T_{n-1}

The problem: ~~is~~ ~~...~~. Consider

$$\cdots \longrightarrow \Lambda^2 T_{n-1} \otimes \mathcal{O}(-2) \longrightarrow T_{n-1} \otimes \mathcal{O}(-1) \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_\lambda \longrightarrow 0$$

which is exact. Then we tensor with $\mathcal{F}$ and get a sequence with homology

$$\mathrm{Tor}_g^{\mathcal{O}}(\mathcal{O}_\lambda, \mathcal{F}).$$

Natural to want $\mathcal{F}(\lambda)$ to be killed by

$\forall \lambda \ \{\mathcal{F} \mid \mathcal{F}(\lambda) = \emptyset\} = U_\lambda$ open in Cart

$\forall \mathcal{F} \ \exists \lambda \ni \mathcal{F}(\lambda) \neq \emptyset$

$$\Longrightarrow \bigcup_\lambda U_\lambda = \text{Cart}$$

hence finite union works already.

$$\tilde{Z} = \{H \supset T_\lambda \otimes E + R$$

i.e. H/R hyp. in M_1/T_λ ?

$$M \otimes_{S(T)} S(T/T_\lambda) = M/T_\lambda \cdot M$$

graded module over $S(T/T_\lambda) = C$

Question: Is $M/T_\lambda M$ by any chance involutive, e.g. is its rank constant in degrees ≥ 1.

~~Hilbert~~

Given T, $P(T^*)$. keep the following.

F flag in T $\quad F = 0 \subset T_1 \subset T_2 \subset \cdots \subset T_n = T$.

Given Cartan integers $s_0, \ldots, s_n \geq 0$.

$$0 \longrightarrow R \hookrightarrow T \otimes E \longrightarrow M_1 \longrightarrow 0.$$

$\mathrm{Im}\,\{T \otimes R \longrightarrow S^2 T \otimes E\}$ must have low dimension

$\{R \hookrightarrow T \otimes E \mid$ ⓐ R

filtration

$$T_i M_0 = T_i \otimes M_0 + R/R.$$

$$K_i = \mathrm{Ker}\left\{t_i : M_0 \longrightarrow M_1/T_{i-1} M_0\right\}.$$

$$M_1/T_{i-1}M_0 \cong T \otimes E/T_{i-1} \otimes M_0 + R$$

$$\therefore \quad \boxed{E/K_i \xrightarrow[\cong]{t_i} T_i \otimes M_0 + R/T_{i-1} \otimes M_0 + R}$$

$$\tilde{X} = \{ (E, \ell) \mid \ell_0 \subset E^2 \supset \ell \}$$

$$\mathbb{P}(V) \leftarrow \tilde{X} \xrightarrow{pr_1} \mathbb{P}(V/\ell_0)$$

$$\mathbb{P}(1 \oplus W) \leftarrow \mathbb{P}_1\{1 \oplus \mathcal{O}(-1)\} \rightarrow \mathbb{P}(W)$$

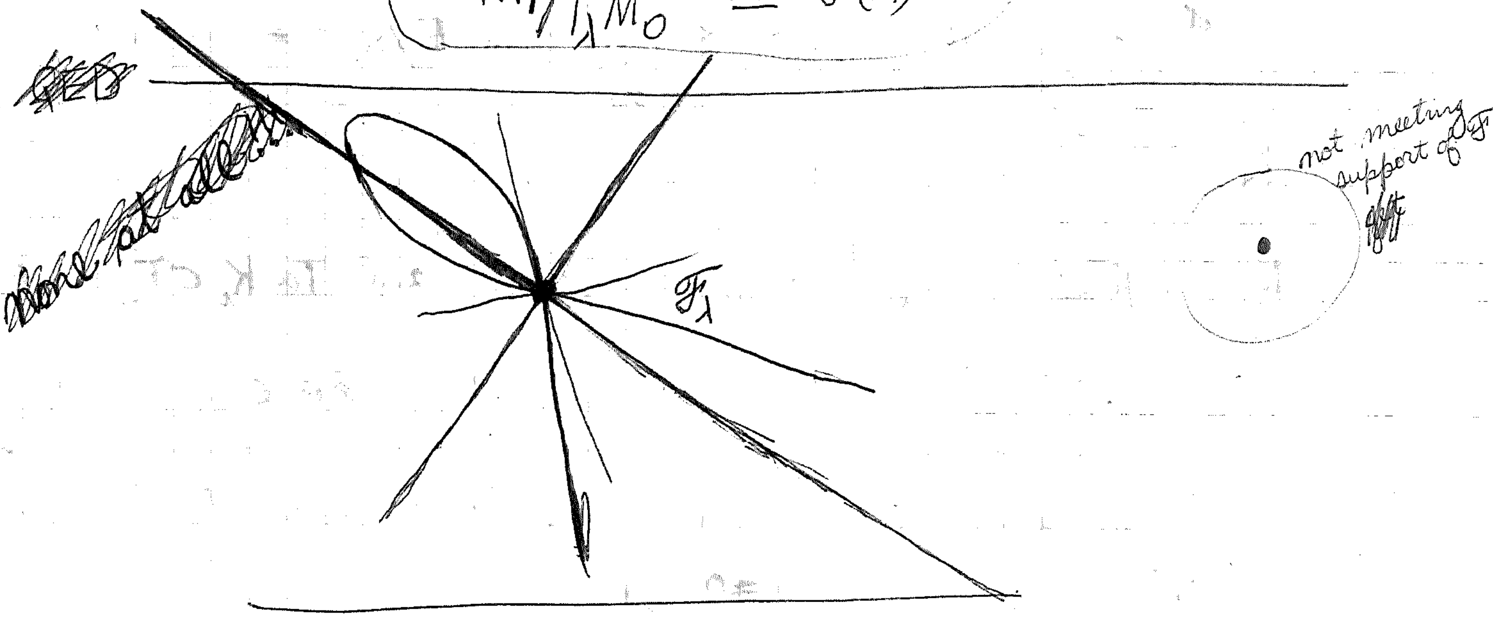
<u>problem</u>: Given $\lambda \in T^*$ describe $R \ni$

