A March 21

Go back over degeneracy structure results for a while. Try to prove conjecture.

$(C_\lambda(A), b, d)$

How about organizing the results into some coherent form.

First part concerns derivations & coderivations.

$R = A * \mathbb{C}[\xi]$ grading $|a| = 0$, $|\xi| = 1$

define derivations

$|b'| = -1$ $b'(a) = 0$, $b'(\xi) = 1$.

$|d| = +1$ $d(a) = [\xi, a]$ $d(\xi) = \xi^2$

$|\delta| = 1$ $\delta(a) = 0$ $\delta(\xi) = \xi^2$

$$d + \delta = ad(\xi)$$

$$b'^2 = 0 = d^2 = \delta^2 = [b, d] = [b, \delta]$$

$$[b', s] = [b', -\lambda^{-1} s] = 1$$

$s(\alpha) = \xi\alpha$ $(-\lambda^{-1}s)(\alpha) = (-1)^{|\alpha|}\alpha\xi$

$$R = A + A\xi A + A\xi A\xi A$$

$$= A \oplus (A \otimes A) \oplus (A \otimes A \otimes A) \oplus \ldots$$

other formulas

$$d = \sum_{i=0}^{n+1} \lambda^i s \lambda^{-i} \quad \text{on } A^{\otimes n+1}$$

$$d = \underbrace{s \quad \delta}_{d''} \underbrace{ + \lambda^{-1} s}_{d'}$$

$B \quad [b', d''] = [b', -d'] = 1$

$$(d'')^2 = (\delta - s)^2 = -[\delta, s] + s^2$$

$$[\delta, s](\alpha) = \delta(\xi\alpha) + \xi\delta(\alpha) = \xi^2\alpha = s^2\alpha$$

$$(d')^2 = (d - s)^2 = -[d, s] + s^2 = 0 .$$

Relative X-complex $\quad X_A(\underbrace{A * \mathbb{C}[\xi]}_{T_A(A\xi A)})$

$$A \, \oplus \bigoplus_{n\geq 1} [E\otimes_A]^{(n)} \underset{N_\sigma}{\overset{1-\sigma}{\rightleftarrows}} \bigoplus_{n \geq 1} [E\otimes_A]^{(n-1)} \partial E \otimes_A$$

$$[A\xi A \otimes_A]^{(n+1)} \ni d(a_0\xi \cdots a_{n-1}\xi a_n \xi)$$

$$\Big\downarrow \ \delta_* = d' \text{ on chains}$$

$$[A\xi A\otimes_A]^{(n)} (A\partial\xi A)\otimes_A \ni \quad a_0\xi \cdots a_{n-1}\xi a_n\partial\xi$$

$\xrightarrow{\ \delta_*\ }$

$$(-\delta)(a_0\xi a_1 \cdots a_{n-1}\xi a_n\partial\xi) + (-1)^n \underbrace{a_0 \xi a_1 \ \xi a_n(-\xi\partial\xi - \partial\xi\xi)}_{(-1)^{n+1}}$$

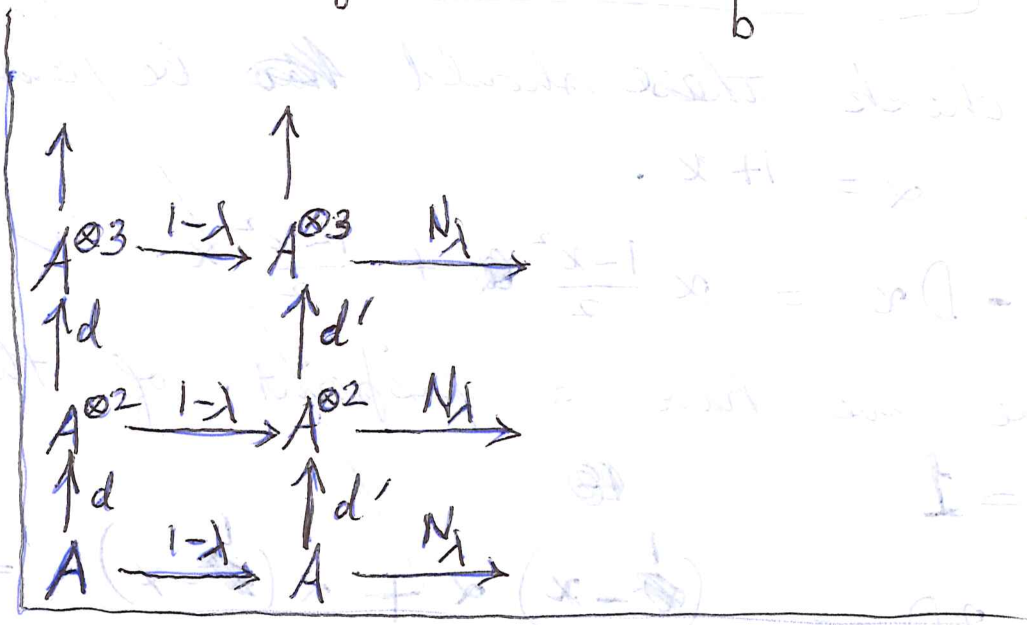
$$= \xi a_0 \cdots \xi a_n \partial\xi \quad + (-1)^{n+1} a_0\xi \cdots a_n\xi d\xi$$

