~~Note~~ January 68.

Harish-Chandra claims $\exists$ canonical $\overset{\text{ring}}{\text{isom}}$

$$S(\mathfrak{g})^{\mathfrak{g}} \longrightarrow U(\mathfrak{g})^{\mathfrak{g}}$$

In the semi-simple case, independent of choice of positive roots.

He chooses $\mathfrak{h} \subset \mathfrak{g}$ and $\mathfrak{n}^+, \mathfrak{n}^-$ and defines

$$\gamma : U(\mathfrak{g})^{\mathfrak{g}} \longrightarrow U(\mathfrak{h})^{W}$$

to be $\beta = \varepsilon_- \otimes 1 \otimes \varepsilon_+$

followed by $\mathfrak{h} \longrightarrow \mathfrak{h}$

$H \mapsto H - \rho H \qquad \rho = \frac{1}{2}\sum_{\alpha \in P}' \alpha$

The claim is that the result is independent of the choice of P. Maybe obvious by char. fmla, e.g. γ is so arranged that

$$\chi_\lambda(z) = \langle \beta z, e^\lambda \rangle$$

$$= \langle \gamma z, e^{\lambda+\rho} \rangle$$

$$= \langle \gamma z, \underbrace{\frac{\det e^{\lambda+\rho}}{\det e^{\rho}} \frac{1}{\dim V_\lambda}} \rangle$$

$$\cancel{=} \; (\gamma z)\left(\frac{1}{\dim V_\lambda} \chi\right)(0).$$

which shows independence if done carefully

(2) Classify maximal ideals in $U(\mathfrak{g})$.

$U(\mathfrak{g}\times\mathfrak{g})$

Problem: What happens when k is ~~[illegible]~~ nilpotent?

$$U(\mathfrak{g})\otimes_k 1 \longrightarrow M$$

So ~~[illegible]~~ M is unipotent as a k module!

Can you show that M^1 is 1-dimensional. Yes because consider

~~$Hom_k(U(\mathfrak{g}), 1)$ this should~~

$Hom\ (U(\mathfrak{g}), 1)$ this is an injective of module of nilpotent as well as k.

So $Homcont.\ (U(\mathfrak{g}), 1)$

is clearly the injective hull of the trivial rep of k!

So consider

$$M \longrightarrow Hom_k(U(\mathfrak{g}), \widetilde{Hom}(U(k), 1))$$

Can you establish a general isomorphism

$$S(\mathfrak{g})^{\mathfrak{g}} \xrightarrow{\sim} U(\mathfrak{g})^{\mathfrak{g}} ?$$

Idea: An irreducible representation of $\mathfrak{g}$ should determine a character on Z and an orbit of $\mathfrak{g}$ in $\mathfrak{g}'$.

Burnsides thm. only holds for f.d. Hopf algs.

Generalize your results to when k is nilpotent in $\mathfrak{g}$.

Let ~~$\mathfrak{g}$~~ $\mathfrak{n} \subset \mathfrak{g}$ be positive root space and let M be a $\mathfrak{g}$ module with a vector killed by $\mathfrak{n}$, hence ~~all [illegible]~~ M is unipotent under $\mathfrak{n}$. ~~Let~~ $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$ is the normalizer of $\mathfrak{n}$. ① Does $\mathfrak{b}$ act on $M^{\mathfrak{n}}$?

~~$b\in$~~ $b \in \mathfrak{b}$ $\quad m \in M^{\mathfrak{n}}$.

$$n\,bm = \left[\underline{[n,b]} + b\underline{n}\right] m$$

$$= 0$$

Yes.

② Let $N \subset M^{\mathfrak{n}}$ be a $\mathfrak{b}$ submodule.

Problem: Can you show that $U(\mathfrak{g})N \cap M^{\mathfrak{n}} = N$.