R_λ doesn't have minimal dimension.

eg if $R + T_\lambda \otimes E \overset{s_n = 0}{<} T \otimes E$

then consider hyperplanes cont. first e.g hyperplanes

in $M_1 / T_\lambda M_0 \cong \mathcal{F}(\lambda)$



Problem to describe $\{ \mathcal{F}$ such that $\mathcal{F}_\lambda \neq 0 \}$.

Classical method: take a generic direction not in the tangent cone to support $\mathcal{F}$. and blow up

~~$\{R \subset T \otimes M_0$~~

Question is $\quad K_i = \{e \mid t_i \otimes e \in T_{i-1} \otimes M_0 + R\}$

~~first~~ set $\quad L_i = \{e \mid t_i \otimes e \in T_{i-1} \otimes M_0 + R\}$

Then as $\quad t_i \otimes K_i \subset T_{i-1} \otimes M_0 + R$

we have that

$$K_i \subset L_i \qquad T_i \otimes M_0 / T_{i-1} \otimes M_0 + R_i$$

but

$$E/L_i \simeq (T_i \otimes M_0 + R)/(T_{i-1} \otimes M_0 + R).$$

$$\dim M_1 = \sum_{i=1}^{n} \dim E/L_i \leq \sum_{i=1}^{n} \dim E/K_i = \dim M_1$$

$\boxed{\text{OKAY}}$. <u>Conclusion:</u> If

$$\mathcal{R} = \{R \subset T \otimes M_0 \mid \operatorname{codim} R = \sum i s_i \text{ and } T \otimes K_i \subset T_{i-1} \otimes M_0 + R\}$$

then $\quad R \in \mathcal{R} \Longrightarrow \quad K_i = \{e \mid t_i \otimes e \in T_{i-1} M_0 + R\}$

Claim $\mathcal{R}$ is an affine space. In effect choose $P_i \subset K_{i+1}$

$\ni \quad K_i \oplus P_i = K_{i+1} \qquad i = 0, \dots, n.$

Then we find

$$t_i \otimes P_j \subset T_j \otimes M_0 + R$$

hence we can write

$$t_i \otimes v_j = \sum_{\substack{k \leq j \\ k \leq l}} t_k \otimes a^{ij}_{kl}(v_j) + h$$

$\cap P_l$

$$\begin{tikzcd} \mathcal{F} & & \mathbb{P}^{n-1} \arrow[ll, bend right] \arrow[dll] & \widetilde{\mathbb{P}}^{n-1} \arrow[l] \\ B & & & \end{tikzcd}$$

Take universal family over B

blow up. λ

$$\begin{tikzcd} \mathcal{F} \arrow[d, bend right, "flat"'] & \mathbb{P}^{n-1} & \widetilde{\mathbb{P}}^{n-1} \arrow[l, "g"'] & g^*\mathcal{F} \arrow[dll] \\ B & & & \end{tikzcd}$$

---

new family is no longer flat over B so that one obtains a stratification of B.

$$\begin{tikzcd} & \overset{g^*\mathcal{F}}{\widetilde{\mathbb{P}}^{n-1}} \arrow[dl, "g"'] \arrow[dr, "\pi"] & \\ \mathcal{F}\ \mathbb{P}^{n-1} \arrow[dr] & & \mathbb{P}^{n-2} \arrow[dl] \\ & C & \end{tikzcd}$$

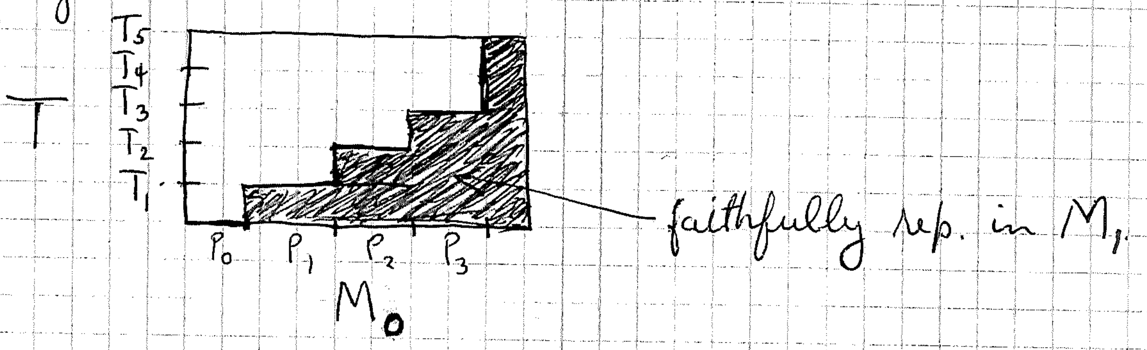
<u>basic questions:</u> Our idea is to take

$$\mathcal{F} \longmapsto \pi_* g^* \mathcal{F}$$

Thereby getting a map from sheaves on $\mathbb{P}^{n-1}$ to sheaves on $\mathbb{P}^{n-2}$.

