$\langle \alpha \rangle$ Consider the following situation. Let $E \in \mathcal{P}(\tilde{A}^{op})$ $F \in \mathcal{P}(\tilde{B}^{op})$. Then consider Enrica stuff

$$F \xrightarrow{g} E \otimes_A Q \xrightarrow{P} F \qquad \text{1-pg } \cancel{\otimes} \text{ invertible}$$

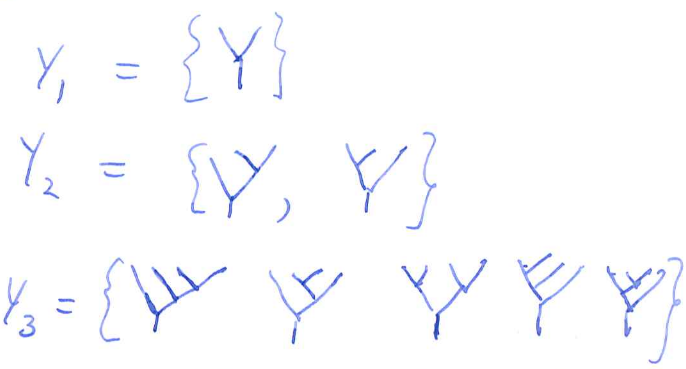
$$\Rightarrow \quad F \otimes_B P \longrightarrow E \longrightarrow F \otimes_B P \longrightarrow E$$

---

dialy homology. first bar type homology, Loday's version. You're given a tree with ~~vertices~~ leaves $0\ 1 \ldots n$ faces delete the leaves $i = 1, \ldots, n-1$ combined with either $*$ or $\bullet$ depending on whether the branch to the leaf comes in left or right. D's go between leaves

$$\overset{0}{\bullet}\ D\ \overset{1}{\bullet}\ D\ \overset{2}{\bullet}\ D\ \overset{3}{\bullet}$$

how many trees.



One of these trees is equivalent to a ~~item~~ system of parentheses.

$$a_0(a_1(a_2 a_3)) \quad a_0((a_1 a_2) a_3) \quad ((a_0 a_1)(a_2 a_3)) \quad ((a_0 a_1) a_2) a_3 \quad (a_0(a_1 a_2)) a_3$$

deleting i-th leaf means removing a_i.

~~letting ∂ ∂ D ∂ D ∂ D~~

Recall axioms. ~~$a_1\ a_2\ a_3$~~

$$a * a' = f(a) a' \qquad a \bullet a' = a f(a').$$

$$a_1 \cdot (a_2 \cdot a_3) = (a_1 \cdot a_2) \cdot a_3$$

$*$

$$a_1 \otimes a_2 \otimes a_3$$

$$a_1 \bullet (a_2 * a_3 - a_2 \cdot a_3) = 0$$

$$(a_1 * a_2 - a_1 \bullet a_2) * a_3 = 0$$

$$(a_1 * a_2) \bullet a_3 = a_1 * (a_2 \bullet a_3)$$

5

$\langle \beta \rangle$ 10/23 adjoint functors.

point: there are various equivalent pictures of the same category situation

- pair of adj functors ~~one of which~~ left adjoint is fully faithful
- ———————— ~~one~~ adj map $1 \to GF$ is iso
- reflection situation (cat, full sub, inclusion has right adj).
- cat $\mathcal{C}$ with endofunctor U and $\eta : U \to 1$ such that $\eta.U = U.\eta : U^2 \xrightarrow{\sim} U$.
- localization $\mathcal{C} \to \mathcal{C}[\Sigma^{-1}]$ admitting left adjoint.

~~what~~

- mult. sys. Σ $\ni$ i) two of $f, g, gf \in \Sigma \Rightarrow$ third does
  ii) $\forall M \; \exists \; M^{\#} \to M \in \Sigma \ni \mathrm{Hom}(M^{\#}, -)$ inv. Σ.