~~Yes because~~ ~~α~~

Problems:

1) Calculate the answer for $sl(2,\mathbb{C})$ or read Borgman

2) The H-C procedure consists in inducing characters from ~~the~~ various Cartan subalgebras. Determine whether these are all.

3) Is there always a k rep. of multiplicity 1? —minimal as in PRV.

4) Determine Ω_Λ and especially $(\Omega_\Lambda)_{ab}$ if 3) is true.

Assuming 3 calculate the character on Ω_Λ by means of a canonical map

$$(\Omega_\Lambda)_{ab} \longrightarrow J.$$

A basic theorem: R f.d. algebra over $\mathbb{C}$, Λ an irred representation of R (nec. of finite dim) $\chi_\Lambda : R \to \mathbb{C}$ its trace. Claim χ_Λ completely determines Λ.

Proof: Let N be the radical of R. Then as trace of a nilpotent transf is 0 we have

$$\chi_\Lambda(r_1 r_2) = 0 \quad \text{if } r_1 \text{ or } r_2 \in N$$

so $\chi_\Lambda(N) = 0$ and $\chi_\Lambda : R/N \longrightarrow \mathbb{C}$. By Wedderburn $R/N = \prod R_i$ where R_i are simple. Then tr non-zero on ~~$\Lambda_1 \neq \Lambda_2$, then~~ ~~$\chi_{\Lambda_1} \neq \chi_{\Lambda_2}$~~ R_Λ and 0 on others

$$\parallel$$

$\dim \Lambda$ on 1.

Proposition:

~~Theorem:~~ Let M be an irred. $\mathfrak{g}$ module ~~[illegible]~~, and let M_1 be a $\mathfrak{g}$ module generated by M_1^Λ. Then

$$\mathrm{Hom}_{\mathfrak{g}}(M_1, M) \xrightarrow{\sim} \mathrm{Hom}_{\Omega_\Lambda}(\mathrm{Hom}_k(\Lambda, M_1), \mathrm{Hom}_k(\Lambda, M)).$$

Proof: Clearly 1-1 because if $\varphi: M_1 \to M$ ~~[illegible]~~ kills every map $\Lambda \to M_1$ over k, then φ kills generators of M_1 and so φ is 0. Conversely given $\psi: \mathrm{Hom}(\Lambda, M_1) \to \mathrm{Hom}_k(\Lambda, M)$ ψ must be onto by irreducibility of $\mathrm{Hom}_k(\Lambda, M)$.

$$\begin{array}{ccc}
(U(\mathfrak{g}) \otimes_k \Lambda) \otimes_{\Omega_\Lambda} \mathrm{Hom}_k(\Lambda, M_1) & \xrightarrow{1\otimes 1\otimes\psi} & (U(\mathfrak{g}) \otimes_k \Lambda) \otimes_{\Omega_\Lambda} \mathrm{Hom}_k(\Lambda, M) \\
\downarrow \rho_1 & & \downarrow \rho \\
M_1 & & M
\end{array}$$

The problem is to show that $(1\otimes 1\otimes\psi) \operatorname{Ker}\rho_1 \subset \operatorname{Ker}\rho$. ~~Ker ρ~~ However we know $\operatorname{Ker}\rho$ is the largest $\mathfrak{g}$ submodule of target of $1\otimes1\otimes\psi$ which is contained in the k-subspace disjoint from $1\otimes\Lambda\otimes\mathrm{Hom}_k(\Lambda, M)$. Thus have to ~~show~~ show that $(1\otimes 1\otimes\psi)\operatorname{Ker}\rho_1$ has no ~~[illegible]~~ k subreps of type Λ. This will follow if $\operatorname{Ker}\rho_1$ has no Λ reps. But we know that

$$\begin{array}{c}
\mathrm{Hom}_k(\Lambda, \underline{(U(\mathfrak{g}) \otimes_k \Lambda)} \otimes_{\Omega_\Lambda} N) \\
\wr\uparrow \\
N,
\end{array}$$

so it's clear.

