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# Summary of work on the Atiyah-Singer theorem

(I). **A simple analytical model:** Consider the following norms on $C_0^\infty(R^n)$.

$$\|u\|_k^2 = \sum_{|\alpha|, |\beta| \le k}' \|x^\alpha D^\beta u\|^2$$

Completing we get a sequence of Sobolev spaces

$$H_k \supset H_{k+1} \supset \cdots$$

where $H_\infty$ = Schwarz space of rapidly decreasing functions.

Consider Fourier integral operators

$$\varphi \cdot u = (2\pi)^{-n} \int e^{i x \cdot \xi} \varphi(x, \xi)\, \hat{u}(\xi)\, d\xi$$

where

$$\left| \left(\frac{\partial}{\partial x}\right)^\alpha \left(\frac{\partial}{\partial \xi}\right)^\beta \varphi \right| \le C_{\alpha\beta} \left(\sqrt{|x|^2 + |\xi|^2}\right)^{m - |\alpha| - |\beta|}$$

for all $x, \xi$. Then we say $\varphi$ is a pseudo-differential operator of order $\le m$. $\varphi$ is elliptic with principal symbol $\varphi_m$, ~~where~~ if $\varphi_m \in C^\infty(R^{2n} - 0)^*$ is homogeneous of degree $m$. and $\varphi - \rho\varphi_m$ is of order $\le m-1$ where $\rho \equiv 0$ near $0$ and $\equiv 1$ near $\infty$. Obvious generalization to systems.

(II) Analytical ~~Index~~ expression for Index: Let $\varphi(x,\xi)$ be elliptic of order 0, an $r\times r$ matrix. Then by the recursive method we can find $\psi$ quasi-inverse to $\varphi$ ie

$$\varphi * \psi \equiv \psi * \varphi \equiv 1$$

modulo smooth operators (i.e. of order $-\infty$). Then

$$\text{index}\, \varphi = \operatorname{Tr}[\varphi, \psi]$$

$$= (2\pi)^{-n} \int \sum_{|\alpha|\le N} \operatorname{tr}\left(P^\alpha \varphi \cdot Q^\alpha \psi - Q^\alpha \varphi \cdot P^\alpha \psi\right) dx\, d\xi$$

where $P_j = \frac{\partial}{\partial \xi_j}$, $Q_j = \frac{1}{i}\frac{\partial}{\partial x_j}$ and $N \ge n$ (terms with $|\alpha| > n$ may be integrated out by parts since they give rise to boundary terms of degree $< -2n+1$ which integrate out to zero at $\infty$).

(III) Topological (i.e. characteristic class) expression for the index. Recall $\varphi : E \to E$ where $E$ is a trivial bundle over $\mathbb{R}^{2n}$. Moreover $\varphi$ is an isomorphism off some compact set, so defines an element $\alpha$ of $K_c(\mathbb{R}^{2n})$. We wish to determine $\operatorname{ch}\alpha$, so we choose the flat connection $d$ on the first copy of $E$ and $d+\theta$ on the second copy. Want

$$(d+\theta)\circ\varphi = \varphi\circ d \qquad \text{for out}$$

i.e. $\qquad \theta = -d\varphi \cdot \varphi^{-1} \qquad$ " "

so if $\rho \equiv 0$ on singular set of $\varphi$ and $\equiv 1$ far out we may take

$$\Theta = - d\varphi \cdot \rho \varphi^{-1}$$

$$K = + d\varphi \cdot d(\rho\varphi^{-1}) + \rho^2 d\varphi \cdot \varphi^{-1} \cdot d\varphi \cdot \varphi^{-1}$$

$$\boxed{K = (\rho^2 - \rho)(d\varphi \cdot \varphi^{-1})^2 - d\rho\,(d\varphi \cdot \varphi^{-1})}$$

As $(d\rho)^2 = 0$, we have

$$\operatorname{tr} K^n = (\rho^2-\rho)^n \overbrace{\operatorname{tr}(d\varphi\cdot\varphi^{-1})^{2n}}^{=0} - n(\rho^2-\rho)^{n-1} d\rho \operatorname{tr}(d\varphi\cdot\varphi^{-1})^{2n-1}$$

Now $\quad n(\rho-\rho^2)^{n-1} d\rho = n\dfrac{(n-1)!^2}{(2n-1)!}\, d\bar{\rho} \quad$ so

$$\cancel{\operatorname{ch}\alpha = \operatorname{tr} e^{\frac{i}{2\pi}K^0} - \operatorname{tr} e^{\frac{i}{2\pi}K^1} = -\sum_{n=1}^{\infty}\left(\frac{1}{2\pi i}\right)^n \frac{(n-1)!}{(2n-1)!} \operatorname{tr}(d\varphi\cdot\varphi^{-1})^{2n-1}}$$

$$\operatorname{ch}\alpha = \operatorname{tr} e^{\frac{i}{2\pi}K^0} - \operatorname{tr} e^{\frac{i}{2\pi}K^1}$$

($K^i$ curvature of connection on $i$th copy)

$$\boxed{\operatorname{ch}\alpha = -d\bar{\rho} \wedge \sum_{n=1}^{\infty}\left(\frac{1}{2\pi i}\right)^n \frac{(n-1)!}{(2n-1)!} \operatorname{tr}(d\varphi\cdot\varphi^{-1})^{2n-1}}$$

(IV) Atiyah – Singer theorem would say in this case that

$$\text{index}\ \varphi = \pm \int_{S^{2n-1}} \frac{1}{(2\pi i)^n} \frac{(n-1)!}{(2n-1)!}\, tr\,(d\varphi\cdot\varphi^{-1})^{2n-1}$$

and one should be able to integrate by parts to obtain this formula from the analytic expressions given in II.

Precise statement of what it means to integrate by parts.
Let

$$B_h(u,v) = \sum_{k=1}^{\infty} h^k B_k(u,v) = \sum_{\alpha} \frac{h^{|\alpha|}}{\alpha!}\, tr\left(P^{\alpha}u\, Q^{\alpha}v - Q^{\alpha}u \cdot P^{\alpha}v\right) \cdot \frac{\omega^n}{n!}$$

where $\quad i\omega = \sum dp_i\, dq_i \qquad \frac{\omega^n}{n!} = \prod_{i=1}^{n} dp_i\, dq_i$

Then the problem is to find a $2n-1$ form $\lambda_h(u,v)$ such that

a) $d\lambda_h(u,v) = B_h(u,v)$

b) $\lambda_h(\varphi,\psi_h)$ involves only first derivatives of $\varphi$

where $\varphi * \psi_h = \psi_h * \varphi = 1$. Actually we may assume here that $h^{n+1} = 0$, because the remaining terms may be integrated out by parts.

$\underline{n=1}$. ~~Then~~ Set $\boxed{\lambda_1(u,v) = -\mathrm{tr}(du\cdot v)\,\dfrac{\omega^{n-1}}{(n-1)!}}$

Then $d\lambda_1(u,v) = \mathrm{tr}(du\cdot dv)\,\dfrac{\omega^{n-1}}{(n-1)!}$

But $\omega = \sum dp_i\, dq_i$ $\qquad P_j = \dfrac{\partial}{\partial \xi_j} \quad Q_j = \dfrac{1}{i}\dfrac{\partial}{\partial x_j}$

$\omega^k = k! \sum_{i_1<\dots<i_k} dp_{i_1} dq_{i_1} \cdots dp_{i_k} dq_{i_k}$ $\qquad dp_j = d\xi_j \quad dq_j = i\,dx_j$

So $d\lambda_1(u,v) = \sum_i \mathrm{tr}(P_i u\cdot Q_i v - Q_i u\cdot P_i v)\,\omega^n = B_1(u,v)$.