Spend a few minutes on an intrinsic K_1

Idea - suppose given A, $Q \otimes P \twoheadrightarrow A$. Objects are something like $P \otimes_A S$ ~~the sub~~ with an auto

$1 - pq$

$$\cancel{P \otimes_A T \xrightarrow{q} B \to P \otimes_A T}$$

$$P \otimes_A S \xrightarrow{q} B \otimes_B T \xrightarrow{p} P \otimes_A S$$

$$A \otimes_A S \xrightarrow{q} Q \otimes_B T \xrightarrow{p} A \otimes_A S$$

$q \in M_{kn}^{?}(Q)$

$p \in M_{nk}(P)$

What do you have? $1 - pq$, $pq \in M_{nn}(B)$.

use right modules

$$S \otimes_A Q \xrightarrow{p\cdot} T \xrightarrow{q\cdot} S \otimes_A Q$$

$$S \xrightarrow{p\cdot} T \otimes_B P \xrightarrow{q\cdot} S$$

$\langle 8 \rangle$ What does this mean? Suppose

$p \in M_{nk}(P) \qquad q \in M_{kn}(Q)$

$M_k A$	q
p	$M_n B$

$$A^k \xrightarrow{p\circ} P^n \xrightarrow{q\circ} A^k$$

$$B^n \xrightarrow{q\circ} Q^k \xrightarrow{p\circ} B^n$$

All I was going to try ~~want~~ was to see if I could make something more elaborate than that (p,q) $\quad p \in M_{nk}P \quad q \in M_{kn}Q \quad \ni \quad 1-pq \in M_n(B)$

It would seem that I can replace n by a free B^{op} module T and Ak by a free A^{op}-modules.

Then

$$S \xrightarrow{p\circ} T \otimes_B P \xrightarrow{q\circ} S$$

$$T \xrightarrow{q\circ} S \otimes_A Q \xrightarrow{p\circ} T$$

Is there another meaning. ~~Need look~~ It's not clear that S, T really help except to allow coordinate changes, really action of $GL_k(A)$, $GL_n(B)$ on the possible (p,q).

So let's consider these matrices $\left(\begin{array}{c|c} 0 & q \\ \hline p & 0 \end{array}\right)$ invertible action by elementary matrices, and maybe some kind of Whitney sums. I'm roughly forming a noncommutative analogue of $P \otimes_A Q \otimes_B$. In fact this is probably not such a bad idea. You might try Lie alg homology.

You might try Lie alg. homology. What sort of this might you hope for? $H_*(gl(A)) \simeq H_*(gl(B))$. There's all this stuff you've forgotten.

$\langle \delta \rangle$ Get back to paper. ~~[illegible]~~

excision: $A \subset B$ ideal

suppose A ideal in B. Then $m(\tilde{A}, A) = m(\tilde{B}, A)$

So in lecture today something funny.

~~$m(A, A)$~~ Start with ~~[illegible]~~ firm(R, A)

The problem is this:

Define $m(R, A)$ to be firm$(R, A) \subset$ mod(R)

Define $m(A)$ to be firm$(\tilde{A}, A) \subset$ mod$(\tilde{A})$.

Prop. $m(A) = m(R, A)$

M unital R-module $\ni$ ~~[illegible]~~ $AM = M$. Then

$$A \otimes_A M \xrightarrow{\sim} A \otimes_R M$$

confused - from this viewpt get

$m(A, A) =$ ~~[illegible]~~ A-mods $M \ni$ $A \otimes_A M \xrightarrow{\sim} M$.

nonunital viewpt: B-modules M which are A firm $A \otimes_B M \xrightarrow{\sim} M$

$$m(A, A) = m(B, A)$$

$M \in$ mod$(\tilde{B})$ $\quad AM = M$ then $A \otimes_A M \xrightarrow{\sim} A \otimes_B M$

$$a \otimes_A b\, a_i m_i = a b a_i \otimes_A m_i = ab \otimes_A a_i m_i$$

~~A~~ $N \in$ mod$(\tilde{A})$ $\quad A \otimes_A N \xrightarrow{\sim} N$. then Y.