Problem: Construct a canonical map

$$\Lambda \longrightarrow \mathrm{Hom}_k(\mathcal{U}(\mathfrak{g}), \Lambda).$$

Two possibilites:

(i) $\mathcal{U}(\mathfrak{g}) = e(S(\mathfrak{p}))\,\mathcal{U}(\mathfrak{k})$

(ii) $\mathcal{U}(\mathfrak{g}) = \cancel{\mathcal{U}(\mathfrak{b})\,\mathcal{U}(\mathfrak{k})}$

$= \mathcal{U}(\underbrace{\mathfrak{a}+\mathfrak{n}}_{\text{solvable}})\cdot\mathcal{U}(\mathfrak{k})$

Example: Take $\Lambda = 1$. Then want a k invariant in

$$\mathrm{Hom}_k(\mathcal{U}(\mathfrak{g}), 1)$$

want an element of $(\Omega_1)'$

$$\begin{array}{ccc} \Omega_1 & \longrightarrow & S(\mathfrak{a})^W \\ & & \Big\downarrow {\scriptstyle \text{evaluation on } \rho} \\ & & \mathbb{C}. \end{array}$$

Canonical linear functional on Ω_1, homomorphism augmentation.

Therefore I get a canonical element of $(\Omega_1)'$ which in fact is a character. Same under (i)+(ii).

Mackey coset formula

$$\begin{tikzcd} & \overset{M}{H} \arrow[dr, "\iota"] & \\ T \arrow[ur, "j'"] \arrow[dr, "\iota'"'] & & G \\ & \underset{L}{K} \arrow[ur, "j"'] & \end{tikzcd}$$

$$\mathrm{Hom}_K(L, j^* \iota_* M) = \mathrm{Hom}_K(L, \mathrm{Hom}_H(G, M))$$

$$= \mathrm{Hom}_{H\times K}(G, \mathrm{Hom}(L,M))$$

$$= \Gamma(G \times_{K\times H} \mathrm{Hom}(L,M))$$

$$\underbrace{\mathrm{Hom}_G(j_! L, \iota_* M)}_{\text{A kind of group of cohomological correspondences!}} = \Gamma(K\backslash G/H,\ G \times_{(K\times H)} \mathrm{Hom}(L,M))$$

Take special case where there is a single coset, i.e. K acts transitively on G/H and assume that $T = K \cap H$. Then get

$$\boxed{\mathrm{Hom}_G(j_! L, \iota_* M) = \mathrm{Hom}_T(\iota'^* L, j'^* M).}$$

Note that

Example: $\Lambda = 1$. Want distributions φ on G biinvariant under K with support in K. Then get a invariant differential operator D on G/K by

$$(Df)\circ\pi = \varphi * (f\circ\pi)$$

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biinvariance under K means

$$\delta_k * \varphi = k^{-1}\circ\varphi$$
$$\varphi * \delta_k = \varphi\circ k^{-1}$$

Proof: for functions

$$(\delta_k * \varphi)_g = \int \delta_k(gx^{-1})\,\varphi_x$$
$$= \varphi_{k^{-1}g} = k^{-1}\varphi_g.$$

Conjecture: Ω_Λ is the subalgebra of those distributions on G with values in $\mathrm{Hom}(\Lambda,\Lambda)$ which satisfy

(i) biinvariant under k: $\delta_k * \varphi = k^{-1}\circ\varphi$, $\varphi * \delta_k = \varphi\circ k^{-1}$

with $\int f\,\delta_k = f(k)\cdot id_\Lambda$

(ii) have support in K.

Conjecture: $\Omega_\Lambda \simeq S(\mathfrak{a})^W \otimes \mathrm{Hom}_M(\Lambda,\Lambda)$

some trouble between M and $\mathfrak{m}$?

$\Longrightarrow$ one gets characters only when Λ contains a multiplicity 1 rep of M.