Also $\lambda_1(\varphi,\varphi^{-1}) = -\mathrm{tr}(d\varphi\cdot\varphi^{-1})\,\dfrac{\omega^{n-1}}{(n-1)!}$

Suppose that $n=1$. Then $\lambda_1(\varphi,\varphi^{-1}) = -\mathrm{tr}\, d\varphi\cdot\varphi^{-1}$

$= -\,d\log(\det\varphi)$

so

$$\text{index}\,\varphi = \cancel{\tfrac{1}{2\pi}\int B_1(\varphi,\psi)} \; \frac{1}{2\pi}\int \mathrm{tr}(P\varphi\cdot Q\psi - Q\varphi\cdot P\psi)\,\underbrace{|d\xi\, dx|}_{(\text{volume})}$$

$$= \frac{1}{2\pi}\int \mathrm{tr}(P\varphi\cdot Q\psi - Q\varphi\cdot P\psi)\cdot i\omega$$ (choose orientation of $\mathbb{R}^2$ so that $dx\,d\xi>0$)

$$= \frac{i}{2\pi}\int d(-\mathrm{tr}\, d\varphi\cdot\psi) = \frac{1}{2\pi i}\oint_{S^1} \mathrm{tr}(d\varphi\cdot\varphi^{-1})$$

$$= \frac{1}{2\pi i}\int_{S^1} d\log\det\varphi = \text{winding no. of } \varphi.$$

n=2. In general if

$$K(u,v) = \sum_{\alpha,\beta} \frac{|\alpha|!}{\alpha!} \frac{|\beta|!}{\beta!} \frac{h^{|\alpha|+|\beta|+1}}{(|\alpha|+|\beta|+1)!} \operatorname{tr}\{P^\alpha Q^\beta u \cdot Q^\alpha P^\beta v\},$$

then

(A) $$\lambda_h(u,v) = -K_h(du, v)\,\frac{\omega^{n-1}}{(n-1)!} \quad \text{satisfies}$$

$$d\lambda_h(u,v) = B_h(u,v).$$

However if $\psi_h$ so that $\varphi *_h \psi_h = 1$ then it is not true that $\lambda_h(\varphi,\psi_h)$ involves only first derivatives of $\varphi$. This is already false for terms in $h^2$. Thus either

(A.) take $$\lambda_2(u,v) = \frac{1}{2}\sum_i \operatorname{tr}\{-dP_i u \cdot Q_i v + Q_i u \cdot dP_i v\} \cdot \frac{\omega^{n-1}}{(n-1)!}$$

in which case you can calculate that

$$\lambda_1(\varphi,\psi_1) + \lambda_2(\varphi,\varphi^{-1})$$

involve only first derivatives of $\varphi$, or

(B.) Start with $\varphi_0 = \varphi$ and invent $\varphi_h = \varphi_0 + h\varphi_1 + \cdots$ so that if $\varphi_h *_h \psi_h = 1$, then $\lambda_h(\varphi_h,\psi_h)$ as in (A) involves only first derivatives of $\varphi$. We checked this worked ~~with~~ ~~$\varphi_1$~~ for $h^2$ terms with $\varphi_1 = \frac{1}{2}\sum_i (P_i\varphi\cdot\varphi^{-1}, Q_i\varphi\cdot\varphi^{-1})$.

Remarks on the Atiyah-Singer index theorem.

$X$ smooth compact manifold dim $n$.

Let $\varphi : \Gamma 1 \longrightarrow \Gamma 1$ be an elliptic pseudo-differential operator of order $k$. If $\psi$ is a PDO with symbol inverse to $\varphi$, then

$$\varphi\psi = 1 - S^1$$
$$\psi\varphi = 1 - S^0$$

where $S^0$ and $S^1$ are operators with smooth kernels (PDO's of order $-\infty$). From this formula one sees that

$$\text{index } \varphi = \operatorname{tr} S^0 - \operatorname{tr} S^1 = \operatorname{tr} \{\varphi\psi - \psi\varphi\}.$$

**Lemma:** If $A, B$ are PDO's of order $p$ and $q$ respectively, and $p+q < -n$, then $\operatorname{tr} AB - BA = 0$.

Note that if $P$ is a pseudo-differential operator of order $<-n$, then

$$P(x,y)dy = \int p(x,y,\xi)\, e^{i\varphi(x,y)\xi}\, d\xi dy (2\pi)^{-n}$$

is an absolutely convergent integral and we can define

$$\operatorname{tr} P = \int P(x,x)dx = \int p(x,x,\xi)\, d\xi\, dx\, (2\pi)^{-n}.$$

Applications of the lemma:

1) <u>Additivity of the index:</u> $\quad$ $\operatorname{index}(AB) = \operatorname{index} A + \operatorname{index} B$.

<u>Proof:</u> Let
$$A\tilde{A} = 1 - S^1 \qquad \tilde{A}A = 1 - S^0$$
$$B\tilde{B} = 1 - T^1 \qquad \tilde{B}B = 1 - T^0$$

Then
$$AB\,\tilde{B}\tilde{A} \equiv \tilde{B}\tilde{A}\,AB \equiv 1 \quad \text{mod smooth ops. and}$$

$$\operatorname{index} AB = \operatorname{tr}(AB\tilde{B}\tilde{A} - \tilde{B}\tilde{A}AB) = \operatorname{tr}\left[A(1-T^1)\tilde{A} - \tilde{B}(1-S^0)B\right]$$
$$= \operatorname{tr}\left[1 - S^1 - \underline{AT^1\tilde{A}} - 1 + T^0 + \tilde{B}S^0B\right]$$
$$= \operatorname{tr} T^0 - \operatorname{tr} S^1 - \operatorname{tr}[\tilde{A}AT^1] + \operatorname{tr}[S^0B\tilde{B}] \qquad \text{(use lemma here)}$$
$$= \operatorname{tr} T^0 - \operatorname{tr} S^1 - \operatorname{tr}(T^1 - \cancel{S^0T^1}) + \operatorname{tr}(S^0 - \cancel{S^0T^1}).$$
$$= \operatorname{index} A + \operatorname{index} B.$$

2) <u>Index depends only on principal part of symbol:</u> If $\varphi$ is of order $k$ and $a$ is of order $< k$, then $\operatorname{index}(\varphi + a) = \operatorname{index} \varphi$.

<u>Proof:</u> Let $\psi$ be a quasi-inverse for $\varphi$ so that $\psi$ is of order $-k$. If $a$ is smooth, then $\psi$ is also a quasi-inverse for $\varphi + a$, so

$$\operatorname{index} \varphi = \operatorname{tr}\, \varphi\psi - \psi\varphi = \operatorname{tr}\,[\varphi + a, \psi] = \operatorname{index}(\varphi + a).$$

This shows conclusion holds if order $a - k < -n$. In particular index $(1+a) = 0$ if order $a < -n$. So if order $a < 0$ then take $a = 1 + \varphi$ $b = 1 - \varphi + \cdots \pm \varphi^N$. Then if $\tilde{b}$ is a quasi-inverse for $1+a$ we have

$$\mathrm{ind}(1+a) = \mathrm{tr}\ a\tilde{b} - \tilde{b}a = \mathrm{tr}\ [1+a, b] + \underbrace{\mathrm{tr}\ [1+a, b-\tilde{b}]}_{\text{order} \ll 0.}$$
$$= 0.$$

Finally $\quad \mathrm{index}\,(\varphi + a) = \mathrm{index}\,(\varphi + \varphi\psi a) \qquad$ (diff smooth)

$$= \mathrm{index}\ \varphi + \mathrm{index}\ 1 + \psi a$$
$$= \mathrm{index}\ \varphi \qquad (\mathrm{order}(\psi a) = -k + \mathrm{order}\ a < 0.)$$

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~~[illegible]~~

Remark: The lemma may be proved by noting that if $A, B$ are smooth it is true and that $\mathrm{tr}\ AB$ is continuous with respect to the $p$-norm on $A$ and $q$-norm on $B$ for $p+q < -n$