$T \otimes K_i \subset T_{i-1} \otimes M_0 + R.$

Diagram



Involutivity means simply that ~~$t\otimes$~~ $\mathrm{Im}\{T \otimes R \to S^2T \otimes M_0\}$ is complementary to $\bigoplus_{k \le l \le j} t_k t_l \otimes P_j$.

Now R is filtered

$$R_\alpha : \quad t_i \otimes v_j - \sum_{\substack{k \le j \\ k \le l}} t_k \otimes \underset{\substack{\cap \\ P_l}}{a^{ij}_{kl}(v_j)} \qquad \begin{array}{l} i > j \\ i \le \alpha. \end{array}$$

Thus $R_\alpha \subset T_\alpha \otimes M_0 \cap R.$

Complementary to $\bigoplus_{i \le j} t_i \otimes P_j \quad i \le \alpha.$

Claim equality since if $\sum a_{ij} t_i \otimes v_j \in R \cap T_\alpha \otimes M_0$

then $\sum_{i \le \alpha} a_{ij} t_i \otimes v_j$

whence modulo R_α can remove all terms with $i > j$

and remaining are then indep. mod R

so get 0.

$(T_\alpha \otimes M_0) \cap R$ generated by

$t_i \otimes v_j - \cdots \quad \alpha \ge i > j$ and i

Therefore if we take the elements

$$\cancel{t_g t^i \otimes v_j}$$

$$t_g\left(t_i \otimes v_j - \sum t_k \otimes a^{ij}_{kl}(v_j)\right)$$

which span TR ~~and if I put in the requirement~~ then ~~clearly~~ I can ~~[illegible]~~ rewrite ~~these~~ expressions so that only terms of type $t_{pg} \otimes P_j$ $\quad p \le g \le j$ occur. ~~The method is~~ The method is to filter $S_2T \otimes M_0$ by

---

Lemme: If $R \subset T \otimes M_0$ is involutive, then
$R \cap T_{n-1} \otimes M_0$ is involutive ~~of [illegible]~~ in $T_{n-1} \otimes M_0$.

~~Proof.~~

$$t_n\left(t_i \otimes v_j - \sum t_k \otimes \underline{a^{ij}_{kl}(v_j)}\right) \qquad i<n$$

$$= t_i\left(t_n \otimes v_j - \sum t_k \otimes a^{nj}_{kl}(v_j)\right) + \left(\sum t_i\left(t_k \otimes a^{nj}_{kl}(v_j)\right)\right.$$

$$-t_k\left(t_n \otimes a^{ij}_{kl}(v_j) - t_g \otimes a^{nl}_{kl}\, a^{ij}_{kl}\, v_j\right) + t_k t_g \otimes a^{nl}_{kl}\, a^{ij}_{kl}\, v_j$$

$$\boxed{t_n \cdot R_{n-1} \subset T \cdot Q_n + T_{n-1} \cdot R_{n-1}}$$

can be reduced by induction to someth[ing]

$$t_k t_l \otimes a \cdots a^{nl}_{kl}\, a^{ij}_{kl}\, v_j$$

modulo $T_{n-1} \cdot R_{n-1}$

$$T_i \otimes E_i \subset T_{i-1} \otimes E + R$$

Then $\dim \dfrac{R \cap (T_i \otimes E)}{R \cap (T_{i-1} \otimes E)}$ $\quad \dim E_i$

Prop: If $T_i \otimes E_i \subset (T_{i-1} \otimes E) + R$, ~~thus~~ and is

$$\dim (T \otimes E / R) = \sum i s_i$$

then $E_i = \{ e \mid F_i \otimes e \subset T_{i-1} \otimes E + R \}$.

$$t_i \otimes e_i - \sum_{j<i} t_j \otimes \psi_{ij}(e_j) \in R \qquad e_j' \text{ basis of } E_i'$$

are lin ind.

Consider

$$\mathcal{H} = \left\{ (\mathcal{F}, \mathcal{E}, R) \;\middle|\; \begin{array}{l} \mathcal{F} \text{ flag in } T \\ \mathcal{E} \text{ flag}' \text{ in } E \\ R \subset T \otimes E \end{array} \;\middle|\; T_i \otimes E_i \subset T_{i-1} \otimes E + R \right\}$$

Then $\mathcal{H} \xrightarrow{pr_3} \mathcal{G}(T \otimes E)$

explicitly R spanned by

$$\boxed{t_i \otimes e_i - \sum_{j} t_j \otimes \psi_{ji}(e_i)}$$

$\psi_{ji} \in Hom(E_i, Q_j)$.

$$t_i \otimes e_i - \sum_{j} t_j \otimes \psi_{ji}(e_i)$$

involutivity equations

~~for ψ_{ji} small~~ for R given form $M_1 = T \otimes E / R$

whence has filtration $T_i \otimes E$ and so can define $E_i = \{ e \mid T_i \otimes e \subset T_{i-1} \otimes E + R \}$.

need formula for E_i in terms of R.

---

need formulas for

$$\underline{t_i} \otimes e_j - t_{i-1} \otimes \psi_{i,i-1}(e_j) - \cdots - t_1 \psi_{i,1}(e_j) \in R$$

where $\psi_{i,k} : E_j \longrightarrow E$.