Two problems

1. Irreducibility of induced representations with dominant weight vector.
$$U(\mathfrak{g}) \otimes_{\mathfrak{b}_+} \lambda_0 \hookrightarrow \mathrm{Hom}_{\mathfrak{b}_-}(U(\mathfrak{g}), \lambda_0)$$

2. Action of $\mathfrak{k}$ on the resulting $\mathfrak{g}$ module

$$\mathrm{Hom}_{\mathfrak{k}}(\Lambda, \mathrm{Hom}_{\mathfrak{b}_-}(U(\mathfrak{g}), \lambda_0))$$
$$\Vert\wr$$
$$\mathrm{Hom}_{\mathfrak{b}_- \times \mathfrak{k}}(U(\mathfrak{g}), \mathrm{Hom}(\Lambda, \lambda_0))$$

Now we shall assume that $\mathfrak{g} = \mathfrak{k} + \mathfrak{b}_-$ or equivalently that K_0 acts transitively on G/B_-. In the complex case this is true. In the principal series $\mathfrak{b}_+ = \mathfrak{m} + \mathfrak{a} + \mathfrak{n}$ is parabolic so $\mathfrak{k} + \mathfrak{a} + \mathfrak{n} = \mathfrak{g} \Rightarrow \mathfrak{g} = \mathfrak{k} + \mathfrak{b}_+ \Rightarrow \mathfrak{g} = \mathfrak{k} + \mathfrak{b}_-$.

The general case not clear

~~The Wang situation is that~~

~~$\mathfrak{g} = \mathfrak{a}$~~

$$\mathrm{Hom}_{\mathfrak{g}}(U(\mathfrak{g})\otimes_k \Lambda_1, U(\mathfrak{g})\otimes_k \Lambda_2) \cong [U(\mathfrak{a}) \otimes \mathrm{Hom}_M(\Lambda_1, \Lambda_2)]^W.$$

<u>Question</u> Is there an analogue of the isom $U(\mathfrak{g})^{\mathfrak{g}} = S(\mathfrak{g})^{\mathfrak{g}}$, i.e. is above isomorphic to

$$\mathrm{Hom}_{\tilde{\mathfrak{g}}}(U(\tilde{\mathfrak{g}})\otimes_k \Lambda_1, U(\tilde{\mathfrak{g}})\otimes_k \Lambda_2)$$

where
$$\tilde{\mathfrak{g}} = \mathfrak{p} \times_\sigma k \qquad \text{semi-direct product.}$$

Thus is there a correspondence between irreducible representations of the homogeneous Lorentz group and irred. reps. of the group of Euclidean motions? In the former things are parametrized by pairs (k_0, c) where k_0 $\frac{1}{2}$ integer + c single cx no. Same is true for the latter by Mackey's theory c playing the role of the radius of the sphere.

For $sl(2,\mathbb{R})$. $\tilde{\mathfrak{g}} = \mathfrak{p} \times_\sigma k$ Euclidean motions in the plane (solvable - not Heisenberg)

Calculate

$$\left[U(\mathfrak{a}) \otimes \mathrm{Hom}_M(\Lambda_1, \Lambda_2)\right]^W \longleftarrow U(\mathfrak{a})^W$$

in the complex case!! and determine when the principal series are irreducible.

In this case $\mathfrak{a} \simeq \mathfrak{m}$ + M is connected and $W =$ Weyl group of k. Everything can be done in the k framework

i.e. Have to calculate

$$\left[U(\mathfrak{h}) \otimes \mathrm{Hom}_{\mathfrak{h}}(\Lambda_1, \Lambda_2)\right]^W \qquad W = \text{ordinary Weyl group.}$$

$$\left[U(\mathfrak{h}) \otimes \Lambda^{\mathfrak{h}}\right]^W$$

Chose a hom. $\chi : U(\mathfrak{h})^W \longrightarrow \mathbb{C}$. & let $U(\mathfrak{h})_\chi$ be the ring $U(\mathfrak{h}) \otimes_{U(\mathfrak{h})^W} \mathbb{C}$. Assume the generic case. then $U(\mathfrak{h}) \otimes_{U(\mathfrak{h})^W} \mathbb{C}$ is the product of fields ie $\mathrm{Hom}(W, \mathbb{C})$