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Find a formula for the character of a complex. Thus let $E^0 \xrightarrow[\varphi]{} E^1 \xrightarrow[\varphi]{} E^2 \to \cdots$ be a complex and assume that each $E^q$ endowed with a flat connection $d$. If $\psi$ is a homotopy operator on the set $A$ where $\varphi$ is acyclic and $\rho \equiv 0$ on $X - A$, then off $A$ can take connection

$$D = \psi d\varphi + \varphi\psi d\varphi\psi$$

so that

$$\theta = \psi d\varphi + \varphi\psi\, d(\varphi\psi)$$

$$K = d\theta + \theta\cdot\theta$$

$$= d\psi\cdot d\varphi + d(\varphi\psi)d(\varphi\psi) + (\psi d\varphi)^2 + [\varphi\psi\, d(\varphi\psi)]^2 + \psi d\varphi\cdot\varphi\psi\, d(\varphi\psi) + \varphi\psi d(\varphi\psi)\cdot\psi d\varphi$$

thus

$$\operatorname{tr} K = \operatorname{tr} d\psi\cdot d\varphi$$

i.e. $$\boxed{\operatorname{tr} K^q = \operatorname{tr} d\psi^{q+1}\cdot d\varphi^q}$$

$$\operatorname{tr}\sigma = \sum(-1)^q \operatorname{tr} d\psi^{q+1}\cdot d\varphi^q$$

why is

$$\operatorname{tr}\sigma = 0 \quad \text{if} \quad \psi\varphi + \varphi\psi = 1$$

$$(P+\bar{P})A(P,Q,h) + (Q+\bar{Q})B(P,Q,h) = e^{hP\bar{Q}} - e^{hQ\bar{P}}$$

$$\varphi \underset{h}{*} \psi_h = 1$$

$$A(P,Q,h)\,\varphi\psi_h$$

~~$e^{hP\otimes Q}$~~

$$\boxed{\begin{aligned} &\mu \cdot e^{hP\otimes Q}(\varphi\otimes\psi_h) = 1 \\ &\mu \cdot A(P,Q,h)(\varphi\otimes\psi_h) = -\mathrm{tr}\{Q\varphi\cdot\varphi^{-1}\} \\ &\mu \cdot B(P,Q,h)(\varphi\otimes\psi_h) = \mathrm{tr}\{P\varphi\cdot\varphi^{-1}\} \\ &(P\otimes 1 + 1\otimes P)A(P,Q,h) + (Q\otimes 1 + 1\otimes Q)B(P,Q,h) = e^{hP\otimes Q} - e^{hQ\otimes P} \end{aligned}}$$

Can one solve these equations for A and B ?

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$$R\otimes R \xrightarrow{\mu} R$$

~~power series~~

$\alpha \in R$ $\quad\quad \alpha = \sum h^i \alpha_i(p,q).$ $\quad\quad \mathbb{C}[[p,q,h]]$

$$\sum (-1)^g \operatorname{tr} e^{K g} - \sum (-1)^g \operatorname{tr} e^{K g} = d\beta$$

$$\beta = \int$$

$$\sum \frac{1}{i!} P^i \varphi \cdot Q^i \psi_{g-i} = 0 \qquad \sum \frac{1}{i!} P^i \psi_{g-i} \cdot Q^i \varphi = 0$$

$$\Downarrow$$

$$\sum \frac{1}{i!} \operatorname{tr}\left(Q^i \varphi \cdot P^i \psi_{g-i}\right) = 0$$

An intermediate result would be

$$\sum \frac{1}{(i+j)!} \operatorname{tr} P^i Q^j \varphi \cdot Q^i P^j \psi_{g-i-j} = 0 \qquad ?$$

Introduce Planck's constant $h$.

$$\sum \frac{h^i}{i!} P^i \varphi \cdot Q^i \psi_{g-i}\, h^{+g-i} = 0 \qquad \psi = \psi_0 + h \psi_1 + h^2 \psi_2 + \cdots$$

becomes $\boxed{\sum \frac{h^i}{i!} P^i \varphi \cdot Q^i \psi_h = 1}$ $\qquad \psi = \sum_{n=0}^{\infty} h^n \psi_n$

$\therefore$

$$\sum \frac{h^{i+j}}{i!\,j!} Q^j P^i \varphi \cdot P^j Q^i \psi_h = e^{+hPQ}\left(e^{-hPQ} \varphi \cdot e^{-hPQ} \psi_h\right)$$

an excellent formula in any case

I need a workalbe first Chern class for ~~the~~ the Weyl algebra of dimension 1. In other words given an operator W we should be able to define on the circle some kind of cohomology class tr $W^{-1}dW$. What is the module of differentails of the Weyl algebra? Clearly generated by dP and dQ. It is important here to be thinking in terms of the cohomology of algebras. Thus the module of differentials of the tensor algebra is free of rank n.

Question: Is it possible to define Chern classes of non-singular matrix valued functions., Thus suppose that u is an elliptic element of the Weyl algebra and that we have constructed a quasi-inverse v for u We regard u as an endomorphism of the trivial ~~line bundle~~ vector bundle and put a connection ~~mmmhm~~ on the beginning and the end so that ~~the~~ far out at least the endomorphism ~~it~~ preserves the connection. ~~Therefore it~~ It seems necessary at this point to undderstand in some way what might be meant by a connection. Suppose that

Here is the geometric problem: Given a topological manifold, can you associate toit some kind of smoothing kernels which represent the simplicial structure? What Milnor does is to define rational Pontyagin classes for a simplicial rational homology n-manifold in terms of intersection theory. othe Thus if M is a rational homology n-manifold and if f:M X is a map then by approximating f simplicially we can take the inverse image of a generic point and then take its index thereby getting a well-defined number ~~I(f)~~ I(f). Actually one might as well assume that X is a sphere. The problem that you should consider is the following: Can you use some varinat of Grothendieck's intersection theory with the property that

Segal's situation:

$L$ v.s. anti-symm. form

$\mathcal{U} = \cancel{E(L)} = $ Weyl alg.

$E \quad E_0$ linear fns on $\mathcal{U}$.

Defn: a renorm of $E$ rel. to $E_0$ is a op. $N$ on $\mathcal{U}$ ∋

$$[N(u), x] = N[u, x] \qquad u \in \mathcal{U} \quad x \in L$$

$$E(Nu) = E_0 u.$$

(N modules map for) adjoint action.

$$\boxed{\text{Thm: } \quad E(1) \neq 0 \implies N \text{ exists + is unique}}$$

$E_0$ arb.

algebra of forms is $\boxed{\mathcal{U} \otimes \Lambda L^*}$

$d$ is a derivation such that

$$(du)(x) = [u, x].$$

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Assertion is that if $\quad L = V \oplus V^* \quad$ then

$$S(V) \otimes V^* \hookrightarrow \mathcal{U} \otimes \Lambda L^*.$$

Proposition: Let $E^\cdot, \varphi^\cdot$ be a complex having a homotopy operator $\psi \ni \psi^2 = 0$. Then

$$\sum (-1)^q \, tr \, d\psi^{q+1} d\varphi^q = 0.$$

Proof:

$$tr \, d\psi^{q+1} \cdot d\varphi^q = tr \, d\psi^{q+1}(\psi^{q+2}\varphi^{q+1} + \varphi^q \psi^{q+1}) d\varphi^q .$$

$$= tr \, \underline{\psi^{q+1} d\varphi^q d\psi^{q+1} \varphi^q} + tr \, \psi^{q+1} d\psi^{q+2} \cdot d\varphi^{q+1} \cancel{\psi}\varphi^q$$

$$=$$

$$d\psi^{q+1} \cdot \varphi^q \cdot \psi^{q+1} + \psi^{q+1} d\varphi^q \psi^{q+1} + \psi^{q+1} \varphi^q d\psi^{q+1}$$

$$= d\psi^{q+1}$$

$$tr \, d\psi^{q+1} d\varphi^q = d\psi^{q+1} \varphi^q \psi^{q+1} d\varphi^q + \psi^{q+1} \varphi^q d\psi^{q+1} d\varphi^q$$

## §5. Hörmander's Localization theorem in the Quantum case

Let

$$(\ )\qquad \Gamma V_0 \xrightarrow{A} \Gamma V_1 \xrightarrow{B} \Gamma V_2 \qquad\qquad BA = 0$$

be first order constant coefficient operators and let

$$V_0 \xrightarrow{A(\xi)} V_1 \xrightarrow{B(\xi)} V_2$$

be the symbol homomorphism at ~~at~~ $\xi \in T'$. Let $\xi_\nu$ be a sequence of points in $T'$ such that $|\xi_\nu| \to \infty$ and $\ni$

$$(\star)\qquad \begin{array}{ll} \text{graph } A(\xi_\nu) \in \mathrm{Grass}_m (V_0 \times V_1) & m = \dim V_0 \\ \text{graph } B(\xi_\nu) \in \mathrm{Grass}_n (V_1 \times V_2) & n = \dim V_1 \end{array}$$

converge to $m$ and $n$ planes $\bar{A}$ and $\bar{B}$ resp. Let

$$\begin{array}{ll} Z_0 = pr_1 \bar{A} & B_0 = 0 \\ Z_1 = pr_2 \bar{B} & B_1 = pr_2(\bar{A} \cap 0 \times V_1) \\ Z_2 = V_2 & B_2 = pr_2(\bar{B} \cap 0 \times V_2) \end{array}$$

$$H_i = Z_i / B_i$$

and note that although $\bar{A}$ and $\bar{B}$ are no longer graphs they induce maps

$$H_0 \xrightarrow{S} H_1 \xrightarrow{T} H_2$$

by

$$S(pr_1 x) = pr_2 x + B_1 \qquad \text{if } x \in \bar{A}$$
$$T(pr_1 y + B) = pr_2 y + B_2 \qquad \text{if } y \in \bar{B}.$$

