J. Frank Adams 1

Theorems on ordinary homology + coh. which need generalization

Künneth theorem

Universal Coeff. thm.

E ring spectrum $h_*$ $h^*$ corresponding theories

MU, BU, S $h^r(X) = \varinjlim [S^{n-r}X, E_n]$ (X finite cx)

$h_r(X) = \varinjlim [S^{n+r}, X \wedge E_n]$ (X arb.)

Kunneth should say $\exists$ spec. seq.

$$\mathrm{Tor}^{h_*(pt)}(h_*(X), h_*(Y)) \Longrightarrow h_*(X \times Y)$$

$$(\mathrm{Ext}_{h_*(pt)}(h_*(X), h^*(Y)) \Longrightarrow h^*(X \times Y))$$

$$\mathrm{Tor}^{h^*(pt)}(h^*(X), h^*(Y)) \Longrightarrow h^*(X \times Y)$$

$$(\mathrm{Ext}_{h^*(pt)}(h^*(X), h_*(Y)) \Longrightarrow h_*(X \times Y))$$

Universal coeff. thm: F module spectrum over E corresponding to theories $k_*$, $k^*$. E= MU F= BU (Conner-Floyd)

(i) $\mathrm{Tor}^{h_*(pt)}(h_*(X), k_*(pt)) \Longrightarrow h_*(X)$

(ii) $\mathrm{Ext}_{h_*(pt)}(h_*(X), k_*(pt)) \Longrightarrow k^*(X)$

(iii) $\mathrm{Tor}^{h^*(pt)}(h^*(X), k^*(pt)) \Longrightarrow k^*(X)$

(iv) $\mathrm{Ext}_{h^*(pt}(h^*(X), k^*(pt)) \Longrightarrow k_*(X)$

note that universal coeff thm. $\Rightarrow$ Künneth by

$$k_*(X) = h_*(X \times Y)$$
$$k^*(X) = h^*(X \times Y)$$

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Observe that (iii) + (iv) need finiteness hypotheses. Then (i) + (ii) $\Rightarrow$ (iii) + (iv) by Spanier-Whitehead dual of $X$, assuming $X$ is a finite complex.

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Special case in which theorem is true: Assume $h_*(X)$ flat over $h_*(pt.)$, then

$$h_*(X) \otimes_{h_*(pt)} k_*(pt) \xrightarrow{\sim} k_*(X)$$

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If $E = S$ any $F$ will do

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Atiyah method + S duality $\Rightarrow$ (i) + (ii) for

$BO$, $BU$, $BSp$, $MU$, $MSp$, $S$, $K(\mathbb{Z}_p)$

doesn't work for $K(\mathbb{Z})$

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$S$ is known to be strictly associative, others

(Boardman ~~[illegible]~~ supposed to have a proof)

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Dyer + Daniel Kahn have a proof of Künneth + ~~[illegible]~~ method yields universal coeff. (i) if $E$ ~~strictly~~ associative + $F$ strictly assoc. module spectrum

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Folklore about universal coeffs. + Adams spec. seq.

$$\mathrm{Ext}_A(\tilde{H}^*(X), \tilde{H}^*(Y)) \Longrightarrow [Y, X].$$

Assume $\exists$ spec. sequence

(i) $\mathrm{Ext}_{h^*(E)}(h^*(Z), h^*(X)) \Longrightarrow [X, Z]$.

Requires finiteness assumptions on $Z$.

(ii) $\mathrm{Ext}_{h^*(E)}(h_*(X), h_*(Z)) \Longrightarrow [X, Z]$

Problem is that $A^* = h^*(E)$ is a topologised ring
& $h^*(X)$ ——— module

and that $A_q \neq 0$ for both $q<0$ $q>0$ in general.

(iii) $\mathrm{Ext}_{h_*(E)}(h_*(X), h_*(Z)) \Longrightarrow [X, Z]$.

as an $h_*(E)$ comodule

Need $\boxed{h_*(E) \text{ flat over } h_*(pt.)}$ to define (iii).

True for $BO$, $BU$, $BSp$, $MU$, $MSp$, $S$, $K(\mathbb{Z}_p)$
but not $K(Z)$.

Consider

$$\begin{array}{c} h_*(X) = \pi_*(X \wedge E) \xrightarrow{\text{hurewicz}} h_*(X \wedge E) \\ \uparrow \cong \\ h_*(X) \otimes_{h_*(pt)} h_*(E) \end{array}$$

this should show $h_*(E)$ is an $h_*(E)$ comodule.

assume $Z$ module spectrum over $E$, then

$$h_*(Z) = \pi_*(Z \wedge E) = k_*(E)$$

$$= h_*(E) \otimes_{h_*(pt)} k_*(pt)$$

thus from (ii)

$$\begin{array}{ccc} \mathrm{Ext}_{h_*(E)}(h_*(X), h_*(Z)) & \cong & \mathrm{Ext}_{h_*(pt)}(h_*(X), k_*(pt)) \\ \Downarrow & & \Downarrow \\ [X,Z] & & k(X) \end{array}$$

So Adams spectral sequence yields universal coeff thm part (ii).

Doesn't seem possible to deduce part (i) from Adams sp. seq.

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Adams 2:

$h^*(X)$ gen. coh. theory $h^*(pt)$ countable
$G$ torsion free countable abelian gp. (e.g. $G \subset \mathbb{Q}$).
$h^*(X) \otimes G = [X/\phi, E]$ by Brown

Defn: $h^*(X; G) = [X/\phi, E]$.

$\Omega_u^*(X) = [X/\phi, MU]$

$MU \underset{\text{off } p}{\sim} \prod_i S^{n_i} BP(p)$   BP = Brown-Peterson spectr

Let $Q_p = \mathbb{Z}_{p\mathbb{Z}}$. Then $\Omega_u^n(X, Q_p) \cong \prod_i L^{n+n_i}(X)$
where $L^*(X) = [X, BP(p)]$. Note that $A_L$ much smaller than $A_\Omega$

Adams will now show how to decompose $MU$ canonically by constructing idempotents $e$ in $A_\Omega$

let $S$ be a subring of $\mathbb{Q}$, $A(S)$ = alg. of coh. operations on $K^*(\ , S)$ which are $S$ linear + $\varphi(x+y) = \varphi(x) + \varphi(y)$, $\varphi(0) = 0$.

Lemma 1: If $S_1 \subset S_2$, then $A(S_1) \to A(S_2)$ is mono

Proof: $CP^\infty$ $A(S) \xrightarrow{\cong} K(CP^\infty; S) \cong S[[\xi]]$
$a \longmapsto a(\underset{\eta}{\text{can. line bundle}})$  $\xi = (\underset{\eta}{\text{can. bundle}}) - 1$

Over $\mathbb{Q}$ $K(X, \mathbb{Q}) \xrightarrow[ch]{\cong} \prod_n H^{2n}(X, \mathbb{Q})$
$e_n$ projection on $n$th component
$e_n \in A(\mathbb{Q})$.

Let $\alpha \in \mathbb{Z}/d\mathbb{Z}$, let $E_\alpha \in A(Q)$ $E_\alpha = \sum_{n \in \alpha} e_n \in A(Q)$.

Thm. 2: (i) $E_\alpha \in A(S)$ where $S = \{a/b \mid b$ not divisible by any prime $p \equiv 1 \ (d)\}$

(ii) $E_\alpha E_\beta = 0 \qquad \alpha \neq \beta$

$$E_\alpha^2 = E_\alpha$$

$$\sum_{\alpha \in \mathbb{Z}/d\mathbb{Z}} E_\alpha = 1$$

(iii) $x, y \in K^0(X; S) \Rightarrow \quad E_\alpha(xy) = \sum_{\beta+\gamma=\alpha} E_\beta(x) E_\gamma(y)$

Proof: By lemma 1 have only to prove (i).

Cor 3: (i) $K^0(X; S) = \sum_\alpha K_\alpha(X)$

(ii) $x \in K_\alpha,\ y \in K_\beta \Rightarrow xy \in K_{\alpha+\beta}$

(iii) $K_\alpha$ representable

(iv) $K_0$ has products $\qquad (iv)_{1/2}\ \tilde{K}_\alpha(S^n) = \begin{cases} S & \text{if } \frac{1}{2}n \in \alpha \\ 0 & \text{otherwise} \end{cases}$

(v) $K_\alpha$ is periodic of period $2d$ in sense that $g \in \tilde{K}_0(S^{2d})$

$x \mapsto gx$ gives iso $\tilde{K}_\alpha(X) \longrightarrow \tilde{K}_\alpha(S^{2d} \wedge X)$

(vi) $\tilde{K}_\alpha(X) \xrightarrow{\sim} K_{\alpha+1}(S^2 \wedge X) \qquad g' \in \tilde{K}_1(S^2)$

$x \longmapsto g'x$.