$$U(\mathfrak{h})_\chi \simeq \mathrm{Hom}(W, \mathbb{C}) \qquad \text{so}$$

$$\mathbb{C}_\chi \otimes_{U(\mathfrak{h})^W} \left[U(\mathfrak{h}) \otimes \mathrm{Hom}_M(\Lambda_1, \Lambda_2)\right]^W = \left[U(\mathfrak{h})_\chi \otimes \mathrm{Hom}_M(\Lambda_1, \Lambda_2)\right]^W$$

$$= \left[\mathrm{Hom}(W, \mathrm{Hom}_M(\Lambda_1, \Lambda_2))\right]^W$$

$$= \mathrm{Hom}_M(\Lambda_1, \Lambda_2)$$

So we recover Bruhat's theory when $\{$ in this case χ is not on walls.

Somehow the idea is that the category is now equivalent to ~~#~~ k sheaves on $\mathfrak{p}'$, as follows. Conjecturally

$$\left[U(\mathfrak{a}) \otimes \mathrm{Hom}_M(\Lambda_1, \Lambda_2)\right]^W \simeq \mathrm{Hom}_{k, S(\mathfrak{p})}(S(\mathfrak{p}) \otimes_k \Lambda_1, S(\mathfrak{p}) \otimes_k \Lambda_2).$$

This doesn't hold water because of the integer conditions.

$$\left[U(\mathfrak{a}) \otimes \mathrm{Hom}(\Lambda_1, \Lambda_2)\right]^{N_A} \simeq \left[S(\mathfrak{a}) \otimes \mathrm{Hom}(\Lambda_1, \Lambda_2)\right]^{N_A}$$

$\underline{S(\mathfrak{p})}$

NO NO NO. You get contradiction because you cannot see the integer conditions from this point of view. Nothing in your argument prevents you from applying same argument to get ~~that~~ an isomorphism with the $\tilde{\mathfrak{g}}$ situation where integer conditions do not occur.

Idea: In passing from $\tilde{\mathfrak{g}}$ to $\mathfrak{g}$ involves these integer fudge factors like going from z to e^z.

Conjectural situation: Have established a map

$$\mathrm{Hom}_{\mathfrak{g}}(U(\mathfrak{g})\otimes_k \Lambda_1, U(\mathfrak{g})\otimes_k \Lambda_2) \longrightarrow U(\tilde{\mathfrak{n}}) \otimes \mathrm{Hom}(\Lambda_1, \Lambda_2)$$

namely apply functor $\quad U(\mathfrak{n}) \underset{U(\mathfrak{a}+\mathfrak{n})}{\otimes}$.

This map is compatible with composition and we want to determine its images. Idea is to make N_A = normalizer of A in K act on $U(\mathfrak{n}) \otimes \mathrm{Hom}(\Lambda_1, \Lambda_2)$ in a reasonable way so that the image is the invariants of the action.

~~The issue~~

Go back to Iwasawa decomp.

Fix Λ problem is given an irreducible Ω_Λ module.

Suppose ~~a~~ V is an irreducible Π.

Let ν be an irred. representation of M, let $\lambda \in \mathfrak{a}'$ $\lambda(H_i) \geq 0$ all i. Define the principal series reps $\pi_{\lambda,\nu}$.

Idea somehow is to have k structure

$$\bigoplus_{\Lambda} \Lambda \otimes \mathrm{Hom}_M(\Lambda, \nu)$$

hence for ~~if~~ Λ to occurs with mult 1 means that $\mathrm{Hom}_M(\Lambda, \nu)$ is 1 dimensional. Does this always happen?