Form some sort of set out of $\begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix}$

$$\begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & q' \\ p' & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & q & 0 \\ 0 & 0 & 0 & q' \\ p & 0 & 0 & 0 \\ 0 & p' & 0 & 0 \end{pmatrix}$$

Interp of $\begin{array}{cc} k & n \\ \hline \multicolumn{2}{c}{\begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix}} \end{array}$ is

$$B^n \xrightarrow{q\cdot} Q^k \xrightarrow{p\cdot} B^n$$

$$R^n \xrightarrow{q\cdot} A^k \xrightarrow{p\cdot} P^n$$

<ε> So now you want to introduce elementary ~~matrices~~ equivalence relation. From my calculations the other day I learned that once I fix $p \otimes q = pq \in M_n B$, then ~~the~~ possible $1 - qp$ are ~~were~~ related by elementary matrices.

Now your arguments did not use the Whitney sum.

$$p_1 q_1 = p_2 q_2 \qquad p = \begin{pmatrix} p_1 & p_2 \end{pmatrix} \qquad q = \begin{pmatrix} q_1 \\ -q_2 \end{pmatrix} \qquad pq = 0.$$

$$\begin{pmatrix} 1 - q_1 p_1 & -q_1 p_2 \\ q_2 p_1 & 1 + q_2 p_2 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} p & 0 \\ 0 & p' \end{pmatrix}\begin{pmatrix} q & \\ & q' \end{pmatrix} = \begin{pmatrix} 1-pq & 0 \\ 0 & 1-p'q' \end{pmatrix}$$

$$1 - \begin{pmatrix} q & 0 \\ 0 & q' \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & p' \end{pmatrix} = \begin{pmatrix} 1-qp & 0 \\ 0 & 1-q'p' \end{pmatrix}$$

One relation needed.

Suppose $1 - a_1 \quad 1 - a_2 \quad 1 - a_1 - a_2$ inv.

$$\begin{pmatrix} 1 & 0 \\ -ca^{-1} & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \begin{pmatrix} 1 & -b(d-ca^{-1}b)^{-1} \\ & 1 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & d-ca^{-1}b \end{pmatrix} \qquad \begin{pmatrix} a & 0 \\ 0 & d-ca^{-1}b \end{pmatrix}$$

$$1 + q_2 p_2 + q_2 p_1 (1 - q_1 p_1)^{-1} q_1 p_2 = 1 + q_2 (1 - p_1 q_1)^{-1} p_2$$

$$\left(\begin{array}{c|c} & q_1 \\ & \vdots \\ & q_k \\ \hline p_1 \cdots\cdots p_k & \end{array}\right) \qquad \sum p_i q_i \qquad \begin{pmatrix} & q_1 & q_1 \\ & q_2 & p_2 \\ -p_1 & 0 & -p_2 \end{pmatrix}$$

$$\left(\begin{array}{c|c|c} & q_1 & \\ \hline p_1 & \vdots & p_2 \\ \hline & q_2 & \end{array}\right) \qquad \begin{pmatrix} 0 & 0 & q_1 \\ 0 & 0 & q_2 \\ -p_1 & p_2 & 0 \end{pmatrix}$$

$\langle \xi \rangle$

$$\begin{pmatrix} & q_1 & \\ p_1 & & p_2 \\ & q_2 & \end{pmatrix} \longmapsto \begin{pmatrix} 0 & 0 & q_1 \\ p_1 & p_2 & 0 \\ 0 & 0 & q_2 \end{pmatrix} \longmapsto \begin{pmatrix} & & q_1 \\ & & q_2 \\ p_1 & p_2 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & y \\ x & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & \\ & 1-yx \end{pmatrix} \qquad \begin{pmatrix} 1 & y \\ x & 1 \end{pmatrix} \sim \begin{pmatrix} 1-xy & 0 \\ 0 & 1 \end{pmatrix}$$

So where do I go from here?