Equivalently if $w \in V_0$ (resp. $w \in V_1$) then $v \in Z_0$ (resp $w \in Z_1$) iff $\exists$ sequence $v_\nu \to v$ (resp. $w_\nu \to w$) such that $A(\xi_\nu) w_\nu$ ~~converges~~ (resp. $B(\xi_\nu) w_\nu$) converges to something, say $z$, in which case $Sw = z + B_1$ (resp. $Tw = z + B_2$). From

$$0 = \left(B(\xi_\nu)\, A(\xi) + B(\xi) \underset{\sim}{A(\xi_\nu)}\right) \underset{\sim}{w_\nu},$$

we see that $B(\xi) B_1 \subset B_2$, $A(\xi) Z_0 \subset Z_1$ ~~and~~ so that $A(\xi)$ ~~and~~ $B(\xi)$ induce maps on the $H_i$. Moreover

$$T A(\xi) + B(\xi) S = 0$$
$$TS = 0,$$

so that we get a sequence of first order operators

$$(\ ) \qquad H_0 \xrightarrow{A+S} H_1 \xrightarrow{B+T} H_2$$

with symbols $A(\xi) + S$ and $B(\xi) + T$ at $\xi \in T$! The sequence ( ) will be called the <u>derived sequence</u> of ( ) with resp. to the sequence $\xi_\nu$.

Theorem (Hörmander): The following are equivalent:

(i) $\forall \varepsilon > 0 \ \exists C(\varepsilon) \ni$

$$\|u\|^2 \le (1+\varepsilon)\{\|A^*u\|^2 + \|Bu\|^2\} + C(\varepsilon)\|u\|_{-1}^2 \qquad \text{all } u \in H_\infty \otimes V_1$$

(ii) For every sequence $\zeta_\nu \to \infty$ such that graph $A(\zeta_n)$ and graph $B(\zeta_n)$ converge we have for the derived sequence (1) the estimate

$$\|(A+S)^* w\|^2 + \|(B+T)w\|^2 \ge \|w\|^2 \qquad w \in H_\infty \otimes H_1$$

where the norms on the $H_i$ are induced by those of the $V_i$.

(iii) $\exists N \ge 2$ and a function $\varepsilon(\lambda) \ge 0$ such that $\varepsilon(\lambda) \to 0$ as $\lambda \to \infty$ and such that

$$\|u\|^2 \le \|A(D+\zeta)^* u\|^2 + \|B(D+\zeta)u\|^2 + \varepsilon(|\zeta|)\|u\|_N^2 \qquad \begin{matrix} \zeta \in T' \\ u \in H_\infty \otimes V_1. \end{matrix}$$

Proof: (i) $\Rightarrow$ (ii). Introduce the operator $T_\zeta$ on $H_\infty$ by

$$\cancel{(T_\zeta u)(z) = e^{-\mathrm{Re}(z,\zeta) - |\zeta|^2/2} u(z+\zeta)}$$

$$\cancel{\|T_\zeta u\|^2 = \int |u(z+\zeta)|^2 e^{2\mathrm{Re}(z,\zeta) + |\zeta|^2} e^{-|z|^2} dV}$$

$$\cancel{= \int |u(z)|^2 e^{2\mathrm{Re}(z,\zeta) - 2|\zeta|^2 + |\zeta|^2 - |z|^2 + 2\mathrm{Re}(z,\zeta) - |\zeta|^2}}$$

$$(T_\zeta u)(z) = e^{(z,\zeta) - |\zeta|^2/2}\, u(z-\zeta)$$

$$\|T_\zeta u\|^2 = \int |u(z-\zeta)|^2\, e^{2\,\mathrm{Re}(z,\zeta) - |\zeta|^2}\, e^{-|z|^2}\, dV$$

$$= \int |u(z)|^2\, e^{2\,\mathrm{Re}(z,\zeta) + 2|\zeta|^2 - |\zeta|^2}\, e^{-|z|^2 - 2\,\mathrm{Re}(z,\zeta) - |\zeta|^2}$$

$$= \|u\|^2.$$

so $T_\zeta$ is unitary. Also

$$D T_\zeta u = T_\zeta (D + \bar\zeta) u.$$

$$\boxed{D T_\zeta = T_\zeta (D + \bar\zeta)}$$

Replacing in the estimate (i) $u$ by $T_{-\zeta} u$ we obtain

$$\cancel{\|T_\zeta u\|^2 \le (1+\varepsilon)\{\|A(D+\bar\zeta)^* u\|^2 + \|B(D+\bar\zeta)u\|^2\}}$$

$$\|u\|^2 \le (1+\varepsilon)\left\{ \|A(D+\zeta)^* u\|^2 + \|B(D+\zeta)u\|^2 \right\} + C(\varepsilon)\, \|T_{-\zeta} u\|_{-1}^2$$

and

$$\|T_{-\zeta} u\|_{-1}^2 = \int \frac{1}{1+|z|^2} \left| e^{(z,\zeta) - |\zeta|^2/2}\, u(z-\zeta) \right|^2 e^{-|z|^2}\, dV$$

$$= \int \frac{1}{1+|z+\zeta|^2} |u|^2 e^{-|z|^2}\, dV$$

Lemma: $\dfrac{1+|\xi|^2}{1+|\eta|^2} \le 2(1+|\xi+\eta|^2)$

$$\therefore \ |||T_\zeta u|||_{-1}^2 \le \frac{2}{1+|\zeta|^2}\, \|u\|_1^2$$

Now let $\zeta_\nu$ be a sequence in $T'$ going to $\infty$ such that graph $A(\zeta_n)$ and graph $B(\zeta_n)$ converge. Realize $H_i$ as the orthogonal complement of $B_i$ in $Z_i$. Then if $w \in H_1$ $(w, Tw) \in \bar{B}$ where $Tw \perp B_2$. ~~Note that graph $A(\zeta_n)^*$ converges also to $\bar{A}^*$ and that if $v \in B_1^\perp$ then $(S^*v, v) \in \bar{A}^*$ where $S^*v \perp Z_0$. The orthogonal complement of graph $A(\zeta_n)$ in $V_0 \times V_1$ is $(-A^*(\zeta)w, w)$, hence $(-S^*w, w)$ the orthogonal complement of $\bar{A}$ is $\{(S^*w, w)\}$~~ Similarly if $v \in Z_0$, then $(v, Sv) \in \bar{A}$ and $Sv \perp B_1$. Note that the orthogonal complement of graph $A(\zeta_\nu) \subset V_0 \times V_1$ is $\{(-A(\zeta_\nu)^* w, w),\ w \in V_1\}$. and ~~therefore~~ that if $w \in H_1$, then $(-S^*w, w) \in \bar{A}^\perp$ and $S^*w \in H_0$. Therefore if $w \in H_1$ there is a sequence $w_\nu \to w$ such that $A(\zeta_\nu)^* w_\nu \to S^* w$.

Given $v \in H_\infty \otimes H_1$ choose $v_\nu$ and $w_\nu$ $\ni$

$$\left(w_\nu, [B(\zeta_\nu) + A(\zeta_\nu)^*]w_\nu\right) \text{ closest to } \left(0, \pi[B(D) + A(D)^*]v\right)$$

$$\left(v_\nu, [B(\zeta_\nu) + A(\zeta_\nu)^*]w_\nu\right) \text{ ——— } \left(v, (S^* + T)v\right)$$

where $\pi$ is the orthogonal projection on $B_2 \oplus Z_0^\perp$. We know that there are such sequences $w_\nu, v_\nu$ converging to RHS's. ~~so we choose~~ This is because $B(\zeta_\nu)A(\zeta_\nu) = 0$ so that can write $w_\nu = w_\nu^1 + w_\nu^2$ $\ni$ $A(\zeta_\nu)^* w_\nu^1 = 0 = B(\zeta_\nu) w_\nu^2$. Then $w_\nu$ and $v_\nu$ are given by linear operators in terms of $\pi(B(D) + A(D)^*)v$ and $(S^*+T)v$ so $w_\nu \to 0$ and $v_\nu \to v$ in $H_\infty$. Thus

$$u_\nu = v_\nu - w_\nu \longrightarrow v$$

$$\left[A(D+\zeta_\nu)^* + B(D+\zeta_\nu)\right] u_\nu \to (A(D)^* + B(D))v + (S^* + T)v$$
$$- \pi\,(A(D)^* + B(D))v$$

$$= (A^* + S)^* v + (B+T)v$$

in $H_0 + H_1$.