Consider $\mathrm{Hom}_M(K, \nu)$

$$K/M \overset{\text{s.s.}}{\simeq} \text{reg orbit in } \mathfrak{a}'$$

Basic questions Is M real!

$$\left(U(\mathfrak{g})\otimes_k \Lambda\right)\otimes_{\Omega_\Lambda}\Lambda = \left(\bigoplus_{\Lambda_1} U(\mathfrak{a})^W \otimes \Lambda_1 \otimes \mathrm{Hom}_M(\Lambda_1,\Lambda)\right) \underset{U(\mathfrak{a})^W\otimes \mathrm{Hom}_M(\Lambda,\Lambda)}{\otimes} \mathrm{Hom}_M(\mu,\Lambda)$$

$$= \bigoplus_{\Lambda_1} \Lambda_1 \otimes \mathrm{Hom}_M(\Lambda_1,\Lambda) \otimes_{\mathrm{Hom}_M(\Lambda,\Lambda)} \mathrm{Hom}_M(\mu,\Lambda)$$

variance is wrong.

new defn

$$\Omega_\Lambda = \mathrm{Hom}_k(\Lambda, U(\mathfrak{g})\otimes_k \Lambda) = \mathrm{End}_{\mathfrak{g}}(U(\mathfrak{g})\otimes_k\Lambda)$$

$$\Omega_\Lambda \to U(\mathfrak{a})^W \otimes \mathrm{End}_M(\Lambda).$$

$N = \mathrm{Hom}_k(\Lambda, M)$ is a right Ω_Λ module
$= \mathrm{Hom}_M(\Lambda,\mu)$ where $U(\mathfrak{a})^W$ acts as it should

$$N \otimes_{\Omega_\Lambda} \left(U(\mathfrak{g})\otimes_k \Lambda\right)$$

$$\cong$$

$$\mathrm{Hom}_M(\Lambda,\mu) \underset{U(\mathfrak{a})^W\otimes \mathrm{Hom}_M(\Lambda,\Lambda)}{\otimes} \bigoplus_{\Lambda_1} \Lambda_1 \otimes U(\mathfrak{a})^W \otimes \mathrm{Hom}_M(\Lambda_1,\Lambda)$$

$$\|$$

$$\bigoplus_{\Lambda_1} \Lambda_1 \otimes \left[\mathrm{Hom}_M(\Lambda,\mu) \otimes_{\mathrm{Hom}_M(\Lambda,\Lambda)} \mathrm{Hom}_M(\Lambda_1,\Lambda)\right].$$

$$\mathrm{Hom}(W,1) \otimes_{\mathrm{Hom}(W,W)} \mathrm{Hom}(V,W) = \mathrm{Hom}(V,1).$$

clear ✓

choose $\lambda_0 \in W'$. $w \in W \ni \lambda_0(w) = 1$

$$\lambda_0 \otimes (\alpha \otimes w) \longleftarrow \alpha$$
$$\longrightarrow$$

Conclude that

$$N \otimes_{\Omega_\Lambda} \left(U(\mathfrak{g}) \otimes_k \Lambda\right) \cong \bigoplus_{\Lambda_1} \Lambda_1 \otimes \mathrm{Hom}_M(\Lambda_1, \mu)$$

as K modules

In particular we get $\mathrm{Hom}_M(\Lambda_1, \Lambda_1)$ action which I must ~~show~~ determine when irred.

Problem: Started with N coming from $\mathrm{Hom}_M(\Lambda, \mu)$, then calculate that in some degree

$$\mathrm{Hom}_k\left(\Lambda_1, N \otimes_{\Omega_\Lambda} \left(U(\mathfrak{g}) \otimes_k \Lambda\right)\right) \cong \mathrm{Hom}_M(\Lambda_1, \mu).$$

we have to determine when this is irreducible over $\mathrm{Hom}_M(\Lambda_1, \Lambda_1)$. Looks like it always is ~~expt~~ except that we haven't analyzed how char. on $S(\mathfrak{a})^W$ interferes.