10/24

Review $\quad A < B \qquad A = A^2,\ ABA \subset A,\ BAB = B$

$ABA = A,\ B = B^2$

$$\begin{pmatrix} A & AB \\ BA & B \end{pmatrix} \subset M_2 B \qquad \begin{array}{ccc} m(A) & \simeq & m(B) \\ M & \longmapsto & BA \otimes_A M \\ AB \otimes_B N & \longleftarrow & N \end{array}$$

$$\begin{pmatrix} eBe & eB \\ Be & B \end{pmatrix} \qquad m(eBe) \simeq m(B)$$

$\quad \shortparallel$

$\mathrm{mod}(eBe)$

$A = B/K,\ BKB = 0$

$$\begin{pmatrix} B/K & B/KB \\ B/BK & B \end{pmatrix} = \begin{pmatrix} B & B \\ B & B \end{pmatrix} \Big/ \begin{pmatrix} K & KB \\ BK & 0 \end{pmatrix}$$

$$\begin{pmatrix} B & B \\ B & B \end{pmatrix}\begin{pmatrix} K & KB \\ BK & 0 \end{pmatrix} \subset \begin{pmatrix} BK & 0 \\ BK & 0 \end{pmatrix}$$

$$\begin{array}{ccc} m(B/K) & \rightleftarrows & m(B) \\ M & \longmapsto & B/BK \otimes_A M \\ B/KB \otimes_B N & \longleftarrow & N \end{array}$$

$\langle \eta \rangle$ 10/24 ~~[illegible]~~

Question. Given ~~[illegible]~~ (p_1, q_1) (p_2, q_2)

Is there some notion of addition?

Idea is this: Want a non-comm analogue of $P \otimes_A Q = B$. Consider pairs (p, q) s.t. $\begin{pmatrix} 1 & q \\ p & 1 \end{pmatrix}$ inv. Inverse doesn't have the same form.

Morita converse. $m(A) \xrightarrow{F} Ab(B)$ rt cont

$$\mathrm{mod}(\tilde{A}) \xrightarrow{j^*} m(A) \xrightarrow{F} Ab$$

$$F j^*(M) \simeq F j^*(\tilde{A}) \otimes_A M$$

$$\begin{tikzcd}
m(A) \arrow[r, "F"] & m(B) \arrow[d, "j_! \text{ inclusion}"] \\
\mathrm{mod}(\tilde{A}) \arrow[u, "j^*"] & \mathrm{mod}(\tilde{B})
\end{tikzcd}$$

$$F(j^* M) \xleftarrow{\sim} \underbrace{F(j^* \tilde{A})} \otimes_A M$$

B, A - bimodule firm on either side.

Thm. $m(B \otimes A^{op}) \sim \mathrm{rtcontfun}(m(A), m(B))$

Idea when walking to school. Consider

$$\begin{tikzcd}
A \otimes Q \otimes P \arrow[d, "\wr"'] \arrow[dr, "\mu \otimes 1"] & \\
(A \otimes A) \otimes_A (Q \otimes P) \arrow[r] \arrow[d] & Q \otimes P \arrow[d] \\
A \otimes A \arrow[r] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
a_1 \otimes a_2 q \otimes p & \\
(a_1 \otimes a_2) \otimes (q \otimes p) \arrow[u, leftrightarrow] \arrow[r, mapsto] \arrow[d, mapsto] & a_1 a_2 q \otimes p \\
a_1 \otimes a_2 q p &
\end{tikzcd}$$

$$\begin{tikzcd}
A \otimes Q \otimes P \arrow[r] \arrow[d] & Q \otimes P \arrow[d] \\
A \otimes A \arrow[r] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
a \otimes q \otimes p \arrow[r, mapsto] \arrow[d, mapsto] \arrow[dr, leftrightarrow] & aq \otimes p \arrow[d, mapsto] \\
a \otimes qp \arrow[r, mapsto] & aqp
\end{tikzcd}$$

$\langle\theta\rangle$ So this is interesting. You have the A-bimodule map $A\otimes Q\otimes P \longrightarrow A$, $a\otimes q\otimes p \longmapsto aqp$. You can split this into $(A\otimes Q)\otimes P$ or $A\otimes(Q\otimes P)$ former leads to the Morita context) latter to

$$\begin{pmatrix} A & A\otimes Q \\ P & P\otimes_A(A\otimes Q) \end{pmatrix} \qquad\qquad \begin{pmatrix} A & A \\ Q\otimes P & (Q\otimes P)\otimes_A A \end{pmatrix}$$

$$P\otimes_A(A\otimes Q)\cong P\otimes Q \qquad\qquad (Q\otimes P)\otimes_A A \cong Q\otimes P$$