~~Thus from the estimate~~

$$\cancel{(1+\varepsilon)\left(\|A(D+\zeta)^* u\|^2 + \|B(D+\zeta)u\|^2\right) \geq \|u\|^2 - \frac{2C(\varepsilon)}{1+|\zeta|^2}\|u\|_{-1}^2}$$

~~Putting in $\zeta_\nu$, $u_\nu$ we derive (ii).~~

~~Remark: We have shown (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii).~~

~~(ii) $\Rightarrow$ (iii). Assume~~

Thus putting in $\zeta_\nu, u_\nu$ into the estimate

$$\|u\|^2 \leq (1+\varepsilon)\left\{\|A(D+\zeta)^* u\|^2 + \|B(D+\zeta)u\|^2\right\} + C(\varepsilon)\frac{2}{1+|\zeta|^2}\|u\|_{-1}^2$$

and letting $\nu \to \infty$ we get

$$\|v\|^2 \leq (1+\varepsilon)\left\{\|(A+S)^* v\|^2 + \|(B+T)v\|^2\right\} \qquad v \in H_\infty \otimes H_1$$

and since $\varepsilon > 0$ is arbitrary we get (ii).

(ii) $\Longrightarrow$ (iii). ~~Suppose that~~ Let $N$ be fixed $\geq 2$.
Suppose that no $\varepsilon(|\zeta|)$ exists going to zero as $\zeta \to \infty$ such that (iii) holds. Then there is an $\varepsilon_0 > 0$ and a sequence ~~of~~ $\zeta_\nu \in T_1'$ [illegible] $u_\nu \in H_\infty \otimes V_1$ such that

$$\|u_\nu\|^2 = 1, \qquad \|A(D+\zeta_\nu)^* u_\nu\|^2 + \|B(D+\zeta_\nu)u_\nu\|^2 + \varepsilon_0 \|u_\nu\|_N^2 < 1$$

As $N \geq 2$ may assume $u_\nu$ converge in 1-norm to $v$ and also that the graphs of $A(\zeta_\nu)$ and $B(\zeta_\nu)$ converge. ~~the $A(\zeta_\nu)^*$ Then~~ Set $P(D) = A(D)^* + B(D)$. Then $P(D)u_\nu \to P(D)v$. Look at $P(\zeta_\nu)u_\nu$. $(u_\nu(z), P(\zeta_\nu)u_\nu(z))$. Want to write

$$u_\nu = w_\nu \oplus v_\nu$$

where $w_\nu$ is in the infinite eigenspace part of $P(\zeta_\nu)^* P(\zeta_\nu)$
$v_\nu$ ——— finite ——— of ——— .

Then $P(\zeta_n) v_\nu \to S v$. $\|P(\zeta_\nu) w_\nu\| < \text{Const.} \Rightarrow \|w_\nu\| \to 0$. because eigenvalues go toward infinity. ~~Follows that~~ As $w_\nu$ bdd in 2 norm $\Rightarrow$ ~~can assume $w_\nu \to 0$ in 2 norm assume~~ $w_\nu \to 0$ in 1-norm $\Rightarrow$ can replace $u_\nu$ by $v_\nu$. Also can replace $A(D)v$ by $A(D)v - \pi A(D)v$. and then we get a contradiction of (ii).

(iii) $\Longrightarrow$ (i)

Problem: Define smooth modules over a reg. loc. ring $A$.

example: Suppose $M$

not true for $A \oplus A/I$ $\qquad$ $A/I$ non-sing. but not a divisor.

because $\mathbb{P}(A \oplus A/I) =$ cone on $\mathbb{P}(A/I)$.

And for cone to be non-singular

I need a good notion!!

Thus let $R$ be a complete reg. loc. ring!

want same definition.

$\boxed{\text{BOTT's insight}}$

Suppose we consider the set consisting of

Consider the matrices of maps $\boxed{V \to W}$ of ~~[illegible]~~ rank $\leq r$.
this is generally a singular variety which may be resolved as follows:
~~[illegible]~~ Let $n = \dim V$ and consider the set $Z$ of all pairs $(A, H)$ where
$A \in \mathrm{Hom}(V, W)$ and $H$ is an $n-r$ plane in $V$ such that $AH = 0$. Then
$Z$ is non-singular because one can project

$$Z \xrightarrow{pr_2} \cancel{\mathrm{Hom}(V,W)} \; G_{n-r}(V)$$

and the fibers are ~~grassmann bundles~~ vector bundles. Note also
that $Z \xrightarrow{pr_1} \mathrm{Hom}_{\mathrm{rank} \leq r}(V, W)$ is proper and birational.

Now I am given a bundle

$\Omega_n$ expanding sequence of closed balls.

given $u$ to approx.

$u|\Omega_n \approx P v_n$ extends $v_n$.

Then $P v_{n+1} \approx u - P v_n$

on $\Omega_{n+1}$

want $v_{n+1}$ small on $\Omega_n$.

---

begin the conversion of an idea into a theorem

<u>Definition:</u> $A$ regular local ring.

$M$ module f.t. over $A$.

Say that $M$ smooth if $\operatorname{Proj} SM$ regular.

<u>Key lemma:</u> $0 \to \underset{\text{loc. free}}{M'} \to M \to \underset{\text{smooth}}{M''} \to 0$

$\Longrightarrow M'$ smooth ?

$$P \oplus R \to P \oplus R$$

$$\begin{array}{ccccccccc} 0 & \to & R & \to & P & \to & M & \to & 0 \\ & & \psi\uparrow\downarrow\varphi & & \psi\uparrow\downarrow\varphi & & \parallel & & \\ 0 & \to & R & \to & P & \to & M & \to & 0 \end{array}$$

$\psi\varphi = \mathrm{id} - th.$

Note that by <u>Shanuel</u>

$M' \to M' \oplus \underline{M} \simeq \underline{M}' \oplus M$

so ~~if~~ $M'$ smooth $\Leftrightarrow \underline{M}'$ smooth.

form a system of parameters.

$$S^A M = \underline{k[x_1,\dots,x_n][e_1,\dots,e_r]} = k[x_1,\dots,e_r]$$

basic point sends $x_i \to 0$ $\forall i$

$e_j \to 0$ $j>1$.

What can we now say about $M$! We are given the elements ~~$f_A$~~ $\alpha_1,\dots,\alpha_k \in M'$ ie.

$$\alpha_i = \sum_{j=1}^{n} f_{ij} e_j \qquad f_{ij} \in A. \qquad \begin{matrix} 1\le i\le k \\ 1\le j\le n \end{matrix}$$

~~...~~ We are told that this is part of a system of parameters for $(SM)$ at the point $\left| \begin{matrix} x_i \to 0 & \\ e_j \to 0 & j>1 \end{matrix} \right.$

In other words : $f_{i1} \in$ max ideal.

$$\alpha_i = \sum_{j>1} f_{ij} e_j + \left( \sum_{g=1}^{n} f_{i1g}\, x_g \right) e_1$$

now consider the matrix.

$$\underbrace{f_{ij}(0)} \qquad \underbrace{f_{i1g}(0)} = \frac{\partial f_{i1}}{\partial x_g}(0)$$

and this must be of rank $k$.

Therefore we have ~~made an amusing discovery~~ found:

Suppose that $SM''$ regular

$$\boxed{M' \otimes_A SM \longrightarrow SM \longrightarrow SM''} \to 0$$

reg. cause $A$ is

assumed to be regular except at irrelevant ideal.

elements of $M'$ in $S_1 M$ generate $\boxed{\text{ideal.}}$

Thus at the prime $\mathfrak{p} \in SM''$ which represents a ~~place in M~~ 1-dimensional quotient $M'' \to k$, we have that $SM''$ is non-singular. i.e. there are elements $\alpha_1, \dots, \alpha_k \in M'$ which form part of a system of parameters for $SM$ at the point $M \to M'' \to k$ and which generate the Kernel of $SM \to SM''$ at that point.

(Suppose $I$ ideal in a local $A$ and that $I = (f_1, \dots, f_N)$ and that $I$ is generated by a regular sequence. Can one then take a subset of the $f$'s? Yes because let $I = (x_1, \dots, x_r)$ where $x_i$ are reg. sequence. Then $x_i = \sum a_{ij} f_j$ assume $N = r = \dim I \otimes_A k$ then $a_{ij}$ invertible so $f_j$ is a reg sequence!)