II. Suppose Ω_Λ known. When is

$$N \otimes_{\Omega_\Lambda} (U(\mathfrak{g}) \otimes_k \Lambda) \quad \text{irreducible.}$$

I have to calculate the right Ω_{Λ_1} module

$$\begin{gathered} \operatorname{Hom}_k(\Lambda_1, N \otimes_{\Omega_\Lambda} (U(\mathfrak{g}) \otimes_k \Lambda)) \\ \downarrow \wr \\ N \otimes_{\Omega_\Lambda} (\operatorname{Hom}_k(\Lambda_1, U(\mathfrak{g}) \otimes_k \Lambda)) \end{gathered}$$

$$\begin{tikzcd} & U(\mathfrak{g}) \otimes_k \Lambda \arrow[d] \\ \Lambda_1 \arrow[ur, dashed] \arrow[r] & N \otimes_{\Omega_\Lambda} (U(\mathfrak{g}) \otimes_k \Lambda) \end{tikzcd}$$

if ~~I true,~~ then

$$N \otimes_{\Omega_\Lambda} \operatorname{Hom}(\ , \ )$$

But in any case have only to decide on irred of

$$N \otimes_{\cancel{\operatorname{End}_{\mathfrak{g}}(j_! \Lambda)}} \operatorname{Hom}_{\mathfrak{g}}(j_! \Lambda_1, j_! \Lambda)$$

as a $\operatorname{End}_{\mathfrak{g}}(j_! \Lambda)$ module. Depends only on the formula for these rings so calculate

Conjecture: $\Omega_\Lambda = \mathrm{End}_{\mathfrak{g}}(U(\mathfrak{g}) \otimes_k \Lambda)$

$$\simeq S(\mathfrak{a})^W \otimes \mathrm{Hom}_M(\Lambda, \Lambda).$$

Application to irreducibility.

Let N be an irreducible right Ω_Λ module. From the conjecture there is a hom $\rho : S(\mathfrak{a})^W \to \mathbb{C}$ and an irreducible $\mathrm{Hom}_M(\Lambda,\Lambda)$ module structure on N such that

$$n(j \otimes \varphi) = \rho(j)\, n\varphi \qquad j \in S(\mathfrak{a})^W \quad n \in N$$
$$\varphi \in \mathrm{Hom}_M(\Lambda,\Lambda)$$

Now M is ~~a reductive~~ an algebraic group reductive in K so

$$\Lambda \simeq \bigoplus_\mu \mu \otimes \mathrm{Hom}_M(\mu, \Lambda)$$

where μ runs over the irred fin. reps of M. Hence N irreducible over $\mathrm{Hom}_M(\Lambda,\Lambda) \Longleftrightarrow$ ~~$N \simeq \mathrm{Hom}_M(\mu,\Lambda)$ with compositions~~

$$N \simeq \mathrm{Hom}_M(\Lambda, \mu)$$

with composition action. Look at induced module

$$N \otimes_{\Omega_\Lambda} (U(\mathfrak{g}) \otimes_k \Lambda)$$

as a k module.

$$N \otimes_{\Omega_\Lambda} (U(\mathfrak{g}) \otimes_k \Lambda) = \bigoplus_{\Lambda_1}^{\Lambda_1 \otimes} \mathrm{Hom}_k \left(\Lambda_1, N \otimes_{\Omega_\Lambda} (U(\mathfrak{g}) \otimes_k \Lambda)\right)$$

$$= \bigoplus_{\Lambda_1} \Lambda_1 \otimes \underbrace{\left[N \otimes_{\Omega_\Lambda} \mathrm{Hom}_k (\Lambda_1, U(\mathfrak{g}) \otimes_k \Lambda) \right]}_{\displaystyle \shortparallel \atop \displaystyle N \otimes_{S(\mathfrak{a})^W \otimes \mathrm{Hom}_M(\Lambda, \Lambda)} \left(S(\mathfrak{a})^W \otimes \mathrm{Hom}_M(\Lambda_1, \Lambda) \right)}$$

$$= \bigoplus_{\Lambda_1} \Lambda_1 \otimes \mathrm{Hom}_M(\Lambda_1, \mu).$$

$\exists$ a canonical k module isom.