<u>Hubert's idea</u>. clearly span R this way.

so now want

$t_g(t_i \otimes e_i$

$t_i \otimes e_j$
$i \leq j$ good

~~bas for TR shou~~ $T.R$ should be spanned by elements starting with

~~$t_g t_i \otimes e_i$~~

~~better g~~

$t_g(t_i \otimes e_i$

~~Choose~~

Choose a flag of subspaces $Q_i \subset E$
of dimension compl. to e_i.

$$Q_i = \bigoplus_{j \geq i} P_j$$

Then should get local equations for $P_j \subset Q_i$

---

having chosen Q_i
have new basis

$Q_i \subset E$ $\qquad M_1 \cong \bigoplus_{i \leq j} t_i \otimes P_j$

modulo r have that

$$\left(\bigoplus_{i=1}^{n} t_i \otimes Q_i\right) \oplus R = T \otimes E. \qquad \{R \text{ is open}$$

want

$$\left(\bigoplus_{j \leq i} t_j t_i \otimes Q_i\right) \cap T \cdot R = 0.$$

Therefore R has basis $0 \supset Q_1 \supset \cdots Q_{n-1} \supset Q_n \supset 0$

therefore choose complement $\bigoplus_{i=1}^{n} t_i \otimes E_i$ $\qquad \bigoplus_{i,j} \mathrm{Hom}(E_i, Q_j)$
of the sort that all nearby R are of the form $\mathrm{Hom}(\oplus t_i \otimes E_i, \oplus t_i \otimes Q$

$\mathfrak{g} \subset T \otimes E$

suppose $t_1, \ldots, t_n$ generic

let $\mathfrak{g}^i = \mathfrak{g} \cap \{T_i \otimes E\}$

$E_1 \subset T \otimes M_0$

$K_i = \operatorname{Ker}\{M_0 \xrightarrow{t_{i+1}} M/T_i M_0\}$

$$\overset{P_0}{0 \subset K_0} \overset{P_1}{\subset K_1} \subset \cdots \overset{P_n}{\subset K_n} = M$$

Then

$$t_{i+1} \otimes K_i \ \cancel{} \subset E_1 + T_i \otimes M_0$$

$$\therefore \quad t_{i+1} \otimes (v_i) - \cancel{t_i \otimes \psi} \ t_i \otimes \psi_{i+1,i}(v_i) - \cdots - t_1 \otimes \psi_{i+1,1}(v_i) \in E$$

where $\psi_{i+1,a} : K_i \longrightarrow M_0$ not unique

but ~~are~~ are modulo K_{a-}

$\mathfrak{g} \subset T \otimes E \qquad T = \bigoplus_{i=1}^{n} \mathbb{C} t_i \qquad 0 \subset E_1 \subset \cdots \subset E_n \overset{P_n}{\subset} E$

$$E_i = \{ v \in E \mid t_i \otimes e \in \mathfrak{g} + T_{i-1} \otimes E \}.$$

Then

$$t_i \otimes v - t_{i-1} \otimes \psi_{i,i-1}(v) - t_{i-2} \otimes \psi_{i,i-2}(v) \cdots - t_1 \otimes \psi_{i,1}(v) \in \mathfrak{g}$$

where $v \in E_i \quad \psi_{i,j} : E_i \longrightarrow E.$

ψ_{ij} may be made unique by requiring ~~that~~

$$\boxed{\psi_{i,j} : E_i \longrightarrow \bigoplus_{a \geq j} P_a \simeq E/E_j} \ ?$$

~~my scheme~~: Given $\mathcal{F}$ and $\mathcal{E}$ I describe space of all R for which $T_i \otimes E_i \subset T_{i-1} \otimes E + R$ as an affine space given a splitting of $\mathcal{E}$, then I calculate involutivity equations.

$(\mathcal{F}, \mathcal{E}, R)$

$$t_j \; P_i = \sum_{k \leq j i} t_k \otimes \quad \in P_l$$

$$j > i \qquad P_i \subset E_{i+1} \qquad k \leq l$$

$Z \quad \boxed{\mathrm{Flag}(T) \times \mathbf{Cart}}$

$$P_i \subset E_{i+1}$$

$$H \longrightarrow SL_2(k) * \mathrm{Bor}\; k[X] \longrightarrow$$

$$H = \cancel{SL_2(k) *} \; \mathrm{Bor}^\circ k[X]$$

Suppose $F_i \ni$

E_i

$$E_i \oplus F_i = V$$

then if E_i' near to E_i

still $E_i \oplus F_i = V$.

$\underline{n=2}$:

$t_1^2 \otimes v_1$ ✓

$t_1 t_2 \otimes v_1$

$t_1 t_2 \otimes v_2 - t_1^2 \otimes \psi_{21}(v_2)$

$t_2^2 \otimes v_2 - t_1 t_2 \otimes \psi_{21}(v_2)$.

$$\underline{t_1} t_2 \otimes v_1 \equiv t_1(t_1 \otimes \psi_{21}(v_1))$$

$$t_1^2 \otimes \psi_{21}(v_1)$$

$$\therefore \psi_{21}(v_1) \subset E_1 \qquad \therefore \psi_{21}(E_1) \subset E_1$$

$\underline{n=3}$:

$t_1^2 \otimes v_1$ ✓

$t_2 t_1 \otimes v_1$ ✓

$t_3 t_1 \otimes v_1 \equiv t_1 t_2 \otimes \psi_{32}(v_1) + t_1^2 \otimes \psi_{31}(v_1)$

~~$t_i \otimes v$~~ ~~$\psi_{ij}(v_j)$~~

$$\psi_{ij}(v_k) = 0 \quad \text{if} \quad i > j \geq k$$

ie $\quad T \otimes v_i \subset T_{i-1} \otimes E \oplus V$.

$$\psi_{ij} : E_i \longrightarrow E/E_j \qquad i > j$$

and $\psi_{ij}(E_k) = 0$ if $j \geq k$.