To prove Thm (2) $\qquad e_n(\eta) = \ ?$ $\qquad$ require $\eta$ = can. line bundle

$$ch\, \eta = \sum_0^\infty \frac{x^n}{n!} \qquad x = c_1(\eta) \qquad y = \eta - 1.$$

$$ch\, e_n(\eta) = \frac{x^n}{n!}$$

$$ch \operatorname{Log}(1+y) = \log ch(1+y) = \operatorname{Log} e^x = x$$

$$\therefore\ ch \frac{\operatorname{Log}(1+y)^n}{n!} = \frac{x^n}{n!}$$

$$\therefore \quad e_n(\eta) = \frac{\{\log(1+\xi)\}^n}{n!}$$

$$\sum_n t^n e_n(\eta) = \sum_n \frac{t^n [\log(1+\xi)]^n}{n!}$$

$$= e^{t\log(1+\xi)} = (1+\xi)^t$$

$$= 1 + t\xi + \frac{t(t-1)}{2!}\xi^2 + \cdots$$

Since $p \equiv 1 \ (d)$ we can find in $\hat{\mathbb{Z}}_{p\mathbb{Z}}$ a primitive $d$th root of $1$, say $\omega$. Let $m \in \{1, \cdots, d\}$ + let $\rho = \omega^m$

$$\sum_\alpha \rho^\alpha E_\alpha(\eta) = \sum_n \rho^n e_n(\eta)$$

$$= 1 + \rho\xi + \frac{\rho(\rho-1)}{2}\xi^2 + \cdots \quad = f_m(\xi)$$

power series with $p$-adic integer coeffs. (binomial coeffs. are continuous fns from $\mathbb{Z}$ to $\mathbb{Z}$ for $p$ topology)

$$\therefore \quad E_\alpha(\eta) = d^{-1}\sum_{m=1}^{d} \omega^{-m\alpha} f_m(\xi) \in \hat{\mathbb{Z}}_{p\mathbb{Z}}[[\xi]].$$

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Lemma 4: $B(S) =$ degree 0 ~~stable~~ stable operations on $\Omega^*(X; S)$

If $S_1 \subset S_2$, then $B(S_1) \hookrightarrow B(S_2)$.

$$B(\mathbb{Q}) = \mathrm{Hom}_{\mathbb{Q}}\left(H_*(MU;\mathbb{Q}), H_*(MU;\mathbb{Q})\right)$$

Choose an integer $d$ and let $E_\alpha$ be as before

$$\begin{array}{ccc} H_*(MU;\mathbb{Q}) & \xrightarrow{\varepsilon} & H_*(MU;\mathbb{Q}) \\ \uparrow\cong & & \uparrow\cong \\ H_*(BU,\mathbb{Q}) & \xrightarrow{(E_0)_*} & H_*(BU,\mathbb{Q}) \end{array}$$

then $\varepsilon^2 = \varepsilon$.

Theorem 5: (i). $\varepsilon \in B(S)$  $S$ as before

(ii). $\varepsilon^2 = \varepsilon$

(iii) $\varepsilon(xy) = \varepsilon(x)\cdot\varepsilon(y)$  any $x, y \in \Omega^*_u(X;S)$

Cor 6: (i) Define $\Omega^*_0(X) = \varepsilon\,\Omega^*_u(X;S)$.

Then $\Omega^*_0(X)$ is a cohomology theory with products and $\Omega^*_0(pt)$ is a poly ring with gen. in dims $-2d, -4d, \ldots$

(ii) $\Omega^*_u(X;S)$ is a direct product of theories $\cong$ to $\Omega^*_0(X)$.

(Splitting not canonical but the injection + projection on it are canonical)

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$BU_0$  reps. space for $K_0$

$MU_0$ ——— $\Omega_0$

(i) To what extent is $MU_0$ a Thom spectrum for $BU_0$?

can one prove a Conner-Floyd thm.

$$K_0(X) \cong \Omega_0(X) \otimes_{\Omega_0(pt)} K^*_0(pt.)$$

(ii) can one prove a Novikov thm. generalizing

C-F calculation of $\Omega^*_u(BU)$

Novikov " $\Omega^*_u(MU)$

(iii) Tom Dieck has ~~studied~~ $h_* \longrightarrow$ bordism theory  TDT $h_*$

Adams conjecture this operation is idempotent.

Let $\quad \underset{\varphi_H^{-1}\, ch_{*}\, \varphi_K}{\underset{\|}{\mathrm{Todd}}} \in H^*(BU, \mathbb{Q})$

First prove $\quad \dfrac{(E_s)_* \mathrm{Todd}}{\mathrm{Todd}} \in K^*(BU, S)$

this entails $\quad \varepsilon(K^*(MU, S)) \subset K^*(MU, S)$

Then Hattori thm. entails $\varepsilon$ carries $MU$ into $MU \otimes S$

(Hattori shows that if $\alpha: Sph \to MU \otimes \mathbb{Q}$ carries $H_*(Sph)$ into $H_*(MU) \otimes S$, then $\alpha: Sph \to MU \otimes S$.)

ADAMS ~~#~~ 3: 1

<u>Theorem</u> (i): $X$ finite cx $\Omega_u^*(X)$ f.g. $\Omega_u^*(pt.)$ (Novikov)

(ii) $X$ spectrum $\begin{cases}\pi_r(X)=0 & \text{a.e. } r\\ \pi_r(X) & \text{f.g.}\end{cases} \Rightarrow H^*(X;\mathbb{Z}/p\mathbb{Z})$ is finitely presented over Steenrod algebra $A$.

(iii). $X$ spectrum $\pi_r(X)=0$ a.e. $r \Rightarrow H^*(X;\mathbb{Z}_p)$ can be presented with generator + relations in finitely many dimensions

(iv) $X$ <u>space</u> $\tilde{H}_*(X;\mathbb{Z}/p)\neq 0 \Rightarrow \pi_r^s(X)\neq 0$ inf many $r$ (in fact contains $\mathbb{Z}$ or $\mathbb{Z}/p$ ~~inf dims~~ in infinitely many dimensions) (this generalization to stable case of old thm of Serre due to Joel Cohen)

<u>Proof:</u> (iii) $\Rightarrow$ (iv) because of relations like $Sq^n x = 0$ $n > \dim x$ for a space

<u>Defn:</u> A ring $R$ is <u>coherent</u> if f.g. left ideals are finitely presented.

Then the finitely presented $R$ modules form a full abelian subcategory of all $R$ modules.

<u>Proof of (i):</u> $\Omega_u^*(pt)$ is coherent, use $H^*(X,\Omega_u^*(pt))\Rightarrow\Omega_u^*(X)$ or else induction on number of cells.

Same method works for (ii).

Suppose given $\mathcal{C}$ class of projectives / $R$ closed under $\oplus$.

Examples:
1) $\mathcal{F}$ f.g. free modules
2) $\mathcal{D}$ free modules with generators in only finitely many dimensions
3) $\mathcal{E}$ free module $\ni$ no. of gen of dim $< n$ finite all $n$.
4) $0$ zero " .

Definition: $M$ has $\mathcal{C}$-type $n$ if it has a projective resolution

$$0 \leftarrow M_0 \leftarrow C_0 \leftarrow \cdots \leftarrow C_r \leftarrow \cdots \qquad \text{with } C_r \in \mathcal{C} \quad r \leq n$$

$M$ has $\mathcal{C}$-cotype $n$. if $\exists$ res with $C_r \in \mathcal{C}$ $r > n$.

Lemma: $0 \to M' \to M \to M'' \to 0$

(1) $M'$ type $n-1$, $M$ type $n \Rightarrow M''$ type $n$

(2) $M'$ type $n$, $M''$ type $n \Rightarrow M$ type $n$

(3) $M$ type $n$, $M''$ type $n+1 \Rightarrow M'$ type $n$

Proof: (2) easy

(1) by mapping cone

(3) if $M$ projective use Shanuel to convert $M'$ to a kernel. The general case reduces to this special case.

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| | | |
|---|---|---|
| $R$ noetherian | type $0 \Rightarrow$ type $\infty$ | for $\mathcal{C} = \mathcal{F}$ |
| $R$ coherent | type $1 \Rightarrow$ type $\infty$. | |

Thm: Given $\mathcal{C}$, integer $n \geq 0$. TFAE

(i) $F \in \mathcal{C}$ $\ni$ $P \subset F$, $P$ type $n-1 \Rightarrow P$ type $n$.

(ii) $M$ type $n$, $P \subset M$, $P$ type $n-1 \Longrightarrow P$ type $n$

(iii) $0 \to K \to C_n \to C_{n-1} \to \cdots \to C_0 \to M \to 0$ exact, $C_r$ type $n$ all $r$
$\Longrightarrow K$ type $n$.

(iv) $C_n \to C_{n-1} \to \quad \to C_0 \to M \to 0$ $C_r \in \mathcal{C}$ $\forall r \Rightarrow$ can extend it
$C_{n+1}$ with $C_{n+1} \in \mathcal{C}$

(v). Each module of type $n$ is of type $\infty$.