What are a system of parameters for $SM$ at the point $M \to k$?
$M = A^r$ say and let $x_1, \dots, x_n$ be parameters for $A$ at the max ideal $A \to k$. Assume $e_1 \to 1$ $e_j \to 0$ $j > 0$ $j = 1, \dots, r$ $e_j$ base for $M$. Then $x_i$ $i = 1, \dots, n$
$e_j$ $j = 2, \dots, r$

Suppose $M$ ~~is a quotient of a free module with~~ is an $A$ module such that $SM$ is regular at a point $\lambda : M \to k$. Then choosing $m_1, \dots, m_\ell \in M$ (minimal) generating system for $M$ near the point $A \to k$ such that $\lambda(m_i) = 0 \quad i > 1$, $\lambda(m_1) = 1$, there are elements

$$\sum_{j=1}^{\ell} f_{ij} e_j \qquad \begin{array}{l} i = 1, \dots, r \\ j = 1, \dots, \ell \end{array}$$

such that
$$\sum f_{ij} m_j = 0 \qquad \text{all } i$$

and such that the vectors

$$v_i = \left( \frac{\partial f_{ij}}{\partial x_g}(0) \; g = 1, \dots, n \; ; \; f_{ij}(0) \quad j = 2, \dots, \ell \right)$$

are independent.

---

To set up a bit differently

Suppose that $m_1, \dots, m_\ell$ generate $M$ near $A \xrightarrow{0} k$ and that $\lambda : M \to k$ lies over $0$. Then $SM$ non-singular at $\lambda$ and of dimension $\mathfrak{f}$ if there are elements

$$\sum_{j=1}^{\ell} f_{ij} e_j \qquad \begin{array}{l} i = 1, \dots, r \\ j = 1, \dots, \ell \end{array} \qquad ?$$

---

conclude transversality of symbol + linear part is incorrect!

next part

form some kind of projective bundle out of the ~~[illegible]~~ kernel. Consider set of all pairs $(\xi, L)$ where $\xi \in T^*$ and $L$ is a line in $E$ such that $\varphi(\xi)L = 0$.

This won't work because adding a trivial $\mathcal{O}$ is not good.

Victor's suggestion!!!

a variety has a dimension and ~~[illegible]~~ we may always (choose a) generic ~~[illegible]~~ linear space transversal to the variety in which case everything is decomposed into normal and tangential operators!

# Note on a theorem of Ehrenpreis-Guillemin-Sternberg

by Daniel G. Quillen

Let $S$ be the ring of polynomial functions $f(z)$ on $\mathbb{C}^n$ and let $P$ (resp. $Q$) be an $r_0 \times r_1$ (resp. $r_1 \times r_2$) matrix of homogeneous linear polynomials such that the sequence

(1) $$S^{r_2} \xrightarrow{Q} S^{r_1} \xrightarrow{P} S^{r_0}$$

is exact where if $f=(f_1,\dots,f_{r_1}) \in S^{r_1}$ $(Pf)_j = \sum_i f_i P_{ij}$ and ~~the~~ $Q$ acts similarly. ~~If $u$ is a polynomial hom $u(z) = \sum_{|\alpha|=m} a_\alpha z^\alpha$ is a homogeneous polynomial of degree $m$, then define $\|u\|^2 = \sum \alpha!\, |a_\alpha|^2$~~

~~Introduce~~ Define a pre-Hilbert space structure on $S$ by defining the norm of a polynomial $u(z) = \sum a_\alpha z^\alpha$ to be

$$\|u\|^2 = \sum \alpha!\, |a_\alpha|^2$$

Then in [ ] it was established that this norm is exponentially equivalent to other reasonable families of norms on polynomials in the sense that there is are constants $C, N$ such that

(2) $$(C^{-1}N^{-k}) \|u\| \leq |||u||| \leq CN^k \|u\|.$$

if $u$ is homogeneous of degree $k$. In particular this holds if

(3) $|||u||| = \max\{|a_\alpha|\}$.

In addition it was shown that (1) is exact in norm in the sense that $\exists\ C, N$

(4) $u$ of degree $k$ $\quad Pu = 0 \Rightarrow u = Qv \quad$ where $\quad \|v\| \le CN^k \|u\|$.

Combining (2), (3), (4) it follows that if $\mathcal{O}$ is the germs of analytic functions on $\mathbb{C}^n$ near $0$ then

(5) $$\mathcal{O}^{r_2} \xrightarrow{Q} \mathcal{O}^{r_1} \xrightarrow{P} \mathcal{O}^{r_0}$$

is exact. Conversely if (5) holds ~~the closed graph theorem yields~~ we obtain (4). (In effect let $U$ nbd. of $0$ and consider for each nbd. of $0$ $V \subset U$ the pairs $Z_V = \{(v, f) \mid v \in \mathcal{O}^{r}(V)\ f \in \mathcal{O}^{r}(U)\ Pv = f\}$. $Z_V$ and $\mathcal{O}^{r}(U)$ are Frechet spaces so if $pr_2 : Z_V \to \operatorname{Ker} Q(U)$ is not onto its image has first category and so by Baire category thm. $\bigcup_V pr_2 Z_V \neq \operatorname{Ker} Q(U)$ which contradicts exactness of (5). Hence $pr_1 Z_V = \operatorname{Ker} Q(U)$. Hence by open mapping theorem there is a nbd. $W$ of $0$ in $\mathcal{O}(U)^{r_2}$ such that $f \in \operatorname{Ker} Q(U)$ & $\|f\| \le 1 \Rightarrow$ $Pv = f$ for some $v \in W$. Now assume $U$ large enough so that $\|f\| \le 1 \Rightarrow f \in \mathcal{O}^{r_1}(U)$ and choose $C, N$ such that $W \in \{V \mid \|V_k\| \le CN^k\}$. Then (4) clearly holds.)

(Weierstrass prep.)

However (5) is an easy consequence of the fact that $\mathcal{O}$ is noetherian and that completion for a local noetherian ring is faithfully flat. (Krull + Artin-Rees). $\mathcal{O}$ is flat over $S$ because if $S_m$ is the localization at $0$ Then have

$$\begin{tikzcd}
S \arrow[r, "\text{flat}"] & S_m \arrow[rr, "\text{f. flat}"] \arrow[dr] & & \hat{S}_m \arrow[d, equal] \\
& & \mathcal{O} \arrow[r] \arrow[ur, "\text{faithfully flat}"'] & \hat{\mathcal{O}}
\end{tikzcd}$$

$\Rightarrow$ $\mathcal{O}$ flat over $S$.

Hence (4) is proved rather simply. ~~[illegible]~~ The purpose of this note however is to prove the following strengthening of (4) from an exponential to a polynomial estimate.

Conjecture ~~Theorem.~~ There exists a constant $C$ and an integer $n$ such that if $u \in S^{r_1}$ is of degree $k$ and $Qu = 0$, then $u = Pv$ where $\|v\| \leq Ck^n \|u\|$.

*Proof:* The idea is to use an isomorphism well known to the physicists to reduce this to a theorem of Malgrange. Introduce the norms

$$\|u\|_k^2 = \sum (n + |\alpha|)^k \, |a_\alpha|^2 \, \alpha!$$

let $H_k$ be the resulting Hilbert space and let $H_{-\infty}$ be the inductive limit of the $H_k$ $k \to -\infty$, and $H_\infty$ the inverse limit ~~By the category theory argument~~ Then our estimate is equivalent to the exactness of the sequence

$$H_\infty^{r_2} \xrightarrow{Q} H_\infty^{r_1} \xrightarrow{P} H_\infty^{r_0}$$

which by duality to the exactness of

$$H_{-\infty}^{r_2} \xleftarrow{Q^*} H_{-\infty}^{r_1} \xleftarrow{P^*} H_{-\infty}^{r_0}.$$

Thus we want to know that $H_{-\infty}$ is an injective module over $S$. However (see Bargmann) $H_{-\infty} \cong \mathcal{S}'$, the space of ~~[illegible]~~ distributions on $\mathbb{R}^n$ with light increase, in such a way that ~~[illegible]~~ action of $D$ in $H_{-\infty}$ corresponds to action of $+D+x/2$ on $\mathcal{S}'$ HELL it doesn't work.

MOSER

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The iterative method required for Kolmogoroff-Arnold's stability theorem!