~~[illegible]~~

Ψ_i

~~A~~

$T \otimes E_i \subset T_{i-1} \otimes E + R$

Let's suppose that in fact

$$\Psi_i \in Hom(T \otimes E_i, T_{i-1} \otimes E)$$

such that

(i) $\Psi_i \mid T_{i-1} \otimes E_i$ is inclusion

(ii) $\Psi_i \mid T \otimes E_j = \Psi_j$ if $j < i$

T_{i-}

$T \otimes E_i$

~~If so then~~ this should be an iterated affine with group

choice for Ψ_i over Ψ_{i-1} is

$Hom(T \otimes E_i / T_{i-1} \otimes E_{i-1}, T_{i-1} \otimes E)$

$\tilde{\Psi}_i \in Hom(T \otimes E_i, T_{i-1} \otimes E)$

$\ni \tilde{\Psi}_i(T_{i-1} \otimes E_i) = 0$

$\tilde{\Psi}_i(T \otimes E_{i-1}) = 0$

~~is $Hom(T/T_{i-1} \otimes E_i, E/E_{i-1})$~~

$\tilde{\Psi}_i(T \otimes E_{i-1}) = 0$

One Cartan integer s_r

$$t_i \otimes E_i \subset T_{i-1} \otimes E + R.$$

$$t_i \otimes e_i$$

$$\begin{cases} t_1 \otimes e_1 \\ t_2 \otimes e_2 - t_1 \otimes \psi_{21}(e_2) \\ t_3 \otimes e_3 - t_2 \otimes \psi_{32}(e_3) - t_1 \otimes \psi_{31}(e_3) \\ t_4 \otimes e_4 - t_3 \otimes \psi_{43}(e_4) - \quad - t_1 \psi_{41}(e_4) \end{cases}$$

Then everything in TR must be a linear combination of the element

$$t_i(t_k \otimes e_k - \quad \cdots) \quad \text{with} \quad i \leq k$$

independent and have correct dimension

$\cancel{\Psi_i \in \mathrm{Hom}(T \otimes E_{i0})}$

$$j > i \qquad t_j \otimes v_i = \sum_{\substack{k \le j \\ k \le l}} \underbrace{t_k \otimes a^{ji}_{kl}(v_i)}_{\mathrm{Hom}(P_i, P_l)}$$

$$\sum_{j>i} \sum_{\substack{k \le j \\ k \le l}} s_i s_l$$

$$= \sum_k \Big(\sum_{l \ge k} s_l \Big) \Big(\sum_{\substack{j \ge k \\ i < j}} s_i \Big)$$

$$= \sum$$

$$T/T_i \otimes \bigoplus_{j \le i} P_j) \qquad \frac{\bigoplus_{j \ge i} P_j}{E/E_i)}$$

$$t_3\left(t_2\otimes e_2 - t_1\otimes\psi_{21}(e_2)\right) = t_1(t_1\otimes e_1')$$

$$t_1\left(t_2\otimes e_2' - t_1\otimes\psi_{21}(e_2')\right)$$

$$t_2\left(t_2\otimes e_2'' - t_1\otimes\psi_{21}(e_2'')\right)$$

$$t_1\left(t_3\otimes e_3' - t_2\otimes\psi_{32}(e_3') - t_1\otimes\psi_{31}(e_3')\right)$$

$$t_2\left(t_3\otimes e_3'' - t_2\otimes\psi_{32}(e_3'') - t_1\otimes\psi_{31}(e_3'')\right)$$

$$t_3\left(t_3\otimes e_3''' - t_2\otimes\psi_{32}(e_3''') - t_1\otimes\psi_{31}(e_3''')\right)$$

t_1^2	$0 = e_1' - \psi_{21}(e_2') - \psi_{31}(e_3') = 0$
t_1t_2	$0 = e_2' - \psi_{21}(e_2'') - \psi_{32}(e_3') - \psi_{31}(e_3'')$
t_2^2	$0 = e_2'' - \psi_{32}(e_3'')$
t_1t_3	$\psi_{21}(e_2) = e_3' - \psi_{31}(e_3''')$
t_2t_3	$e_2 = e_3'' - \psi_{32}(e_3''')$
t_3^2	$0 = e_3'''$

Assertion 1: Can always solve recursively for $\boxed{e_{ijk}}$ hence involutivity expressible as Hilbert said e_i

$$\boxed{\psi_{21}E_2 \subset E_3}$$

$$\boxed{\psi_{32}E_2 \subset E_2}$$

$$\boxed{e_3''' = 0}$$

$$\boxed{\begin{aligned} e_3' &= \psi_{21}(e_2) \\ e_3'' &= e_2 \end{aligned}}$$

$$\boxed{e_2'' = \psi_{32}(e_3'')} \boxed{= \psi_{32}(e_2)}$$

$$e_2' = \psi_{21}(\psi_{32}e_2) - \psi_{32}\psi_{21}(e_2) - \psi_{31}(e_2)$$

$$e_1' = \psi_{21}(\psi_{32}e_2) + \psi_{31}(\psi_{21}e_2)$$

$\hat{E}_1$

$$\boxed{\psi_{31}\psi_{21}E_2 \subset E_1}$$

$$\left(\psi_{21}\overset{E_1}{\psi_{32}} - \psi_{32}\psi_{21} - \psi_{31}\right)E_2 \subset E_2$$

The formulas are liable to be very complicated, therefore best thing is to set up induction.

