(O: The sticky point is still ~~that~~ to show that the resulting element of $K_1 A$ ~~is well~~ depends only on b. I think I have a way to do this. This is my injectivity argument. It applies to $A \subseteq \begin{pmatrix} A & Q \\ P & B \end{pmatrix} = C$ use ${}_A Q$ flat which amounts to B being left flat.

I should still write out my arguments. I know that $C = \begin{pmatrix} A \\ P \end{pmatrix} \otimes_A (A \;\; Q)$ so $K_1 A \twoheadrightarrow K_1 C$.

~~Other technique. If A is left + right A-flat ...~~

$A \xrightarrow{u} B$ homom. + M eq., B left flat say

$$K_1(A) \longrightarrow K_1\begin{pmatrix} A & Q \\ P & B \end{pmatrix} \xrightarrow{\sim} K_1\begin{pmatrix} B & B \\ B & B \end{pmatrix} = K_1(B)$$

SAVE

$A \to B$ a hom. + M.eq. $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$

$$\begin{pmatrix} A & A \\ A & A \end{pmatrix} \longrightarrow \begin{pmatrix} A & Q \\ P & B \end{pmatrix} \longrightarrow \begin{pmatrix} B & B \\ B & B \end{pmatrix}$$

$$\begin{pmatrix} A & Q \\ P & B \end{pmatrix} = C$$

$$\begin{tikzcd}
 & K_1 A \arrow[r] \arrow[dl, "\sim"'] \arrow[d] & K_1(B) \arrow[d, "\cong"] \\
K_1(M_2 A) \arrow[r] & K_1 C \arrow[r] & K_1(M_2 B) \\
 & K_1 B \arrow[u] \arrow[ur, "\simeq"'] &
\end{tikzcd}$$

reduces ~~to~~ surj to subrings.

Also find $K_1 B \twoheadrightarrow K_1 C$

Next idea: $C = \begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ need $P \otimes_A Q \xrightarrow{\simeq} B$ need $AQ = Q$

assume Q A-flat $\Rightarrow$ $Q = \varinjlim F_i = \varinjlim AF_i$

$$\begin{matrix} A & AF_i \\ P & P \otimes_A AF_i \end{matrix} \qquad \begin{matrix} A & AF_i \\ F_i^* A & F_i^* A \otimes_A AF_i \end{matrix} \longrightarrow$$

(i $\qquad P \longrightarrow F_i^* A$

$$Q \otimes P \to A \qquad P \longrightarrow \mathrm{Hom}_A(Q, A) \longrightarrow \mathrm{Hom}_A(F_i, A) = F_i^* A.$$

The actual pairing $Q \otimes P \to A$ can be 0. I think in this argument? But I want A, C to be idempotent, until I find another way to define $K_1 A$. Thus $A = A^2 + QP$ can be 0

$P = PA + \underset{PQP}{\underline{BP}}$ $\qquad Q = AQ + \underset{QPQ}{\underline{QB}}$

$$A = A^2, \quad P = PA, \quad Q = AQ, \quad B = P \otimes_A Q$$

Take $Q = 0$. $\qquad C = \begin{pmatrix} A & 0 \\ P & 0 \end{pmatrix}$ $\qquad$ ~~claim $K_* A$~~

assume $A^2 = A$, $PA = P$ claim $K_* A \to K_* P$ is injective.

$$\begin{matrix} A & A \\ P & P \end{matrix}$$

Can you build up by surjective things

Try $\quad A \otimes A \longrightarrow A$

$$(A \oplus X) \otimes_{\mathbb{Z}} A \longrightarrow A$$

$$\begin{tikzcd} & A \arrow[d] \\ X \arrow[r] \arrow[ur, dashed] & \mathrm{Hom}_{A^{op}}(A, A) \end{tikzcd}$$