$$\begin{aligned} &(p_1a_1\otimes q_1)(p_2a_2\otimes q_2) \\ \cong\ &(p_1\otimes_A a_1\otimes q_1)(p_2\otimes_A a_2\otimes q_2) \\ &p_1\otimes_A\langle a_1\otimes q_1, p_2\rangle a_2\otimes q_2 \\ &p_1\otimes_A a_1q_1p_2a_2\otimes q_2 \\ \cong\ &p_1a_1q_1p_2a_2\otimes q_2 \end{aligned}$$

$$(p_1'\otimes q_1)(p_2'\otimes q_2) \longmapsto p_1'q_1p_2'\otimes q_2$$

$$\begin{aligned} &(q_1\otimes p_1a_1)(q_2\otimes p_2a_2) \\ \cong\ &(q_1\otimes p_1\otimes_A a_1)(q_2\otimes p_2\otimes_A a_2) \\ =\ &q_1\otimes p_1\langle a_1, q_2\otimes p_2\rangle\otimes_A a_2 \\ =\ &q_1\otimes p_1a_1q_2p_2\otimes_A a_2 \\ =\ &q_1\otimes p_1a_1q_2p_2a_2 \end{aligned}$$

$$(q_1\otimes p_1')(q_2\otimes p_2') \overset{\alpha}{\longmapsto} q_1\otimes p_1'q_2p_2'$$

natural ~~$(p_1'\otimes q_1)(p_2'\otimes q_2) \neq p_1'q_2p_2'\otimes q_1$ NO~~.

$$(p_2'\otimes q_2)(p_1'\otimes q_1) \longrightarrow (q_1\otimes p_1')(q_2\otimes p_2') \xrightarrow{\alpha} q_1\otimes p_1'q_2p_2' \to p_1'q_2p_2'\otimes q_1$$

I think we learn something, namely $P\otimes Q = Q\otimes P$ has ~~many~~ four products

$$(p_1\otimes q_1)(p_2\otimes q_2) \mapsto p_1q_1p_2\otimes q_2 \qquad (q_2\otimes p_2)(q_1\otimes p_1) \mapsto q_2\otimes p_2q_1p_1 \qquad (q_1\otimes p_1)(q_2\otimes p_2) \mapsto q_1\otimes p_1q_2p_2$$

<ι> So what happens? You've managed to construct a bimodule surj $\underbrace{A\otimes Q\otimes P}_{M} \longrightarrow A$ such that $[M\otimes_A]^{(*)}$ can be interpreted as the precyclic module attached to two rings, namely

$P\otimes Q$ which maps to A and
$Q\otimes P$ ——— B.

$$\cancel{(p_1a_1\otimes q_1)(p_2a_2\otimes q_2) = (p_1\otimes_A a_1\otimes_A q_1)(p_2\otimes a_2\otimes_A q_2)}$$

$$(p_1\otimes a_1q_1)(p_2\otimes a_2q_2) = (p_1\otimes a_1\otimes_A q_1)(p_2\otimes a_2\otimes_A q_2)$$

$$= p_1\otimes a_1(q_1p_2a_2)\otimes_A q_2 = p_1\otimes \underbrace{a_1q_1p_2a_2q_2}$$

$$(a_1q_1\otimes p_1)(a_2q_2\otimes p_2) = (a_1\otimes_A q_1\otimes p_1)(a_2\otimes_A q_2\otimes p_2)$$

$$= a_1q_1p_1a_2\otimes_A q_2\otimes p_2$$

$$= a_1q_1p_1a_2q_2\otimes p_2$$

So my idea when walking to school didn't pay off yet. What can I do?? Good idea in principle. But still you might try to do some surjectivity argument to reduce to Suslin's theorem. At the moment what did I do.

$$\begin{tikzcd} A\otimes Q\otimes P \arrow[r] \arrow[d] & Q\otimes P \arrow[d] \\ A\otimes A \arrow[r] & A \end{tikzcd}$$

I would guess nothing can be salvaged from this.