$\mathcal{O}^2$

~~Quot~~

Thm 1: Let $P(n) = \sum_{i=1}^{n} s_i \binom{n+i-1}{i-1}$ with $s_i \geq 0$. Then

$s_i = 0 \ i > 1 \quad s_1 \geq 2.$

$\mathrm{Quot}_P^{P(0)} = \mathrm{Cart}_{s_1, \dots, s_n}$ false ~~false (take ...)~~

In other words if $\mathcal{F}$ is a sheaf on $\mathbb{P}^{n-1}$ with $\mathcal{O}^{P(0)} \longrightarrow \mathcal{F} \longrightarrow 0$ and Hilbert poly P then $\Gamma(\mathcal{F}(*))$ is involutive.

~~Proof~~ Let t be a generic element of $T = \Gamma(\mathbb{P}^{n-1}, \mathcal{O}(1))$, so that

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}^{P(0)}(-1) & \longrightarrow & \mathcal{O}^{P(0)} & \longrightarrow & \mathcal{O}_{H_t}^{P(0)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F}(-1) & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F} \otimes \mathcal{O}_{H_t} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

is exact. Then ?

Given two sheaves $\mathcal{F}, \mathcal{G}$ on $\mathbb{P}^{n-1}$ with same Hilb. poly are they in a flat family?

Defn: ~~Cart~~ $\mathrm{Cart}_{s_1, \dots, s_n} = \left\{ T \otimes \mathbb{C}^{s_1 + \dots + s_n} \longrightarrow M_1^{\sum i s_i} \;\middle|\; \forall \text{ generic flags } T_i \ni \dim T_i M / T_{i-1} M \right\}.$

for g to be involutive means that

$$\mathrm{Im}\{T \otimes g \longrightarrow S^2T \otimes E\}$$

has small dimension.

$$\begin{cases} t_1 \otimes v_1 & \in g \\ t_2 \otimes v_2 - t_1 \otimes \psi_{2,1}(v_2) & \in g \\ t_3 \otimes v_3 - t_2 \otimes \psi_{3,2}(v_3) - t_1 \otimes \psi_{3,1}(v_3) & \in g \end{cases}$$

Involutivity for the complement

The bases is $t^k t^l \otimes e_i$
where $e_i \in P_i$ and $k, l \leq i$

Thus you must consider ~~the~~ a typical terms of the form

$$t^k t^l \otimes v_i \quad \text{where} \quad k, l \leq i$$

in the image of $T \otimes g \longrightarrow S^2T \otimes E$.

Image is spanned by

$$t_k t_i \otimes v_i - \sum_{j<i} t_k t_j \otimes \psi_{i,j}(v_i)$$

Bad terms are those for which either k, l exceeds the filtration.

~~Hubert's argument is correct replace t_j by $\lambda^j t_j$ and divide by a scalar. Doesn't change anything and replaces $\psi_{i,j}$ by $\lambda^{i-j}\psi_{i,j}$~~
~~Now let $\lambda \to 0$~~

$$0 = \frac{\partial}{\partial x_3}\left\{\frac{\partial u_1}{\partial x_2} - a_{11}^{21}\frac{\partial u_1}{\partial x_1} - \cdots - a_{1n}^{21}\frac{\partial u_n}{\partial x_1}\right\}$$

$$= \frac{\partial}{\partial x_2}\left\{a_{11}^{31}\frac{\partial u_1}{\partial x_1} + \cdots + a_{1n}^{31}\frac{\partial u_1}{\partial x_1}\right\} - a_{11}^{21}\frac{\partial}{\partial x_1}\left\{a_{11}^{31}\frac{\partial u_1}{\partial x_1} + \cdots + a_{1n}^{31}\frac{\partial u_n}{\partial x_1}\right\}$$

$$- a_{12}^{21}\frac{\partial}{\partial x_1}\left\{a_{11}^{32}\frac{\partial u_1}{\partial x_1} + \cdots + a_{1n}^{32}\frac{\partial u_n}{\partial x_1} + a_{22}^{32}\frac{\partial u_2}{\partial x_2} + \cdots + a_{2n}^{32}\frac{\partial u_n}{\partial x_2}\right\}.$$

$$- \cancel{a_{13}^{21}\frac{\partial^2 u_3}{\partial x_3 \partial x_1}} - \cdots$$

cannot appear.

$$\therefore \quad a_{13}^{21} = \cdots = a_{1n}^{21} = 0.$$

$$a_{13}^{31}\frac{\partial^2 u_3}{\partial x_1 \partial x_2} - a_{11}^{21}\, a_{13}^{31}\frac{\partial^2 u_3}{\partial x_1^2}$$

$$- a_{12}^{21} \cdot a_{13}^{32}\frac{\partial^2 u_3}{\partial x_1^2} - a_{12}^{21}\, a_{23}^{32}\frac{\partial^2 u_3}{\partial x_1 \partial x_2}$$

$$a^{31}_{11}\frac{\partial_1}{\partial x_1}\left(a^{21}_{11}\frac{\partial u_1}{\partial x_1}+a^{21}_{12}\frac{\partial u_2}{\partial x_1}\right)$$

$t_i \otimes e_j$

$\cancel{-a^{21}_{11}}$

$$\boxed{a^{31}_{11}\,a^{21}_{12}-a^{21}_{11}\,a^{31}_{12}-a^{21}_{12}\,a^{32}_{12}=0}$$

$\tau^{ki}a^{ij}_{k\ell}$

$$\boxed{a^{31}_{11}\,a^{21}_{11}-a^{21}_{11}\,a^{31}_{11}-a^{21}_{12}\,a^{32}_{11}=0}$$

a^{31}_{12} a^{31}_{13} functions of others satisfying quadratic relations

$$a^{31}_{14}\frac{\partial^2 u_4}{\partial x_1\partial x_2}-a^{21}_{12}a^{32}_{24}\frac{\partial^2 u_4}{\partial x_1\partial x_2}$$

$$-a^{21}_{11}a^{31}_{14}\frac{\partial^2 u_4}{\partial x_1^2}-a^{21}_{12}a^{32}_{14}\frac{\partial u_4}{\partial x_1}$$