$$B = A \otimes_A (A \oplus X) \simeq A \oplus X$$

---

translate.
$$\begin{pmatrix} A \\ P \end{pmatrix} \otimes_{\begin{pmatrix} A \\ P \end{pmatrix}} \begin{array}{cc} A' & X \\ \left(\begin{pmatrix} A \\ P \end{pmatrix} \right. & \left. \begin{pmatrix} Q \\ P \otimes_A Q \end{pmatrix} \right) \end{array} = \begin{pmatrix} A \\ P \end{pmatrix} \otimes_A (A \quad Q)$$

you would like to write the pairing $X \otimes_{\mathbb{Z}} A' \longrightarrow A'$
$$\begin{pmatrix} Q \\ P \otimes_A Q \end{pmatrix} \otimes_{\mathbb{Z}} \begin{pmatrix} A \\ P \end{pmatrix}$$

(K waste a little time.

~~Let~~ Consider the buildup

$$(A, A, \mu) \longrightarrow \left(A, \begin{matrix} A \\ \oplus \\ P \end{matrix}, \mu \oplus 0 \right) \longrightarrow \left(A \oplus Q, \begin{matrix} A \\ \oplus \\ P \end{matrix}, \begin{pmatrix} \mu \\ \psi \end{pmatrix} \right)$$

$$\begin{pmatrix} A \\ P \end{pmatrix} \oplus \begin{pmatrix} A \\ \oplus \\ P \end{pmatrix} \otimes Q$$

$$(A, A, \mu) \subset (A, A \oplus P, \mu \oplus 0) \subset (A \oplus Q, A \oplus P, \mu \oplus \psi)$$

$$Q \otimes_{\mathbb{Z}} P \longrightarrow A$$

---

so I have $Q \otimes_{\mathbb{Z}} (A \oplus P) \longrightarrow A$

and I would like it to ~~comp~~ come from

$$Q \otimes_{\mathbb{Z}} (A \oplus P) \longrightarrow A \otimes_{\mathbb{Z}} (A \oplus P) \longrightarrow A$$

completely impossible.

Consider (Q, P, ψ) and suppose you want to enlarge Q to $Q \oplus Q'$, where $\psi' : Q' \otimes_{\mathbb{Z}} P \to A$ is given. Good case is where

$$\begin{tikzcd} & Q \arrow[d, "\psi"] \\ Q' \arrow[ur, dashed, "f"] \arrow[r, "\psi'"'] & \mathrm{Hom}_{A^{op}}(P, A) \end{tikzcd}$$

In fact you want more generally $Q \subset Q'$

$$\begin{tikzcd} & Q' \arrow[dl, dotted, shift left] \arrow[d, "\psi'"] \\ Q \arrow[ur, shift left] \arrow[r, "\psi"'] & \mathrm{Hom}_{A^{op}}(P, A) \end{tikzcd}$$

(2) So what seems to happen is that there are good points (Q, P, ψ) where

$$Q \xrightarrow{\sim} A \otimes_A \mathrm{Hom}_{A^{op}}(P, A).$$

~~That ...~~ I think all you have found out is that if you ~~try to~~ can enlarge (Q, P, ψ) by $Q \subset Q_1^{\oplus}$, but keeping P fixed, then you have

$$\begin{tikzcd} & Q_1^{\oplus} \arrow[dr, "\psi_1"] & \\ Q \arrow[ur, phantom, "\subset"] \arrow[rr, "\psi"] & & A \otimes_A \mathrm{Hom}_{A^{op}}(P, A) \\ & & {} \arrow[u] \end{tikzcd}$$

So if ψ yields $\cong \uparrow$, then

$Q_1^{\oplus} = Q \oplus Q'$ where $\psi_1^{\#}(Q', P) = 0$.

Actually for a C^*-alg. A it might be true that

$$A \xrightarrow{\sim} A \otimes_A \mathrm{Hom}_{A^{op}}(A, A)$$

You have the ~~...~~ left module map

$$A \longrightarrow \mathrm{Hom}_A(A, A)$$

You have the right module map

$$A \longrightarrow \mathrm{Hom}_{A^{op}}(A, A)$$

$$a \longmapsto (a' \longmapsto aa')$$

which is a right nil iso.
But it might happen that this is OKAY