To solve

$$\mathcal{F}(f, u) = \Phi \qquad \text{where } \mathcal{F} \ni$$

$$\begin{cases} \mathcal{F}(f, u \circ v) = \mathcal{F}(\mathcal{F}(f,u), v) \\ \mathcal{F}(f, I) = f. \end{cases}$$

i.e. $\mathcal{F}(f,u) = f \cdot u$ so that

$$(f \cdot u) \cdot v = f \cdot (u \circ v).$$

<u>Iteration process:</u> $u_{n+1} = u_n \circ v$ where $v = 1 + \hat{v} \ni$

$$\underbrace{f \cdot u_n}_{f_n} \cdot v = \Phi \qquad \text{mod } (\hat{v}, f_n - \Phi)$$

i.e.

$$\cancel{\mathcal{F}(\Phi, 1)} + \mathcal{F}_1(\Phi, 1) \cdot (f_n - \Phi) + \mathcal{F}_2(\Phi, 1) \cdot \hat{v} = \cancel{\Phi}$$

hence

$$\mathcal{F}_1(\Phi, 1) g = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left\{ \mathcal{F}(\Phi + \varepsilon g, 1) - \mathcal{F}(\Phi, 1) \right\} = g$$

so equation becomes

$$\mathcal{F}_2(\Phi, 1)\, \hat{v} = \Phi - f_n.$$

$$\begin{array}{ccc} P \times_X E & \longrightarrow & E \\ \downarrow & & \downarrow \\ P & \longrightarrow & X \end{array}$$

is trivial in the sense that there exists ~~a section~~ an isomorphism

$$P \times_X E \simeq P \times G .$$

Assuming that equivalence relations are effective, the usual calculations of principal bundles is valid for nonsense reasons!

Is this the situation for DG coalg? ~~Projective objects corresp. to X~~ no projectives, however, as a group is always smooth, a fibration is a smooth map. Therefore if we take contractible entirely primitive coalgebras?

Back to your construction of characteristic classes!
Given $\pi: E \to F$ take tangent bundle to fibers $T_\pi$
form ~~Pontryagin classes~~ some polynomial in the Pontryagin classes $\varphi(T_\pi)$ and integrate over the fibers

$$\pi_*\{\varphi(T_\pi)\}.$$

In example $\underline{T_\pi \text{ dim } 2}$ so all P-classes should be 0. Thus

$$\mathcal{L}(T_\pi) = 1$$

---

My idea was that this should be same as taking fiber cobordism e.g.

$$\begin{tikzcd} E \arrow[r, "j"] \arrow[d, "\pi"'] & X \times \mathbb{R}^{n+N} \arrow[dl] \\ X & \end{tikzcd}$$

$j$ embedding over $X$.

~~This should be same as taking~~ get map

$$X \to \underline{\mathrm{Hom}}(S^{n+N}, MSO(N)).$$

or map $\quad S^{n+N} \wedge X \longrightarrow MSO(N)$

hence an element of $\tilde{H}^0(S^n X, \underrightarrow{MSO}) = \tilde{H}^n(X, \underrightarrow{MSO})$

# Problem: Algebraic Models for manifold theory.

A (smooth) thickening of a finite complex $K$ is a homotopy equivalence $K \to X$ where $X$ is a smooth manifold with boundary. In stable range $\dim M \gg 2 \dim K$

$$\{\text{Thickenings of } K\} \xrightarrow{\sim} [K, BO] = \widetilde{KO}(K).$$

$$\{K \xrightarrow{i} M\} \longmapsto i^*(\tau_M).$$

Similarly $[K, BPL]$ = PL thickenings of $K$. Given a thickening one has by Lefschetz duality a triangle

$$\begin{tikzcd}
A_c(X) \arrow[rr] & & A(X) \simeq A(K) \arrow[dl] \\
& A(\partial X) \arrow[ul, dotted] &
\end{tikzcd}$$

where the upper arrow is zero in the stable range. ~~[illegible]~~ Note that ~~[illegible]~~ $A_c(X)$ is dual to $A(X)$ hence $A(\partial X)$ is a kind of hyperbolic space. In the case of the trivial thickening $A_c(X)$ is the reduced cochain algebra on the ~~[illegible]~~ Spanier-Whitehead dual of $X$. ~~[illegible]~~ One sees that the retraction $r : \partial X \to K$ is a sphere fibration ~~[illegible]~~ of dimension $\dim X - n$ iff $K$ is a Poincaré $n$-complex. If the thickening is trivial, then this is Spivak's ~~[illegible]~~ normal spherical fibre space $\nu$, so in general reasonable to suppose its the sum of $\nu$ + the spherical fibration coming from the thickening.

Given $E^\cdot$ Poincare $n$-complex, $L^\cdot$ perfect

$$f : L^\cdot \longrightarrow E^\cdot \xrightarrow{\phi} \mathrm{Hom}^\cdot(E^\cdot, k[n]) \xrightarrow{f^t} \mathrm{Hom}^\cdot(L^\cdot, k[n])$$

assume first that $f^t \phi f = 0$. Consider

$$L^\cdot[-1] + E^\cdot + \mathrm{Hom}^\cdot(L^\cdot, k[n+1]) = Q$$

with differential

$$\cancel{d(x, y, z) = f(x), y}$$

$$d(x, y, z) = (-dx, f(x) + dy, f^t\phi y - dz)$$

$$d(-dx, f(x) + dy, f^t\phi y - dz) = (d^2x, -f(dx) + df(x) + d^2y, \underbrace{f^t\phi}(f(x) + dy) - d(f^t\phi y - dz))$$

$$\boxed{d(x,y,z) = (-dx, fx + dy, f^t\phi y - dz).}$$

Also we have to define

$$\psi : Q \otimes Q \longrightarrow k[n].$$

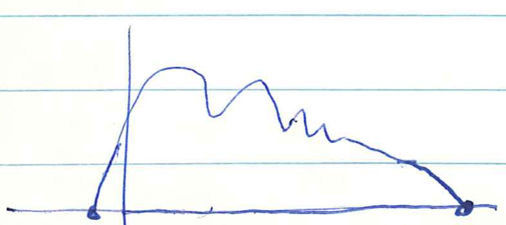
Let $f: X \longrightarrow Y$ be a continuous maps between smooth manifolds. Sard's thm tells us that if $f$ is smooth and proper, then most points of $Y$ are regular and that the index of such an inverse image is independent of the generic point! Moreover we can always smooth out any $f$ uniquely up to smooth homotopy!! Therefore perhaps if we smooth $f$ out in a (more or less) canonical way we can avoid Sard's thm. and get an explicit fmla for $I(f)$.

Nature of smoothing: uses a "linear" structure on $Y$ together with smoothing kernels on $X$. Thus one chooses kernels $k_\varepsilon(x,y)$ form of dim $n$ $\ni$ $k_\varepsilon(x,y) \longmapsto \delta(x,y)$ as $\varepsilon \longrightarrow 0$ and one uses the approximation

$$g_\varepsilon(x) = \int k_\varepsilon(x,y) f(y).$$

Then $$\lim_{\varepsilon\to 0} g_\varepsilon(x) = f(x).$$

What are the regular values of an average?



nowhere diff cont. fn.

Fejer approximation

# Algebraic Models for manifold theory!

Consider the category of **perfect** complexes of $k$ modules, where $k$ is a commutative ring, $D_{perf}(k)$. Then we can form the associated Grothendieck group $K(k)$. Recall this is a universal function $\chi : \mathrm{Ob}\, D_{perf}(k) \longrightarrow K(k)$ which is additive for triangles. Note that the isomorphism classes of objects form an abelian semi-group and that $\chi(\Sigma K^\cdot) = -\chi(K)$.

Prop: $K(k) \simeq K_{naif}(k)$ the Grothendieck group of f.t. projective $k$ modules.

Following Bass, we bring in quadratic forms. Def: A n-diml Poincare complex of $k$ modules is a perfect complex $K^\cdot$ together with a quasi-isomorphism $\phi : K^\cdot \longrightarrow \mathrm{Hom}^\cdot(K^\cdot; \mu[n])$ where $\mu$ is an invertible $k$ modules which is symmetric in the sense that the associated pairing $K^\cdot \otimes K^\cdot \longrightarrow \mu[n]$ $\alpha \otimes \beta \longmapsto \int \alpha\beta$ is symmetric. We will call $K, \phi, \mu$ orientable if $\mu \simeq k$ and oriented if an isomorphism $\mu \simeq k$ is given. From now on we shall work only with oriented Poincare complexes.