Need procedure for reworking $\frac{\partial u_g}{\partial x_k\,\partial x_\ell}$ with $k,\ell<n$ into a standard form

Coherent system

$$\frac{\partial}{\partial x_n}\left\{ \frac{\partial u_j}{\partial x_i} - \sum_{\substack{k\leq j\\ k\leq l}} a^{ij}_{kl}\, \frac{\partial u_l}{\partial x_k} \right\}$$

$$\frac{\partial}{\partial x_n} \sum_{k\leq j} a^{ij}_{kn}\, \frac{\partial u_n}{\partial x_k}$$

$n > i > j$

$$\equiv \frac{\partial}{\partial x_i}\left\{ \sum_{\substack{p\leq j\\ p\leq q}} a^{nj}_{pq}\, \frac{\partial u_q}{\partial x_p} \right\} - \sum_{\substack{k\leq j\\ k\leq l<n}} a^{ij}_{kl}\, \frac{\partial}{\partial x_k}\left\{ \sum_{\substack{p\leq l\\ p\leq q}} a^{nl}_{pq}\, \frac{\partial u_q}{\partial x_p} \right\} -$$

$i<n,\ p<n$ $\qquad k\leq n-2$

involutivity same as requiring that

have to be careful of the term $\sum_{k\leq j} a^{ij}_{kn} \frac{\partial}{\partial x_k}\left\{ a^{nn}_{nn} \frac{\partial u_n}{\partial x_n}\right.$

$t_n R_{n-1} \subset T\mathcal{F}_n +$

Maybe true that

$\boxed{\text{Conjecture: } a^{ij}_{kl} = 0 \text{ if } l > i}$

Probably (Certainly) true that $a^{ij}_{kn} = 0$ if $i < n$

Induction should show that $a^{ij}_{kl} = 0$ if $i < l$ but one must be careful ~~[illegible]~~ to do induction correctly.

$$\left\{ \frac{\partial u_0}{\partial x_1} = 0 \right.$$

$$\left\{ \begin{aligned} &\frac{\partial u_0}{\partial x_2} = 0 \\ &\frac{\partial u_1}{\partial x_2} = a^{21}_{11} \frac{\partial u_1}{\partial x_1} + a^{21}_{12} \frac{\partial u_2}{\partial x_1} \end{aligned} \right.$$

$$\left\{ \begin{aligned} &\frac{\partial u_0}{\partial x_3} = 0 \\ &\frac{\partial u_1}{\partial x_3} = a^{31}_{11} \frac{\partial u_1}{\partial x_1} + a^{31}_{12} \frac{\partial u_2}{\partial x_1} + a^{31}_{13} \frac{\partial u_3}{\partial x_1} \\ &\frac{\partial u_2}{\partial x_3} = a^{32}_{11} \frac{\partial u_1}{\partial x_1} + a^{32}_{12} \frac{\partial u_2}{\partial x_2} + a^{32}_{13} \frac{\partial u_3}{\partial x_1} + a^{32}_{22} \frac{\partial u_2}{\partial x_2} + a^{32}_{23} \frac{\partial u_3}{\partial x_2} \end{aligned} \right.$$

$$a^{21}_{11} \frac{\partial u_1}{\partial x_1} + a^{21}_{12} \frac{\partial u_2}{\partial x_1}$$

$$0 = \frac{\partial}{\partial x_3} \left\{ \frac{\partial u_1}{\partial x_2} - a^{21}_{11} \frac{\partial u_1}{\partial x_1} - a^{21}_{12} \frac{\partial u_2}{\partial x_1} \right\} = \frac{\partial}{\partial x_2} \left\{ a^{31}_{11} \frac{\partial u_1}{\partial x_1} + a^{31}_{12} \frac{\partial u_2}{\partial x_1} + a^{31}_{13} \frac{\partial u_3}{\partial x_1} \right\}$$

$$a^{31}_{14} \frac{\partial u_4}{\partial x_1}$$

$$- a^{21}_{11} \frac{\partial}{\partial x_1} \left\{ a^{31}_{11} \frac{\partial u_1}{\partial x_1} + a^{31}_{12} \frac{\partial u_2}{\partial x_1} + a^{31}_{13} \frac{\partial u_3}{\partial x_1} \right\} \quad a^{31}_{14} \frac{\partial u_4}{\partial x_1}$$

$$- a^{21}_{12} \frac{\partial}{\partial x_1} \left\{ a^{32}_{11} \frac{\partial u_1}{\partial x_1} + a^{32}_{12} \frac{\partial u_2}{\partial x_1} + a^{32}_{13} \frac{\partial u_3}{\partial x_1} + a^{32}_{22} \frac{\partial u_2}{\partial x_2} + a^{32}_{23} \frac{\partial u_3}{\partial x_2} \right\}.$$

$$a^{32}_{14} \frac{\partial u_4}{\partial x_1} \qquad a^{32}_{24} \frac{\partial u_4}{\partial x_2}$$

$$\boxed{a^{31}_{13} = a^{21}_{12} a^{32}_{23}} \qquad \boxed{a^{21}_{11} a^{31}_{13} + a^{21}_{12} a^{32}_{13} = 0} \qquad \boxed{a^{31}_{12} - \cancel{a^{21}_{11} a^{31}_{12}} - a^{21}_{12} a^{32}_{22} = 0}$$