Defn: $R$ $(n, \mathcal{C})$-coherent if above true
$(0, \mathcal{F})$ coherent $\Leftrightarrow$ noeth
$(1, \mathcal{F})$-coherence is coherence because

Lemma: $0 \to M' \to M \to M'' \to 0$, $M'$ $\overset{M''}{\text{have}}$ has type $0 \Rightarrow$ type 1 for submodules $\Rightarrow$ $M$ also.

$R$ finite diml $\Rightarrow$ $R$ $(0, \mathcal{D})$ coherent

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two cases $\qquad R \supset R^\alpha$ ~~direct systems~~

$\underline{\mathcal{C} = \mathcal{F}}$ $\qquad \underline{\mathcal{C} = \mathcal{D}}$

$\forall r_i \in R \quad i = 1, \ldots, n$
$\exists \alpha \ni r_i \in R^\alpha$

$\forall n \exists \alpha, R_1 \cdots R_n \subset R^\alpha$

$R$ free over each $R^\alpha$ (in application $R^\alpha$ is a sub Hopf algebra)

Thm: (1) $R$ module of type $n$ $\quad n \geq 1$ are of form
$R \otimes_{R^\alpha} M^\alpha$ some $\alpha$ $\quad M^\alpha$ type $n$ over $R^\alpha$

(2) $R^\alpha$ $n$-coherent $\forall \alpha \Rightarrow R$ is $n$-coherent.

# Adams' 4:

E ring spectrum, $h_*(X) = \pi_*(X \wedge E)$.

CLAIM: If $h_*(E)$ flat over $h_*(pt.)$, we can make $h_*(X)$ a comodule with respect to $h_*(E)$. Consider classical case: $E = K(\mathbb{Z}_p)$, $A^* =$ Steenrod alg. $A^* \otimes H^* \longrightarrow H^*$. $A^* \otimes A^* \longrightarrow A^*$. Dually $A_*$ is a coalgebra and $H_*$ is a comodule over $A_*$.

PROGRAM: General case: (i) to define

$$\eta : h_*(E) \to h_*(pt.)$$
$$\Delta : h_*(E) \to h_*(E) \otimes h_*(E)$$
$$\Delta : h_*(X) \to h_*(E) \otimes h_*(X)$$

$\otimes$ over $h_*(pt.)$

(ii) If $E = K(\mathbb{Z}/p)$ regain classical defns.

(iii) correct algebraic properties ($\Delta$ assoc. etc).

(iv) diagonal to be obtained by specializing coaction to $X = E$.

Defn of (i). $h_*(E) = \pi_*(E \wedge E) \longrightarrow \pi_*(E) = h_*(pt)$. $\mu : E \wedge E \to E$ the ring structure of $E$.

(ii): $h^*(E) = [E, E]^*$ acts to left of $h^*(X) = [X, E]$ & to ~~left~~ of $h_*(X) = [S, E \wedge X]$

(Note in classical case $A^*$ $\overset{\text{right}}{\text{acts}}$ on $H_*$ by $\langle ay, x\rangle = \langle y, xa\rangle$)

Let $a \in h^*(E)$ $\qquad \mu^* a = \sum a_i' \otimes a_i''$, let $x \in h_*(X)$

$$\sum \langle a_i' y, a_i'' x\rangle = a\langle y, x\rangle$$

For $E = K(\mathbb{Z}/p)$ can define $\chi$ by

$$\langle ay, x\rangle = \langle y, \chi(a) x\rangle$$

Conclude that $x \cdot a = \chi(a) \cdot x$. These formulae do not directly generalize because general cohomology operations ~~do not~~ ~~generally commute with~~ are not $h_*(pt)$ linear.

Thus $\langle ay, x \rangle$ $h_*(pt)$ linear in $x$ and

$\langle y, ax \rangle$ ———————— $y$, so unlikely they be $=$.

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| | | | |
|---|---|---|---|
| Let $a \in [E,E]$. | acts left on | $h^*(E)$ | $b \mapsto ab$ |
| | —— right —— | | $b \mapsto ba$ |
| | acts left on | $\underbrace{h_*(E)}_{}$ $h_*(X)$ ∴ on | straight action |
| | left | $h_*(E)$ any $k_*$ ∴ on | twisted action |

The straight and twisted action related by $\tau : E \wedge E \to E \wedge E$

$$\begin{tikzcd}
h_*(X) = \pi_*(E \wedge X) \arrow[r] \arrow[drr, bend right, "\Delta"] & h_*(E \wedge X) & h_*(E) \otimes_{h_*(pt)} h_*(X) \arrow[l, "\cong"'] \arrow[d, "\cong"', "\tau \otimes 1"] \\
 & & h_*(E) \otimes_t h_*(X)
\end{tikzcd}$$

straight

Then $\Delta$ is ~~an~~ $h_*(pt)$ linear and associativity axiom is tedious

**Prop:** $x \in h_*(X)$ $\quad \Delta x = \sum_i \alpha_i \otimes x_i \quad a \in h^*(E)$. Then

$$ax = \sum \underbrace{\langle a, \tau_* \alpha_i \rangle}_{\text{Kronecker product}} x_i$$

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In classical case $E = K(\mathbb{Z}/p)$

$$\langle ay, x \rangle = \langle y, xa \rangle = \langle y, X(a) \cancel{x} \rangle$$

$$= \sum_i \langle \cancel{X}a, \tau_* \alpha \rangle \langle y, x_i \rangle$$

$$= \sum_i \langle a, \alpha_i \rangle \langle y, x_i \rangle$$

In particular if $X = E$ $y = b$ get good fmla

$$\langle ab, x \rangle = \sum_i \langle a, \alpha_i \rangle \langle b, y_i \rangle = \langle a \otimes b, \Delta x \rangle .$$

# ADAMS 5.

$$\begin{tikzcd} h_*(E) \otimes_{\mathbb{Z}} h_*(E) \arrow[rr, "\mu"] \arrow[d, "\Delta\otimes\Delta"] & \text{comm.} & h_*(E) \arrow[d, bend left, "\Delta"] \\ {[h_*(E)\otimes_{h_*(pt)} h_*(E)] \otimes [\qquad]} \arrow[rr, "m"] & & h_*(E)\otimes_{h_*(pt)} h_*(E) \end{tikzcd}$$

$$m(a\otimes b\otimes c\otimes d) = (-1)^{\dim b.\dim c} (ac)\otimes(bd)$$

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Setting up the Adams spectral sequence

$$h: (\text{stable category}) \xrightarrow[\text{covariant}]{} (\text{abelian cat})$$

$$\begin{tikzcd} Y = Y_0 \arrow[d] & Y_1 \arrow[l, "f_0"'] \arrow[d] & Y_2 \arrow[l, "f_1"'] & \arrow[l, "f_2"'] \\ Z_0 \arrow[ur] & Z_1 \arrow[ur] \end{tikzcd}$$

call such a diagram a filtration of $Y$. Assume $f_q$ are inclusions as we may in the Boardman or Puppe theory.

(1) $h(Y_{r+1}) \to h(Y_r)$ zero so get

$$0 \to h(Y) \to h(Z_0) \to h(Z_1) \to \cdots \qquad \text{exact}$$

Definition: $Z$ is weakly injective wrt $h$ if $f: W \to Z$ a $h(f)=0$ then $f \sim 0$. Example: if $h = [\ ,E]$ then $E$, $S^n E$, $\prod S^{n_\alpha} E$ are weakly injective

(2) Each $Z_r$ is weakly injective.

It is now possible to prove a comparison theorem that any filtration of type (1) maps to one of type (2).

Sometimes can't construct filtrations ~~both weakly injective~~ ~~and~~ satisfying (1) & (2). Thus must assume

a) MU $Y$ finite cx.

b) $h^*(Y)$ free over $h^*(pt)$ for MSp

~~Adams~~ Anderson has tried to take an inverse limit of spectral sequences arising from resolutions of type (2). Require in practice that $h_*(pt)$ be finite. Adams proposes to take direct limits over resolutions of type (1).

So Adams introduces category of filtrations (of types of $Y$) & proves it is directed ala Grothendieck.

<u>Objects</u>: filtrations with $f_r$ injective satisfying (1).
<u>maps</u>: $g: Y_0 \to Y_0' \ni g(Y_r) \subset Y_r'$
homotopy moves $Y_r$ thru $Y_{r-1}'$.

Adams proves directed so gets spectral sequence by taking direct limit.

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Formal properties

1). $k = Th$ e.g. $k = K^*$ $h = \Omega_U^*$ get map

$$E^{**}(X,Y;h) \longrightarrow E^{**}(X,Y;k)$$

2) $k = Th$ $h = Uk$

$$E^{**}(X,Y;h) \xleftarrow{\cong} E^{**}(X,Y;k)$$

Conj: 3) $E^{**}[DY, DX; h] \cong E^{**}[X, Y, hD]$

4) (pairings) $E[Y,Z] \otimes E[X,Y] \longrightarrow E[X,Z]$

for these need projective resolutions of $X$ injective resolutions in $Y$.