$(A\otimes Q)\otimes P$ or $A\otimes(Q\otimes P)$

ring $P\otimes Q\to B$ ring $Q\otimes P\to A$

$\langle K \rangle$ Let's go back to the idea of making some ~~quotient~~ out of $\coprod_{k,n}\left(M_{nk}(P) \times M_{kn}(Q)\right)'$

$$1-pq \in GL_n(B) \qquad 1-qp \in GL_k(A).$$

I think I want to mimic

$$P \otimes_A Q \otimes_B . \qquad M_{nk}(P) \otimes_{M_k(A)} M_{kn}(Q) \otimes_{M_n(B)} .$$

So I propose to consider

$$E_k(A) \Big\backslash \left(M_{nk}(P) \times M_{kn}(Q)\right)' \Big/ E_n(B)$$

to take the direct limit as $k, n \longrightarrow \infty$.

$E_k(A)$ acts via $\quad (p, q) \longmapsto (p\alpha^{-1}, \alpha q)$

$E_n(B)$ ——— $\quad (p, q) \longmapsto (\beta p, q\beta^{-1})$

Why not try

$$\varinjlim_k E_k(A) \Big\backslash \left(\overset{p}{M_{nk}(P)} \times \overset{q}{M_{kn}(Q)}\right)' \longrightarrow \overset{1-pq}{GL_n(B)}$$

surjectivity is clear. Injectivity. Given $(p_1, q_1), (p_2, q_2)$ both in $\left(M_{nk}(P) \times M_{kn}(Q)\right)'$ such that $p_1 q_1 = p_2 q_2$ in $M_n B$.

$$p = (p_1 \ \ p_2) \in M_{n,2k} P \qquad q = \begin{pmatrix} q_1 \\ -q_2 \end{pmatrix} \in M_{2k,n} Q, \quad pq = 0.$$

$$1 - qp = \begin{pmatrix} 1-q_1p_1 & -q_1p_2 \\ q_2p_1 & 1+q_2p_2 \end{pmatrix} \in GL_{2k}(A)$$

$$\sim \begin{pmatrix} 1-q_1p_1 & 0 \\ 0 & (1-q_2p_2)^{-1} \end{pmatrix}$$

$d - ca^{-1}b$

$1 + q_2p_2$

$+ q_2p_1(1-q_1p_1)^{-1}q_1p_2$

$1 + q_2(1 - p_2q_2)^{-1}p_2$

$(1-q_2p_2)^{-1}$

⟨7⟩ So what next?

You need to go from $p_1 q_1 = p_2 q_2$ to

$$(p_1\ p_2\ p_3)\begin{pmatrix} q_1 \\ -q_2 \\ 0 \end{pmatrix} = 0$$

$$\begin{pmatrix} q_1 \\ -q_2 \\ 0 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} q' \qquad (p_1\ p_2\ p_3)\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0.$$

How does this help? I have $(p_1, q_1), (p_2, q_2) \in (M_{nk}(P) \times M_{kn}(Q))'$. What I want to do is to ~~move them via~~ show that by enlarging k, n they become related via $E_k(A)$.

Special case 1. Assume $pq = 0$ where $q = aq'$ and $pa = 0$.

Then have $1 - pq = 1$ in $GL(B)$.

What do I do to (p, q). I need $\alpha \in E(A)$ such that $p\alpha$

$$X_{nk}(P,Q) = \left\{ (p,q) \;\middle|\; p \in M_{nk}(P),\ q \in M_{kn}(Q),\ \begin{pmatrix} 1 & q \\ p & 1 \end{pmatrix} \text{ invertible} \right\}$$

Then you want to take $\varinjlim_{n,k}$. But first you need an equivalence relation on $X_{nk}(P, Q)$ ~~[illegible]~~ arising from A. ~~It sho~~ There is the map

$$X_{nk}(P,Q) \longrightarrow GL_n(B)$$
$$(p, q) \qquad\quad 1 - pq$$

The equiv. relation should contract the fibres in the limit as $k \to \infty$. Original idea namely $(p, q) \sim (p\alpha^{-1}, \alpha q)$ $\alpha \in E_k A$ is inadequate since $(p, q) \sim (0, 0)$ will not happen