$$\mathrm{I}\ \left\{ \frac{\partial u_0}{\partial x_1} = 0 \right.$$

$$\mathrm{II}\ \left\{ \begin{aligned} \frac{\partial u_0}{\partial x_2} &= 0 \\ \frac{\partial u_1}{\partial x_2} &= a^{21}_{11} \frac{\partial u_1}{\partial x_1} + a^{21}_{12} \frac{\partial u_2}{\partial x_1} + \cdots + a^{21}_{1n} \frac{\partial u_n}{\partial x_1} \end{aligned} \right.$$

$$\mathrm{III}\ \left\{ \begin{aligned} \frac{\partial u_0}{\partial x_3} &= 0 \\ \frac{\partial u_1}{\partial x_3} &= a^{31}_{11} \frac{\partial u_1}{\partial x_1} + \cdots + a^{31}_{1n} \frac{\partial u_n}{\partial x_1} \\ \frac{\partial u_2}{\partial x_3} &= a^{32}_{11} \frac{\partial u_1}{\partial x_1} + \cdots + a^{32}_{1n} \frac{\partial u_n}{\partial x_1} + a^{32}_{22} \frac{\partial u_2}{\partial x_2} + \cdots + a^{32}_{2n} \frac{\partial u_n}{\partial x_2} \end{aligned} \right.$$

$$\mathrm{IV}\ \left\{ \begin{aligned} \frac{\partial u_0}{\partial x_4} &= 0 \\ \frac{\partial u_1}{\partial x_4} &= a^{41}_{11} \frac{\partial u_1}{\partial x_1} + \cdots + a^{41}_{1n} \frac{\partial u_n}{\partial x_1} \\ \frac{\partial u_2}{\partial x_4} &= a^{42}_{11} \frac{\partial u_1}{\partial x_1} + \cdots + a^{42}_{1n} \frac{\partial u_n}{\partial x_1} + a^{42}_{22} \frac{\partial u_2}{\partial x_2} + \cdots + a^{42}_{2n} \frac{\partial u_n}{\partial x_2} \\ \frac{\partial u_3}{\partial x_4} &= a^{43}_{11} \frac{\partial u_1}{\partial x_1} + \cdots + a^{43}_{1n} \frac{\partial u_n}{\partial x_1} + a^{43}_{22} \frac{\partial u_2}{\partial x_2} + \cdots + a^{43}_{2n} \frac{\partial u_n}{\partial x_2} \\ &\quad + a^{43}_{33} \frac{\partial u_3}{\partial x_3} + \cdots + a^{43}_{3n} \frac{\partial u_n}{\partial x_3} \end{aligned} \right.$$

Cartan's principle: $\frac{\partial}{\partial x_4}$ applied to I, II, III should be a consequence of IV

Point of involutivity is that ~~derivatives of i — i deriv~~

$$\frac{\partial}{\partial x_i}\left(\frac{\partial u_j}{\partial x_k} - \sum_{j<k\le i} \qquad \right) \equiv 0$$

~~expressible as a~~ modulo derivatives of nth equations

$$u = (u_0, \dots, u_n) \in \prod_{i=0}^{n} P_i^* = E$$

$$\Big\langle \sum_{i>j} t_i \otimes v_j - \sum_{\substack{k\le j\\ k\le l}} t_k \otimes \underset{P_l}{\underset{\cap}{a^{ij}_{kl}(v_j)}}\ ,\ \sum_{h,s} dx_h \otimes \frac{\partial u_s}{\partial x_h} \Big\rangle \equiv 0$$

$$\sum_{i>j}\Big\langle v_j, \frac{\partial u_j}{\partial x_i}\Big\rangle = \sum_{\substack{k\le j\\ k\le l}} \Big\langle a^{ij}_{kl}(v_j), \frac{\partial u_l}{\partial x_k}\Big\rangle$$

$$= \Big\langle v_j, (a^{ij}_{kl})^t \frac{\partial u_l}{\partial x_k}\Big\rangle$$

$$\therefore\ \boxed{\cancel{\frac{\partial u_j}{\partial x_i} = \sum_{\substack{k\le j\\ k\le l}} a^{ij}_{kl} \frac{\partial u_l}{\partial x_k} \quad i>j}}$$

$$\frac{\partial u_i}{\partial x_j} = \sum_{\substack{k\le i\\ k\le l}} a^{ji}_{kl} \frac{\partial u_l}{\partial x_k} \qquad j>i$$

$$\frac{\partial}{\partial x_n}\frac{\partial u_j}{\partial x_i} \equiv \sum_{p<i}' a \frac{\partial}{\partial x_p}\left(\sum_{r<n}' A \frac{\partial u}{\partial x_r}\right)$$

modulo nth equations

$$\equiv$$

$$\frac{\partial}{\partial x_i}\left(\sum_{q<n} A^{nj} \frac{\partial u_\ell}{\partial x_q}\right)$$

$$t_n \otimes u_i$$

~~Grothendieck~~

Duality between equations and $R \hookrightarrow T \otimes M_0$.

$$i<j \qquad \frac{\partial u_i}{\partial x_j} = \sum_{\substack{k<j \\ k\le \ell}} a^{ij}_{k\ell} \frac{\partial u_\ell}{\partial x_k}$$

I seem to be able here to take $a^{ij}_{k\ell}=0$ if already $k>i$ not just $k<j$

Involutivity by induction

$$\frac{\partial u_0}{\partial x_1} = 0$$

$$\frac{\partial u_0}{\partial x_2} =$$

$$\frac{\partial u_1}{\partial x_2} =$$