5) $E_2^{**} \longrightarrow \operatorname{Ext}^{**}(h(X), h(Y))$. when isom?

hence have to choose the correct category to take the Ext in.

Assume $h_*(X) = \pi_*(E \wedge X)$ $h_*(E)$ flat / $h_*(pt)$

The category of comodules w.r.t. $h_*(E)$ is an abelian category. Sufficient ~~[illegible]~~ for calculating

$\operatorname{Ext}^*(L, M)$

to resolve $L$ by modules proj. over $h_*(pt)$ and ~~by~~ $M$ by extended comodules of the form $h_*(E) \otimes_{h_*(pt)} (\cdots)$. Can realize geometrically, e.g.

$$\begin{tikzcd} Y \arrow[dr] & & Y_1 \arrow[ll] \arrow[dr] & & Y_2 \arrow[ll] \\ & E \wedge Y \arrow[ur] & & E \wedge Y_1 \arrow[ur] & \end{tikzcd}$$

filtrations of $Y$ satisfying (1) with $Z_r = E \wedge W_r$ are cofinal.

For other need $E = \varinjlim E_\alpha$ $\ni$ $h^*(E_\alpha)$ proj over $h^*(pt)$

$BO$, $BU$, $BSp$, $MU$, $MSp$, $S$.

Can work Atiyah trick + S-duality and so given $X$ can construct

$$\begin{tikzcd} X = X_0 \arrow[dr] & \cdot \arrow[l] \\ & U_0 \arrow[u] \end{tikzcd}$$

where $h_*(U_r)$ projective over $h_*(pt)$. cofinal

Thus get spectral sequence with correct $E^2$.

Convergence? If $\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \\ & W \arrow[ur] & \end{tikzcd}$ $h(W) = 0$ then $f$ is of infinite filtration. $Y \leftarrow W \leftarrow W \leftarrow \cdots$

So can't detect. [for K theory $\exists$ spaces with $K(W) = 0$ W a finite cx.)

For $MU$ convergence clear because get 2-connected each $1/n$ since $\pi_0 MU = \mathbb{Z}$ $\pi_1 MU = 0$.

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# ADAMS 6

$BU(1) = CP^\infty$ $\qquad H^*(BU(1), \mathbb{Z})$ $\quad 1, x, x^2, \cdots$

$\qquad\qquad\qquad\qquad H_*(BU(1), \mathbb{Z})$ $\quad b_0, b_1, b_2 \cdots$

$BU(1) \longrightarrow BU.$ $\qquad BU(n) \times BU(m) \longrightarrow BU(n+m)$

$H_*(BU)$ $\quad \mathbb{Z}$-base $\quad b_1^{\nu_1} \cdots$ $\qquad \nu = (\nu_1, \cdots)$

$H^*(BU)$ has dual base $\quad c_\nu \in H^{2|\nu|}(BU)$

where $\quad |\nu| = \nu_1 + 2\nu_2 + \cdots$

If $\nu = (i, 0, \cdots)$ then $c_i = i$th Chern class.

Thom isom. $\quad \varphi : H_*(BU) \xrightarrow{\cong} H_*(MU)$ ← poly ring on $b_i'$.

$\qquad b_i \longrightarrow b_i'$

$$H_{2i+2}(MU(1)) \ni b_i' \qquad\qquad MU(1) \sim BU(1)$$
$$\uparrow \qquad\qquad\qquad \text{recall}$$
$$H_{2i}(BU(1)) \ni b_i \qquad\qquad b_i' \sim b_{i+1}$$

---

Conner Floyd Chern classes

Let $\xi$ be a $U(n)$ bundle over $X$ to define classes in $\Omega_u^*(X)$.

<u>Theorem</u>: (C-F) To each such $\xi$ $\exists$

$cf_\alpha(\xi) \in \Omega_u^{2|\alpha|}(X)$

(i) $cf_0(\xi) = 1$

(ii) Natural

(iii) $cf_\alpha(\xi \oplus \eta) = \sum_{\beta+\gamma=\alpha} cf_\beta(\xi) \cdot cf_\gamma(\eta)$

(iv) $\xi$ a $U(1)$ bundle classified by $X \xrightarrow{f} BU(1) \sim MU(1)$ representing $\omega$ in $\Omega^2_u(X)$.

$$cf_\alpha(\xi) = \sum_{i \geq 0} \underbrace{(c_\alpha, b_i)}\, \omega^{i}$$

this is usually zero and non-zero iff $\alpha = (0, \ldots, \underset{i}{1}, \ldots)$

Sketch of proof: $\varprojlim_p{}^1 \; \Omega^*_u(BU(n)^p) = 0.$

so enough to handle fin. dim. cxs where one uses Groth. const. This defines $cf_i$. Then write $c_\alpha = P_\alpha(c_1, \ldots, \ldots)$ in $H^*(BU)$ and define $cf_\alpha = P_\alpha(cf_1, \ldots\ldots)$.

---

Now use analogy

Steenrod squares —— Stieffel whitney classes

Novikov operations —— cf classes

<u>Theorem (Novikov)</u>: For each $\alpha$ $\exists$

$$S_\alpha : \Omega^q_u(X, Y) \longrightarrow \Omega^{q+2|\alpha|}_u(X, Y)$$
$$(S_\alpha \in \Omega^{2|\alpha|}_u(MU)).$$

(i) $S_0 = 1$

(ii) Natural

(iii) stable

(iv) additive

(v) $S_\alpha(xy) = \sum_{\beta+\gamma=\alpha} S_\beta(x) \cdot S_\gamma(y)$

(vi) Suppose $\omega \in \Omega^2(X)$ are represented by $X \xrightarrow{f} MU(1)$ (not all $\omega$ are) $S_\alpha(\omega) = \sum_{i \geq 0} (c_\alpha, b_i)\, \omega^{i+1}$

(vii) If $\xi$ $U(n)$ bundle

$$\begin{array}{ccc} \Omega_U^{2n}(E, E_0) & \xrightarrow{s_\alpha} & \Omega_U^{2(n+|\alpha|)}(E, E_0) \\ \uparrow \scriptstyle{SS} & & \uparrow \scriptstyle{SS} \\ \Omega_U^0(X) & & \Omega_U^{2|\alpha|}(X) \\ \mathbin{\rotatebox[origin=c]{90}{$\in$}} & & \\ 1 & \longmapsto & cf_\alpha(\xi) \end{array}$$

Define: $\quad s_\alpha = \varphi(cf_\alpha)$, where $\varphi : \Omega_U^{2|\alpha|}(BU) \xrightarrow{\cong} \Omega_U^{2|\alpha|}(MU)$.

use $\varprojlim^1 = 0$. Need some properties of Thom isom.

---

Technical digression:

$$s_\alpha : MU \to S^{2|\alpha|} MU$$

$$(s_\alpha)_* : H_q(MU) \to H_{q-2|\alpha|}(MU) \quad ?$$

Thm: (i) $x, y \in H_*(MU)$

$$s_\alpha(xy) = \sum_{\beta+\gamma=\alpha} s_\beta(x)\, s_\gamma(y)$$

(ii) set $b' = \sum_{i=0}^{\infty} b_i'$

$$s_\alpha(b') = \sum_{i \geq 0} (c_\alpha, b_i)(b')^{i+1}$$

These follow from (v) & (vi) of preceding thm.

Cor: $\quad s_\alpha : H^0(MU) \longrightarrow H^{2|\alpha|}(MU)$

$$s_\alpha \varphi(1) = \varphi c_\alpha$$

Given $c \in \Omega_u^{-d}(pt)$, let $t: \Omega_u^p(X) \to \Omega_u^{p-d}(X)$

$x \longmapsto c \cdot x$

Fix $d$, For each $\alpha$ choose $c_\alpha \in \Omega^{d-2|\alpha|}(pt)$ giving $t_\alpha$

$\sum_\alpha t_\alpha s_\alpha$ operation of dim $d$.

<u>Theorem (Novikov)</u>: This sum converges to ~~an~~ unique element of $\Omega_u^d(MU)$. Every element of $\Omega_u^d(MU)$ can be written uniquely in this form.

<u>Proof</u>: $H^*(MU, \Omega_u^*(pt)) \Longrightarrow \Omega_u^*(MU)$.

Corollary says that $s_\alpha$ form an $\Omega_u^*$ base for $E^2$ terms so spec. seq. degenerates (same argument works for symplectic cob.).