It is clear that the direct sum of n-diml Poincare cxs is again an Poincare cx. Call $K^\cdot, \phi : K^\cdot \longrightarrow \mathrm{Hom}^\cdot(K^\cdot; k[n])$ a boundary if $\exists$ triangle

Check this: $V$ is hyperbolic if $\exists$ map $W \xrightarrow{\varphi} V$ such that

$$0 \longrightarrow W \xrightarrow{\varphi} V \xrightarrow{\varphi^t} W^* \longrightarrow 0$$

is exact.

Thus we can form a Grothendieck group

---

Simplest situation: Let ~~[illegible]~~ $\pi$ be a group and consider perfect complexes of $k[\pi]$ modules ~~[illegible]~~ which satisfy P.D. for a given dimension $n$, i.e.

$$K^\cdot \otimes K^\cdot \longrightarrow k[n]$$
$$a \quad b \longmapsto \int ab$$

or better as quasi-isomorphism

$$K^\cdot \longrightarrow \mathrm{Hom}^\cdot(K^\cdot, \mu[n])$$

where $\mu$ is a 1-dimensional representation of $\pi$.

---

Then we get a possible bordism theory!!

$$E^\bullet + L^\bullet[-1] \longrightarrow E^\bullet + L^\bullet[-1] + \mathrm{Hom}^\bullet(L^\bullet, k[n+1]) \oplus E^\bullet \longrightarrow E^\bullet \oplus \mathrm{Hom}(L^\bullet, k[n+1])$$

~~Question~~

Problems: ① What is surgery?

② Does surgery preserve cobordism?

③ Can every cobordism be achieved by a sequence of surgeries?

④ Is a Poincaré n-cx. over k quasi-isomorphic to one such that ~~..~~ $\phi$ is an isomorphism?

⑤ What is the Clifford algebra of a Poincaré n-complex?

⑥ Classification of the surgery obstructions in the odd-dimensional case.

Probably not. In effect in the odd-dimensional case we believe to have constructed an obstruction in $K(X)/\{(a-a^*) \mid a \in K(X)\}$. This would always be zero if ④ is true

vector space $V$ with an isomorphism $\varphi : V \xrightarrow{\sim} V'$ $\ni$ $\varphi^t = \varphi$.
perfect complex $K^\bullet$ with an isom $\varphi : K^\bullet \xrightarrow{\sim} \mathrm{Hom}(K^\bullet; A)$ or equivalently a pairing $g : K^\bullet \otimes K^\bullet \longrightarrow A$. Cobordism.

<u>Problem</u>: Form a Grothendieck group out of such $K^\bullet$, $g$.

<u>Defn</u>: $(K^\bullet; \int)$ will be considered <u>trivial</u> if there is a triangle

$$
\begin{tikzcd}
L^\bullet \arrow[rr] & & M^\bullet \arrow[dl, "i"] \\
& K^\bullet \arrow[ul, "\delta"] &
\end{tikzcd}
$$

and a quasi-isomorphism $L^\bullet \longrightarrow \mathrm{Hom}(M^\bullet; A)$ given by

$$\cancel{\varphi}\; L^\bullet \otimes M^\bullet \longrightarrow A$$

such that

$$\cancel{M \otimes K \to K \otimes K}$$

$$
\begin{array}{ccccccc}
L & \longrightarrow & M & \longrightarrow & K & \longrightarrow & L \\
 & & \times & & \times & & \times \\
 & & L & \longleftarrow & K & \longleftarrow & M \\
 & & & & \downarrow & & \\
 & & & & A & &
\end{array}
$$

$\delta$ is adjoint to $i$.

# Algebraic Cobordism theory.

Let $k$ be a commutative ring containing $\frac{1}{2}$. By an *n-dimensional Poincare complex* over $k$ we mean a perfect complex of $k$ modules $E^\cdot$ together with a quasi-isomorphism $\phi: E^\cdot \to \mathrm{Hom}^\cdot(E^\cdot, k[n])$ which is symmetric in the sense that the associated pairing

$$\begin{aligned} E^\cdot \otimes E^\cdot &\xrightarrow{p} k[n] \\ x \otimes y &\longmapsto \phi(x)(y) \end{aligned}$$

satisfies $pT = p$, where $T$ is the interchange map. Note that if we are given a homotopy-symmetric pairing $p$, we may average it ($\frac{1}{2} \in k$) and obtain a symmetric pairing.

$$\begin{tikzcd} L^\bullet \arrow[rr, "f"] & & \mathrm{Hom}(L^\bullet, k[n+1]) \arrow[dl, "j"] \\ & K^\bullet \arrow[ul, dashed, "d"] & \end{tikzcd}$$

such that $f$ is symmetric and such that $d$ is adjoint to $j$.
More precisely this means that

$$\int x \cdot j\tilde{y} = \langle dx, \tilde{y} \rangle.$$

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One should be able to define cobordism groups and calculate them by surgery. It seems then that the cobordism groups should be related to the Grothendieck group of quadratic forms constructed by Bass. I can see how to surgery away a Poincare complex for which the pairing is non-degenerate on the chain level. But it even works in general! Thus if $H^q(K^\bullet) = 0$ for $q > m$ and $\neq 0$ for $q = m$ and $m > n/2$, then can take $L^\bullet = K^\bullet$ in dim $> n/2$

$L^\bullet \to K^\bullet$ $\qquad\qquad$ $K^\bullet \to \mathrm{Hom}(L^\bullet, k[n])$

$$\begin{tikzcd} \mathrm{Hom}(L^\bullet, k[n]) \arrow[rr, dashed, "?"] & & L^\bullet \arrow[dl] \\ & K^\bullet \arrow[ul] & \end{tikzcd}$$

so gives the cobordism.

Surgery: $L^\bullet \xrightarrow{f} E^\bullet \xrightarrow{f^t} \mathrm{Hom}(L^\bullet; k[n])$

$$+ \quad h : f^t \phi f \sim 0$$

to construct a new $\overset{\text{Poincare}}{\text{complex}}$ $K^\bullet$. $\qquad$ $\mathrm{Ker}\, f^t / \mathrm{Im}\, f$.

$$\cancel{E^\bullet \oplus \left(L^\bullet \oplus \mathrm{Hom}^\bullet(L^\bullet; k[n])\right)}$$

$$\cancel{\oplus\ L^\bullet[-1]\ \oplus}$$

$$E^\bullet \oplus L^\bullet[-1] \oplus \mathrm{Hom}^\bullet(L^\bullet; k[n+1]).$$

$$d\{x, \ y, \ z\} = \{dx, \overset{\pm fx}{dy}, dz \pm f^t \phi x\}.$$

$$\text{if} \quad f^t \phi f = 0.$$

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$$E^\bullet \oplus L^\bullet[-1]$$

$$E^\bullet \longrightarrow E^{\bullet\prime}$$

$$E^\bullet$$

$$E^\bullet \twoheadrightarrow I \hookrightarrow E^{\bullet\prime}$$

$$0 \to K(R) \to K(R[X]) \to K(R[X], R) \to 0.$$

Ask following question: ~~[illegible]~~ Suppose $M$ is projective over $R[X]$ and $N$ is an $R$ submodule of $M$ generating $M$. Then

$$M = \bigcup F_r N$$

where $F_r N = \sum_{s \le r} X^s N$

and for large $r$ we have the exact sequence

$$0 \to R[X] \otimes F_{r-1} N \to R[X] \otimes F_r N \to M \to 0$$

~~Then we assume that~~ One therefore sees that $F_r N / F_{r-1} N$ for large $r$ is an $R$ module. We would like to know whether

(a) $F_r N / F_{r-1} N$ depends only on $M$ or on the choice of $N$.

(b) whether $F_r N$ projective over $R$ for large $r$.

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hence $F_r N / F_{r-1} N$ proj $r$ large $\Rightarrow$ $F_r N$ proj. $r$ large

$$0 \to I \to A \to A/I \to 0$$

$\cancel{F_r N}$

$\cancel{F_r N \otimes I \to A \to F}$

$$\to Tor_1^I(F_r N, A/I) \to F_r N \otimes I \to F_r N \otimes A \to F_r N \otimes A/I \to 0$$

$$0 \to F_{r-1} N \to F_r N \to F_r N / F_{r-1} N \to 0$$

let $r \to \infty$

$$\to Tor_1^A(F_{r-1} N, A/I) \to Tor_1^A(F_r N, A/I) \to Tor_1^A(F_r N / F_{r-1} N, A/I) \to$$