$$\boxed{\bigoplus_{\Lambda_1} \Lambda_1 \otimes \mathrm{Hom}_M(\Lambda_1, \mu) \cong N \otimes_{\Omega_\Lambda} (U(\mathfrak{g}) \otimes_k \Lambda)}$$

where $N = \mathrm{Hom}_M(\Lambda, \mu)$ and the character on $S(\mathfrak{a})^W$ is chosen.

of PRV

To irreducible representation μ of M there belongs a principal series depending on a character $S(\mathfrak{a})^W \to \mathbb{C}$.

of PRV

$$P \times \Lambda^{\mathbb{Z}} \longleftarrow A \times \Lambda^{\mathbb{Z}}$$

Claim goes onto each K orbit ✓

$$\begin{array}{ccc} K\backslash P \times \Lambda^{\mathbb{Z}} & \xleftarrow{\varphi} & A \times \Lambda^{\mathbb{Z}} \\ & & \underline{W\backslash A} \times M\backslash\Lambda^{\mathbb{Z}} \\ \downarrow & & \\ K\backslash P & \longleftarrow & W\backslash A \end{array}$$

for regular orbits:

$\varphi(a \times \lambda^{\mathbb{Z}}) = \varphi(a_1 \times \lambda_1)$ i.e. $\exists\, k \ni$

$k a k^{-1}, k\lambda = a_1, \lambda_1$

Let $N \subset K$ be the normalizer of A. Then as a, a_1 are generic it follows that $k \in N$.

So our orbit space is $N\backslash\underline{A} \times \Lambda$.

But $\underline{N = W \times M}$? no.

$$\boxed{0 \to M \to N \to \underline{W} \to 0}$$ exact seq.

$[U(\mathfrak{a}) \otimes \mathrm{Hom}(\Lambda, \Lambda)]^N$.

core

Theorem: $\mathrm{Hom}_{\mathfrak{g}}(U(\mathfrak{g})\otimes_{\mathfrak{k}}\Lambda_1, U(\mathfrak{g})\otimes_{\mathfrak{k}}\Lambda_2) \simeq [U(\mathfrak{a})\otimes \mathrm{Hom}(\Lambda_1,\Lambda_2)]^N$

where $N =$ ~~centralizer~~ normalizer of $\mathfrak{a}$ in $\tilde{K}$.

Exact sequence:

$$\phi \to M \to N \to W \to 1$$

so

$$[U(\mathfrak{a})\otimes \mathrm{Hom}(\Lambda_1,\Lambda_2)]^N \simeq [U(\mathfrak{a})\otimes \underline{\mathrm{Hom}_M(\Lambda_1,\Lambda_2)}]^W$$

There may a good reason for W to act trivially on

$$\mathrm{Hom}_M(\Lambda_1,\Lambda_2).$$

(e.g. in the complex case $\underline{W}$

$$W \xrightarrow{\Delta} W\times W$$

$$0 \to \mathfrak{h} \xrightarrow{\Delta} \mathfrak{h}\times\mathfrak{h} \to \mathfrak{a} \to 0$$

$\mathfrak{m} = \Delta\mathfrak{h}$

$$\{(w_1,w_2) \mid \cancel{(w_1 h, w_2 h)}\ \cancel{w_1 h = w_2} \quad w_1 = w_2\}.$$

usual Weyl gp. of $\mathfrak{k}$.

let W act on $\boxed{\mathrm{Hom}_{\mathfrak{k}}(\Lambda_1,\Lambda_2)}$. $\boxed{\text{Interesting action here}}$

So action is not trivial

Check carefully.