---

The operations $\sum t_\alpha s_\alpha$ can be distinguished by values on $P = CP^n \times \cdots \times CP^n$ $m$ factors for all $m, n$. Let $\omega_1, \ldots, \omega_m$ be cob. gen. for factors

$\Omega_u^*(P)$ $\quad \omega_1^{i_1} \cdots \omega_m^{i_m}$ independent over $\Omega_u^*(pt.)$

$$s_\alpha(\omega_1 \cdots \omega_m) = \sum_{i_1, \ldots, i_m} (c_\alpha, b_{i_1} \cdots b_{i_m})\, \omega_1^{i_1+1} \cdots \omega_m^{i_m+1}$$

---

$$\left(\sum t_\alpha s_\alpha\right)\left(\sum t'_\beta s_\beta\right):$$

(i) $s_\alpha t'_\beta$ must know $s_\alpha$ on $\Omega_u^*(pt)$

(ii) $t_\alpha t'_\beta$ just multiply in $\Omega^*(pt)$.

(iii) $s_\alpha s_\beta$

---

$$\begin{tikzcd} \pi_*(MU) \arrow[r, hook, "\text{Milnor}"] \arrow[d, "s_\alpha"'] & H_*(MU) \arrow[d, "(s_\alpha)_*"] \\ \pi_*(MU) \arrow[r, hook] & H_*(MU) \end{tikzcd}$$

so in principal if we know char no. of an almost

---

<u>Theorem:</u> The set $S$ of $\mathbb{Z}$ linear combinations of $s_\alpha$ is closed under composition. The ring $S$ is a Hopf algebra over $\mathbb{Z}$ whose dual $S_*$ is a polynomial alg. generators $b''_i$ $(s_\alpha, b''_i) = (c_\alpha, b_i)$.

$$\boxed{\underline{\text{Proof:}}\quad \begin{aligned} s_\beta \cdot s_\alpha(\omega_1 \cdots \omega_m) &= \sum (\text{integers})\, s_\beta(\omega_1^{i_1+1} \cdots \omega_m^{i_m+1}) \\ &= \sum (\text{integers})\, \omega_1^{j_1} \cdots \omega_m^{j_m} \end{aligned}}$$

Set $b'' = \sum b''_i$, $b''_0 = 1$ and

$$\Delta b'' = \sum_{i \geq 0} (b'')^{i+1} \otimes b''_i$$

---

Remaining Question is nature of $\Omega_u^*(pt)$. $(\otimes \mathbb{Q} \simeq \mathbb{Q}[CP^1, CP^2, \cdots])$.

<u>Theorem</u> $\underbrace{\phantom{Theorem}}_{}$ (Novikov – Adams): $s_\alpha[CP^n] = (c_\alpha, b^{n-1})[CP^{n-k}]$ $\qquad b = \sum b_i$

<u>Proof:</u> use Hurewicz above.

Suppose $M^n$ almost-cx, normal bundle $\nu$.

$$cf_\alpha(\nu) \in \Omega^{2|\alpha|}(M^m)$$

$c: M^n \to pt.$ induces $c_!: \Omega^{2|\alpha|}(M^m) \to \Omega^{2|\alpha|-m}(pt)$

$$\boxed{c_! \, cf_\alpha(\nu) = s_\alpha[M^m]}$$

follows easily from reln of $cf_\alpha$ and $s_\alpha$.

# Bordism theory

1. <u>How to define bordism groups without using manifolds with boundary.</u> What follows is the result of piecing together various clues I have had concerning Grothendieck's theory of "motives."

We begin with the simplest example. Let $\mathcal{M}$ be the category of smooth compact manifolds and smooth maps. Let $H_*: \mathcal{M} \to (\mathbb{Z}/2 \text{ modules})$ be the homology functor with $\mathbb{Z}/2\mathbb{Z}$ coefficients. If $f: X \to Y$ is a map in $\mathcal{M}$, then there is an induced map

$$f_* : H(X) \longrightarrow H(Y)$$

as well as a Gysin homomorphism

$$f^* : H(Y) \longrightarrow H(X)$$

defined in terms of $f_*$ by Poincare duality e.g. $f^*$ is the composition

$$\begin{tikzcd} H_k(Y) \arrow[r, dashed, "f^*"] & H_{k+m-n}(Y) \\ H^{n-k}(Y) \arrow[u, "s"] \arrow[r] & H^{n-k}(X) \arrow[u, "s"] \end{tikzcd}$$

where $m = \dim X$, $n = \dim Y$ and the vertical maps are the Poincare duality isomorphisms

The rule $X \mapsto H(X)$, $f \mapsto f_*, f^*$ has the following properties:

(1) $(fg)_* = f_* g_*$ , $(id)_* = id$

(2) $(gf)^* = f^* g^*$ , $(id)^* = id$

(3) If $f$ is homotopic to $g$, then $f_* = g_*$

(4) If

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

is transversal-cartesian, that is, the maps $f$ and $g$ are transversal (equivalently ~~[illegible]~~ $f \times g : X \times Y' \longrightarrow Y \times Y$ is transversal to $\Delta_Y$) ~~[illegible]~~ and $X' \xrightarrow{\sim} Y' \times_Y X$ in $\mathcal{M}$, then

$$g^* f_* = f'_* (g')^* .$$

~~[illegible]~~

We now consider the ~~following~~ universal ~~[illegible]~~ gadget with these properties. Thus we ~~[illegible]~~ want a category $\mathcal{B}$ together with ~~[illegible]~~ maps

$$H : \operatorname{Ob} \mathcal{M} \longrightarrow \operatorname{Ob} \mathcal{B}$$

$$\operatorname{Hom}_{\mathcal{M}}(X,Y) \longrightarrow \operatorname{Hom}_{\mathcal{B}}(HX, HY)$$
$$f \longmapsto f_*$$

$$\operatorname{Hom}_{\mathcal{M}}(X,Y) \longrightarrow \operatorname{Hom}_{\mathcal{B}}(HY, HX)$$
$$f \longmapsto f^*$$

for all $X, Y \in \operatorname{Ob} \mathcal{M}$, so that (1)–(4) above hold. Moreover

$(\mathcal{B}, H, (?)_*, (?)^*)$ should have the universal property that given another $(\mathcal{B}', H', (?)_{*'}, (?)^{*'})$ there is a unique functor $Q: \mathcal{B} \to \mathcal{B}'$ such that $H' = Q \circ H$, $(f)_{*'} = Q(f_*)$, $(f)^{*'} = Q(f^*)$. We shall now construct this category $\mathcal{B}$.

~~First we note that $H: Ob\,\mathcal{M} \to Ob\,\mathcal{B}$ is an isomorphism~~

First assume $\mathcal{B}$ exists. Then it is easily seen that $H: Ob\,\mathcal{M} \to Ob\,\mathcal{B}$ is an isomorphism. Thus we may assume that $\mathcal{B}$ has the same objects as $\mathcal{M}$. Next we note that any map in $\mathcal{B}$ from $X$ to $Y$ may be expressed in the form $g_* f^*$ where $f, g$ are maps in $\mathcal{M}$

(5)
$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$

To see this we need only show that the maps in this form are closed under composition or equivalently by (1) and (2) that if $u: X \to Z$ and $v: Y \to Z$, then $v^* u_* = (u')_* (v')^*$ for suitable $u', v'$. By Thom's transversality theorem we may homotop $u$ (without changing $u_*$ by (3)) until $u$ and $v$ are transversal and then obtain a transversal-cartesian square

(6)
$$\begin{tikzcd} X' \arrow[r, "v'"] \arrow[d, "u'"] & X \arrow[d, "v"] \\ Y' \arrow[r, "u"] & Y \end{tikzcd}$$

whence we are done by (4).

Note that in homotoping $u$ in different ways leads to different manifolds $X'$ and therefore different representations of a map in $\mathcal{B}$ in the form $g_*f^*$. However Thom's transversality theorem allows us prove that the bordism class $(v', u'): X' \to X \times Y'$ of $X \times Y'$ independent of the choice of homotopy of $u$. Conversely.

<u>Lemma:</u> If $(f_0, g_0): Z_0 \longrightarrow X \times Y$ and $(f_1, g_1): Z_1 \longrightarrow X \times Y$ represent the same bordism class of $X \times Y$, then for any quadruple $\mathcal{B}, H, (\ )_*, (\ )^*$ satisfying (1)-(4) we have

$$(g_0)_* f_0^* = (g_1)_*(f_1)^*$$

---

<u>Proof:</u> By assumption there is a manifold with boundary $W$ and a map $h: W \longrightarrow X \times Y$ such that $\partial W = Z_0 \cup Z_1$ and $h/Z_0 = (f_0, g_0)$ $h/Z_1 = (f_1, g_1)$. Let $\xi: W \longrightarrow [0,1] = I$ be a smooth function transversal to $\{0, 1\}$ with $\xi^{-1} 0 = Z_0$, $\xi^{-1} 1 = Z_1$. By doubling $W$ and $I$ to obtain closed manifold $\tilde{W}$ and $S^1$ and using the first component $pr_1 h$ we obtain (after a slight smoothing) a map $\varphi: \tilde{W} \longrightarrow X \times S^1$ such that $\varphi$ is transversal to $X \times 0$ and $X \times 1$. Thus

$$\begin{array}{ccc} Z_0 & \longrightarrow & \tilde{W} \\ \downarrow & & \downarrow \\ X & \xrightarrow{i_0} & X \times S^1 \end{array} \qquad \begin{array}{ccc} Z_1 & \longrightarrow & \tilde{W} \\ \downarrow & & \downarrow \\ X & \xrightarrow{i_1} & X \times S^1 \end{array}$$

*Proof*: By assumption there is a manifold with boundary $W$ and ~~...~~ $(h,k): W \to X \times Y$ such that $\partial W \cong Z_0 \cup Z_1$ in such a way that $(h,k)/Z_i = (f_i, g_i)$ $i = 0, 1$. Let $\xi: W \to I = [0,1]$ be a smooth function transversal to $\{0,1\}$ with $\xi^{-1}0 = Z_0$ and $\xi^{-1} Z_1$. Let $\tilde{W}$ be the double of $W$ and extend $(h,k)$ and $\xi$ to $(\tilde{h},\tilde{k}): \tilde{W} \to X \times Y$, $\tilde{\xi}: \tilde{W} \to S^1$ ~~smoothing~~ by doubling and smoothing them out but leaving them fixed on $Z_0 \cup Z_1$. Then we have diagrams

$$\begin{array}{ccccc} Z_0 & \xrightarrow{j_0} & \tilde{W} & \xrightarrow{\tilde{k}} & Y \\ \downarrow f_0 & & \downarrow (\tilde{h},\tilde{\xi}) & & \\ X & \xrightarrow{i_0} & X \times S^1 & & \end{array} \qquad \begin{array}{ccccc} Z_1 & \xrightarrow{j_1} & \tilde{W} & \xrightarrow{\tilde{k}} & Y \\ \downarrow f_1 & & \downarrow (\tilde{h},\tilde{\xi}) & & \\ X & \xrightarrow{i_1} & X \times S^1 & & \end{array}$$

where the squares are cartesian and $i_0$ is homotopic to $i_1$. Thus from (1)-(4) we have

$$(g_0)_* f_0^* = \tilde{k}_* (j_0)_* f_0^* = \tilde{k}_* (\tilde{h},\tilde{\xi})^* (i_0)_* = (\tilde{k})_* (\tilde{h},\tilde{\xi})^* (i_1)_* = (g_1)_* f_1^* .$$

QED.

---

Using this lemma we find that we have a well-defined map

$$(7) \qquad \begin{array}{ccc} \mathcal{N}(X \times Y) & \longrightarrow & B(X, Y) \\ \text{class}\,[(f,g): Z \to X \times Y] & \longmapsto & g_* f^* \end{array}$$

Moreover it is clear that given two bordism classes $(f,g): Z \to X \times Y$ and $(f',g'): Z' \to Y \times Z$ we may define their composition to be the bordism $(fp, g'q): W \to X \times Z$ where $(W, p, q)$ is obtained by taking the fiber product of $f'$ and a suitable map homotopic to $g$

$$\begin{tikzcd}[column sep=small] & & & W \arrow[dl, "p"'] \arrow[dr, "q"] & & \\ & & U \arrow[dll, "f"'] \arrow[dr, "g"] & & U' \arrow[dl, "f'"'] \arrow[dr, "g'"] & \\ X & & & Y & & Z \end{tikzcd}$$

It is easily seen that the bordism class $W \to X\times Z$ depends only on that of $U \to X\times Y$ and $U' \to Y\times Z$, in fact the operation just defined is the composition

$$(8) \qquad \mathcal{N}(X\times Y) \times \mathcal{N}(Y\times Z) \xrightarrow{\text{product}} \mathcal{N}(X\times Y\times Y\times Z) \xrightarrow{(1\times\Delta_Y\times 1)^*} \mathcal{N}(X\times Y\times Z) \xrightarrow{(pr_{13})_*} \mathcal{N}(X\times Z)$$

in bordism. Thus we may define a category $\mathcal{C}$ with same objects as $\mathcal{M}$ with $\mathcal{N}(X\times Y)$ as maps from $X$ to $Y$ and with composition defined by (8). Clearly (7) is a functor. Moreover if $f: X \to Y$ is a map in $\mathcal{M}$ then we may define

(9) $f_*$ in $\mathcal{N}(X\times Y)$ to be the class of the bordism $(id_X, f): X \to X\times Y$ and $f^*$ in $\mathcal{N}(Y\times X)$ to be class of the bordism $(f, id_X): Y \to X\times Y$.

The properties (1)–(4) for the quadruple $(\mathcal{C}, id_{Ob\,\mathcal{M}}, (?)_*, (?)^*)$ are easily verified and as $\mathcal{C}$ maps to any $\mathcal{B}$ it must be the desired universal category. Thus we have proved

<u>Theorem:</u> The universal quadruple $(\mathcal{B}, H, (?)_*, (?)^*)$ is given by

$$Ob\,\mathcal{B} = Ob\,\mathcal{M}$$

$$\mathcal{B}(X, Y) = \mathcal{N}(X\times Y) \quad \text{with composition given by (8).}$$

If $f: X \to Y$ then $f_*$ and $f^*$ are given by (9).

<u>Definition:</u> We shall call $\mathcal{B}$ the unoriented bordism category of closed manifolds.

Remarks. 1. $\mathcal{B}$ is a graded additive category, the degree of a map $(f,g): Z \to X \times Y$ being $\dim Z - \dim X$.

2. The bordism ring is $\mathfrak{N}(pt) = \mathcal{B}(pt, pt)$ with ring structure given by composition. Thus we have defined the bordism groups without using manifolds with boundary.

3. If $f: X \to Y$ is a map in $\mathcal{M}$ which is a homotopy equivalence, then both $f_*$ and $f^*$ are isomorphisms, however it is not immediately clear that $f_*$ and $f^*$ are inverses of each other. In fact when we come to oriented bordism we will see this needn't be the case. Here, however, we can use the theorems of Thom to prove this. It suffices to show that $f_* f^* = 1$ or that the bordism $(f,f): X \to Y \times Y$ is bordant to $\Delta_Y: Y \to Y \times Y$. But a bordism $f: U \to X$ is determined by its "Stiefel-Whitney numbers" defined to be homomorphism

$$H^*(BO) \xrightarrow{\nu^*} H^*(U) \underset{\text{Poincare}}{\xrightarrow{\cong}} H_*(U) \xrightarrow{f_*} H_*(X)$$

where $\nu: U \to BO$ is the classifying map for the stable normal bundle of $U$. The Wu formulas show that $\nu^*$ can be calculated from the Poincare algebra $H^*(U)$ with its structure as a $\mathcal{A}$-module, $\mathcal{A}$ = Steenrod algebra. Consequently if $f: U \to X$ and $g: V \to X$ are two bordisms such that $\exists$ a map $\varphi: U \to V$ with $\varphi g \sim f$ and $\varphi^*: H^*(V) \xrightarrow{\sim} H^*(U)$ then the bordism classes of $f$ and $g$ are equal. Thus (using $\mathfrak{N}_*(X) = \pi_*(X \wedge MO) = H_*(X, \mathbb{Z}/2) \otimes \pi_*(MO)$) we have proved

Prop: If $f: X \to Y$ induces an isomorphism on $\mathbb{Z}/2$ homology, then $f_*$ and $f^*$ are isomorphisms and $f_* = (f^*)^{-1}$.

2. Generalizations of the bordism category construction to handle various orientations and non-compact manifolds.

In defining a Gysin homomorphism for a map $f: X \to Y$ on integral homology $f^*: H_*(Y, \mathbb{Z}) \to H_*(X, \mathbb{Z})$ what one needs is an orientation of the stable normal bundle $\nu_f$ of $f$, as well as the assumption that $f$ is proper in case the manifolds are not compact. The oriented proper maps can be composed in an obvious way and so define a different category structure on the category $\mathcal{M}$ of all smooth (not necessarily compact) manifolds.

Thus the situation we are led to consider is that of a two categories $\mathcal{M}_p$ and $\mathcal{M}$ having the same objects. ~~We shall often call maps in~~ ~~$\mathcal{M}_p$ the "proper" maps in $\mathcal{M}$.~~ We suppose given a class of "special" squares of the form

$$
(1) \qquad
\begin{tikzcd}
X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\
Y' \arrow[r, "g"] & Y
\end{tikzcd}
\qquad
\left\{
\begin{array}{l}
f', f \in \mathcal{M}_p \\
g', g \in \mathcal{M}
\end{array}
\right.
$$

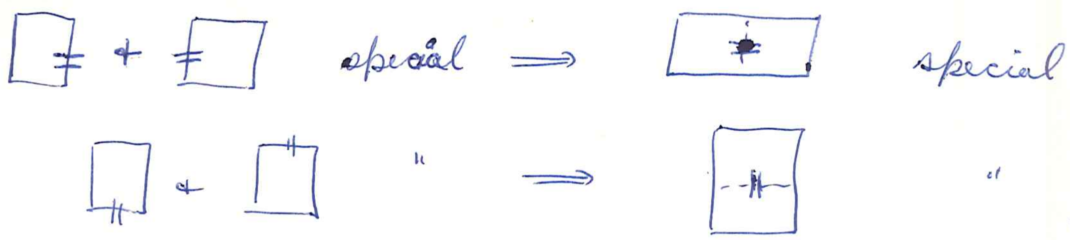
(no commutativity assumed since we can't compose maps of $\mathcal{M}$ and $\mathcal{M}'$)

satisfying the following conditions (abstracted from the preceding section – the example to keep in mind is $\mathcal{M}_p(X,Y)$ proper oriented maps, $\mathcal{M}(X,Y) =$ homotopy classes of maps from $X$ to $Y$.)

A. Given the solid arrows we can complete it to a special square

$$
\begin{tikzcd}
X' \arrow[r, dashed, "g'"] \arrow[d, dashed, "f'"'] & X \arrow[d, "f"] \\
Y' \arrow[r, "g"] & Y
\end{tikzcd}
$$

B. Juxtaposition of special squares is a special square i.e.




We may then consider the universal ~~....~~ problem of finding a quadruple $(B, H, {}_*, {}^*)$ where $H: Ob\,M \to Ob\,B$ ~~....~~ $X \mapsto H(X)$ $f \mapsto f_*$ is a functor $M \to B$ and $X \mapsto H(X)$ $f \mapsto f^*$ is a functor from $M_p^o \to B$ such that ~~....~~ $f^* g_* = (g')_* (f')^*$ whenever ~~..~~ there ~~...~~ is a special square (1). It is easily seen that the universal quadruple has $Ob\, M_p = Ob\, M = Ob\, B$, that any map may be represented in the form $g_* f^*$ where $f: U \to X$ is in $M_p$ and $g: U \to Y$ is in $M$. However it does not seem to be possible to ~~....~~ describe the equivalence class of such a representation of a map in $B$ in this axiomatic situation, so we shall return to the geometric case.

We now consider the case where $M$ = smooth manifolds and homotopy classes of smooth maps and where the $M_p$ = smooth manifolds and proper oriented maps, where ~~oriented~~ an orientation of $f: X \to Y$ is a reduction of the structural group of $\nu_f$ ~~....~~ to a group $G$ mapping to $O$ such as $G$ = Spin, $U$, $Sp$, $1$, etc. What is important is that if $X \xrightarrow{f} Y \xrightarrow{g} Z$ so that $\nu_{gf}$ ~~..~~ is canonically isomorphic to $\nu_f \oplus f^* \nu_g$, then the orientations of $g + f$ should define one of $\nu_{gf}$, and so $M_p$ becomes a category.

By a special square (1) we mean one where $g$ and $g'$ are the homotopy classes of map $\tilde{g}$ and $\tilde{g}'$ such that

$$\begin{array}{ccc} X' & \xrightarrow{\;\;\;} & X \\ \downarrow f' & g' & \downarrow f \\ Y' & \xrightarrow[g]{\;\;\;} & Y \end{array}$$

is transversal cartesian and moreover ~~the~~ $f'$ is oriented by the canonical ~~....~~ isomorphism $(g')^* \nu_f \cong \nu_{f'}$ and the orientation of $f$. Axiom A follows by the Thom transversality thm. and B is clear. By using ~~results of Thom~~ standard arguments of Thom and the same argument as the preceding section we obtain the following description of the ~~co~~bordism category $\mathcal{B}$.

<u>Theorem:</u> Let $\Omega^G_q(X,Y)$ be the bordism classes of pairs

$$X^n \xleftarrow{f} U^{n+q} \xrightarrow{g} Y$$

where $f$ is proper-oriented and $g$ is arbitrary and let $\Omega_* = \bigcup \Omega_q$.
Then

(2)
$$\Omega^G_*(X,Y) \xrightarrow{\simeq} \mathcal{B}_*(X,Y)$$
$$(X \xleftarrow{f} U \xrightarrow{g} Y) \longmapsto g_* f^*$$

moreover

(3)
$$\Omega^G_*(X,Y) \xrightarrow{\simeq} \lim_{\substack{\longrightarrow \\ N\to\infty}} [S^N \wedge X, ~~\ldots~~ MG(N+*) \wedge Y]$$

Remarks: 1. (3) is just the theorem of Thom. Given $u: S^N \wedge X \longrightarrow MG(N+q) \wedge Y$ homotop $u$ to a map which is smooth on the complement of $u^{-1}\{*\}$ and which is transversal to $BG(N+q) \times Y$. Taking the inverse image of this submanifold we obtain a submanifold $Z \xhookrightarrow{(i,f)} R^N \times X^n$ proper over $X$ (of dim $n+q$) with normal bundle reduced to $G(N+q)$, together with a map $g: Z \to Y$. Then $X \xleftarrow{f} Z \xrightarrow{g} Y$ represents the element of $\Omega_q^G(X,Y)$ given by (3).

2. Example:

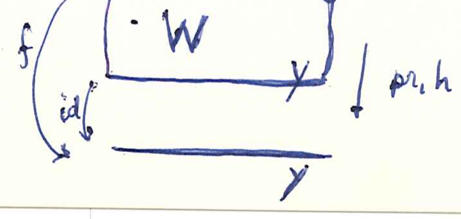
$$B_*(pt, pt) = \Omega_*^G(pt, pt) = \pi_*(MG)$$

$$B_*(X, pt.) = \pi_*(S \wedge X, MG) = \text{cohomology of } X \text{ for the spectrum } MG$$

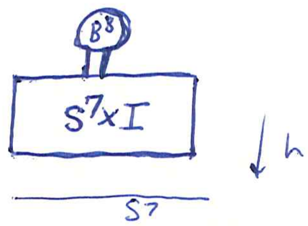
$$B_*(pt, X) = \pi_*(MG \wedge X) = \text{homology of } X \text{ for the spectrum } MG.$$

---

Let us now consider the problem of isomorphism. Suppose $f: X \to Y$ is a proper G-oriented map. Then $f_* f^* = 1$ is equivalent to the G-bordism $f: X \to Y$ being in the same class as $id: Y \to Y$, or equivalently there existing a G-oriented proper map of manifolds with boundary $h: W \longrightarrow Y \times I$ such that $h_0 = f$ and $h_1 = id_Y$ as oriented maps. Picture:



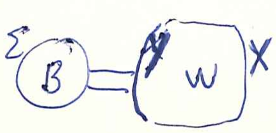
The following is the ultimate counter-example to concluding that $f$ is homotopic to a diffeomorphism. Let $\Sigma^7$ be the Milnor sphere. Then $\Sigma^7 = \partial B^8$ where $B^8$ is stably-parallelizable. ~~Then let $f: \Sigma^7 \to S^7$ be a smooth map of degree 1 and~~ Let $W^8$ be the manifold with boundary




so $\partial W^8 = S^7 \cup \Sigma^7$, let $h: W^8 \to S^7$ be the obvious retraction, and let $f = h|\Sigma^7$. Then $h$ is a framed cobordism from $f: \Sigma^7 \to S^7$ to $id: S^7 \to S^7$ and $f$ is a homotopy equivalence. Thus $f_* f^* = 1$ in any $G$-oriented bordism theory and yet $f$ is not homotopic to a diffeomorphism. This example is typical.

Theorem: (Novikov) If $f: X \to Y$ is a framed homotopy equivalence of smooth closed manifolds (1-conn, dim $\geq 5$) such that $f_* f^* = 1$ in the framed bordism theory, then $X$ is diffeomorphic to $Y \# \Sigma$ where $\Sigma \in \Theta(\partial\pi)$ and $f$ is homotopic to the canonical PL-homeomorphism $Y \# \Sigma \to Y$.

Remark: The framing of $f$ need not be the obvious one. Thus by assumption there is a framed cobordism $h: W \to Y$ with $\partial W = X \cup Y$ $h|X = f$ $h|Y = id$. Do surgery on $W \to Y \times I$ until the middle dimension is reached, then attach suitable Milnor or Kervaire manifold $B$ to $W$

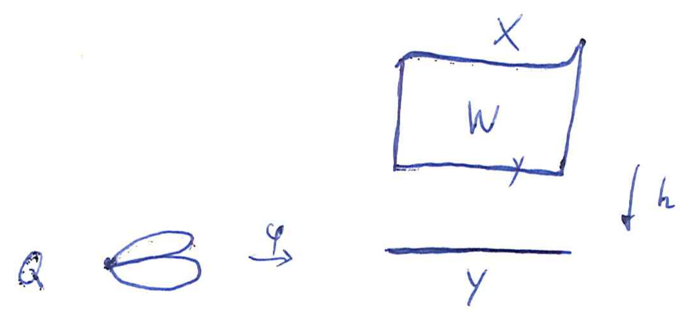



to kill obstruction. By h-cobordism

$X \cong Y \# \Sigma$.

<u>Question:</u> Let $f: X \to Y$ be an oriented homotopy equivalence of two smooth closed manifolds and assume that $f_* f^* = 1$ in the oriented bordism category. Is $f$ homotopic to a PL-homeomorphism and/or is $\nu_f \simeq 0$?

<u>Discussion:</u> If $Y$ is 1-connected and dimension $\geq 5$, then (according to Sullivan) $f$ defines an element of $[Y, F/PL]$ whose vanishing is n.a.s. that $f$ be homotopic to a PL-homeomorphism. Also Sullivan shows that a n.a.s. condition is that for any $\mathbb{Z}/n$-manifold $Q$ and map $\varphi: Q \to Y$ the index of $f^{-1}Q$ should be the same as that of $Q$. However we are given




First we adjust $\varphi$ so that $h$ is transversal to $\varphi|\partial Q$ and then keeping $\varphi|\partial Q$ fixed adjust $\varphi$ so that $h$ is transversal to $\varphi$ on the interior of $Q$. Then $h^{-1}Q$ is a $\mathbb{Z}/n\mathbb{Z}$-manifold boundary joining $\varphi$. But $h^{-1}Q\,|\,X = f^{-1}Q$ so the indices are equal. Thus $f$ is homotopic to a PL-homeomorphism.

<u>Conclusion:</u> In the oriented PL-category if $f$ is a homotopy equivalence such that $f_* f^* = 1$, then $f$ is homotopic to a PL homeomorphism, proved simply-connected dim $\geq 5$.

Summary of work on cobordism theory, July 1968 at Battelle

1. Proposition: Let $W$ be a $G$-oriented manifold with boundary components $X, Y$ and assume that ~~[illegible]~~ there exist $G$-framed retractions $r: W \to X$ and $s: W \to Y$. Let $f =$ composite $X \to W \xrightarrow{s} Y$. Then $f_* = (f^*)^{-1}$ when $f$ is $G$-framed in the canonical way.

Proof: $f_* f^* = 1$ similarly $g_* g^* = 1$ with $g =$ composite $Y \to W \to X$. Now use noetherianness of $\Omega^G_0(T, X)$ and $\Omega^G_0(T, Y)$ to conclude $f^* + g^*$ are isomorphisms.

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2. Difficulty defining cobordism category in the case of algebraic geometry because of the transversality problem, ~~[illegible]~~ e.g. self-intersection of an ~~[illegible]~~ exceptional curves and general positivity considerations that don't happen in the usual cases.

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3. Tried to define the Whitehead torsion of $f$ as determinant of $f_* f^*$ but I failed for manifolds because $f_* f^* \neq 1$ on the chain level.

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4. Fiber cobordism: Ran into troubles with transversality. Only concrete ideas are that the correct answer should be so that if $X, Y$ are smooth over $B$, then $\Omega^{fw}_{/B}(X, Y)_{\mathbb{Q}} = \mathrm{Hom}_{D(B)}(Rf_*(X), Rg_*(Y))$

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5. Bundles on $X$ should be thought of as a category whose objects are bundles and whose morphisms are homotopy classes of isos. Thus we get a groupoid $VB(X)$ together with a $\oplus$ functor which is associative + commutative. The stable ~~[illegible]~~ $VB(X)$ ~~[illegible]~~ has the additional property that its $\pi_0$ ~~[illegible]~~ is a group. (groupoid with $\oplus \ni \pi_0$ is a group) Another example is Pic.

6. A framed surgery problem $f: X \to Y$ is equivalent to an element of $[Y, F]$. At one time hoped that the Wall obstructions

$$[Y^n, F] \longrightarrow L_n(\pi_1(Y))$$

would yield good information, e.g. if $Y^n$ = $n$-torus. However one would have to know how to classify homotopy tori.

7. Calculations of bordism categories. If $C$ closed under fibre products, then $\Omega(X,Y)$ = Iso classes of $C/X \times Y$.

For homotopy theory we invert t.f. $g$ i.e. we want $g^*$ if $g$ t.f. and $f_*$ in general. so that

① $g^* f_* = f'_* g'^*$ in cart. situation

② $g_* g^* = g^* g_* = 1$

however ② $\Rightarrow$ ① so the universal category is just the localization wrt all t.f. and so for spaces + simplicial groups you get the localization wrt all [illegible]'s.

8. Studied some geometry: alg. K theory and role of $\pi_1$

added boundary to open manifold { Browder, Livesay, Levine (1-conn. case)
Siebenmann (in general

fibring over $S^1$ { Browder + Levine (1-conn. case)
Farrell (in general)

All of these are surgery in codimension 1 problems.

Siebenman–Novikov splitting thm.

Mazur's half open h-cobordism theorem.

9. Grothendieck ring of manifolds: Any other $\chi$: Man $\to A$ $\ni$ $\chi(E) = \chi(F)\cdot\chi(B)$ for all $F \to E \to B$ besides usual Euler char.

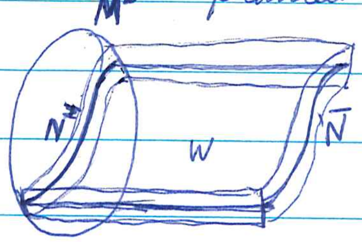
## Conversation with Browder, July 25, 1968

Theorem: Let $f: M \longrightarrow M'$ be a ~~homotopy equivalence~~ (framed, degree 1 map) of $m$-manifolds, $m \geq 6$ (?). Let $N'$ be a submanifold of $M'$. ~~with normal bundle~~ ~~...~~ Assume that $\pi_1(M'-N') = 0$ ~~and that~~. Then $f$ ~~may be~~ is framed cobordant to a map which is ~~deformed so that it is~~ transversal ~~...~~ to $N'$ and so that it induces homotopy equivalences $(M, N, M-N) \longrightarrow (M', N', M'-N')$ (where $N = f^{-1}N'$) if and only if

(i) $f$ is framed cobordant to a heq

(ii) $f|f^{-1}N' \longrightarrow N'$ is framed cobordant to a heq.

Proof: $M' = A' \cup B'$ where $B' =$ ~~...~~ normal tube around $N'$ and $A' =$ complement of the interior of $B'$ in $M'$. By (i) there is a $M^*$ framed cobordism ~~... between M and M̄~~




$h: W \longrightarrow N'$ ~~between~~ $f|f^{-1}N' \to N'$ and $\bar{f}: \bar{N} \to N'$ where ~~...~~ $\bar{f}|\bar{N}$ is a heq. Because of the framing we can thicken out $W$, thus getting a framed cobordism of $f$ to $\bar{f}: \bar{M} \to M'$. Next want to surger $f|\bar{A} \to A'$ modulo boundary to a heq. But the obstruction for this is additive for the sum $\bar{M} = \bar{A} \cup \bar{B}$ ~~and~~ (here use 1-conn. of $A$) and by (i) the total obstruction vanishes + clearly vanishes for $\bar{B}$. QED.

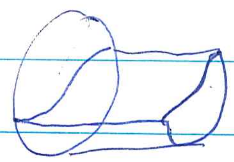
Modifications are necessary in the case with boundary.

Corollaries of the above thm. include:

(1.) Given $f: M \to M'$ $\underset{N'}{\cup}$ codim 1, $f$ a heq. $\pi_1(M) = \pi_1(M') = 0$

then $f$ is homotopic to $\tilde{f}$ transversal to $N'$ $\ni$ $N = \tilde{f}^{-1}N' \to N'$ is a heq.

**Proof:** $M' - N'$ has two components; take the $N'$ of the preceding thm. to be one of them. Then



?

**Lemma:** $i: N \to M$ codim 1. Then $i_*[N] = 0 \Longleftrightarrow M - N$ has two components.

(2.) Reduction of embedding to homotopy data. Let $M^m \subset R^{N+m}$ normal bundle $\nu$, let $A = R^{N+m} - M^m$ and let $f: \underset{\text{Sphere}}{E_0(\nu)} \to A$ be the inclusion. In general consider $(\xi, A, f)$ where $\xi$ is a $k$-vector bundle over $M$ and $f: E_0(\xi) \to A$. Obvious notion of equivalence up to homotopy of such triples as well as suspension. Thm: If $k \geq 3$ (this guarantees $\pi_1$ comp. $= 0$) then if $(\xi \oplus 1, \Sigma A, \Sigma f)$ comes from an embedding, so does $(\xi, A, f)$.

Browder has theorem in Morse volume that if $p_1(M^7) \neq 0$, then $M^7 \# \Sigma \neq M^7$ for $\Sigma$ not $7 \cdot$ ja generator of $\Theta^7(\partial\pi)$. Thus there are homotopy tori which are not smooth tori. Similarly he can construct a diffeomorphism of a torus homotopic but not isotopic to identity.