$$\sum_1^n (-1)^i a_0\xi a_1 \cdot \xi^2 a_i \cdots \xi a_n \partial\xi$$

$\therefore (-\delta)_* = d$ on

C Using this calculations we end up with the following

$$0 \to C_\lambda(A) \xrightarrow{N_\lambda} \underset{b'}{(C(A), d)} \xrightarrow{1-\lambda} \underset{b}{(C(A), d')} \xrightarrow{N_\lambda} (C(A), d) \xrightarrow{1-\lambda}$$



$$d = \sum_{i=0}^{n+1} \lambda^i s \lambda^{-i} \quad \text{on } A^{\otimes n+1}$$

$$d = s + d'$$

$$\lambda d = s + \Big(\underbrace{\sum_{i=0}^{n} \lambda^{i+1} s \lambda^{-i-1}}_{d'}\Big)\lambda$$

$$(1-\lambda)\, d = d'(1-\lambda)$$

Check that $[b, d'] = 0$.

Since $[b', d'] = -1$ this equiv to $[c, d'] = 1$

$$[c, d'] = [c, -\delta + \lambda^{-1} s]$$

equiv to $[c, \delta] = 0$.

D $\mathcal{B}(M, b, B) \cong \mathcal{B}(M, b, 0)$

$$\underset{1}{B} = [\underset{-1}{b}, \underset{2}{k}] \qquad [k, B] = 0$$

$$[b, Sk] = S[b,k] = SB$$

$$[b, e^{Sk}] = \int_0^1 e^{(1-t)Sk} \underbrace{[b, Sk]}_{SB} e^{tSk} \, dt$$

$$= e^{Sk} SB$$

$$\boxed{b e^{Sk} = e^{Sk}(b + SB)}$$

$$e^{-Sk} b e^{Sk} = e^{-S \, ad(k)} b$$

$$-ad(k) b = B$$

So now consider $(C_\lambda(A), b, d)$ and look for an operator k of degree $+2$ ~~...~~ such that $[b, k] = B$ $[B, k] = 0$.

Maybe such things also exist on $(C(A), b', d)$ $\qquad [b', k] = d \qquad [k, d] = 0$

Want k to be ~~of~~ degree $+2$. Look for a derivation $\qquad b'ka = [\xi, a] = \xi a - a \xi$

$\xi a \xi \qquad \xi a_1 a_2 \xi =$

$(\xi a_1 \xi) a_2 \qquad a_1 \xi a_2 \xi$

E $a_1(\xi a_2 \xi) - \xi a_1 a_2 \xi + (\xi a_1 \xi) a_2$

~~$= (a_1 \xi - \xi a_1) a_2 \xi + \xi a_1 \xi a_2$~~

$a_1(\xi^2 a_2)$ − ~~$\xi^2 (a_1 a_2)$~~ + ~~$(\xi^2 a_1)$~~ a_2

~~$\xi a_1 (a_2 \xi^2)$~~ − ~~$(a_1 a_2) \xi^2$~~ $+ (a_1 \xi^2) a_2$

$[b', d'] = -1$ $[b', d] = 0$

$[b', -d'd] = d$

$[b', dd'] = d$

Less trivial example $(C(A), b, d')$

gives

$(\Omega A, b, 0)$

So you can ask for a k of degree $+2$ such that $[b, k] = d'$ $[b, d'] = 0$

$[b, s] =$ $1 - k = [b, d]$

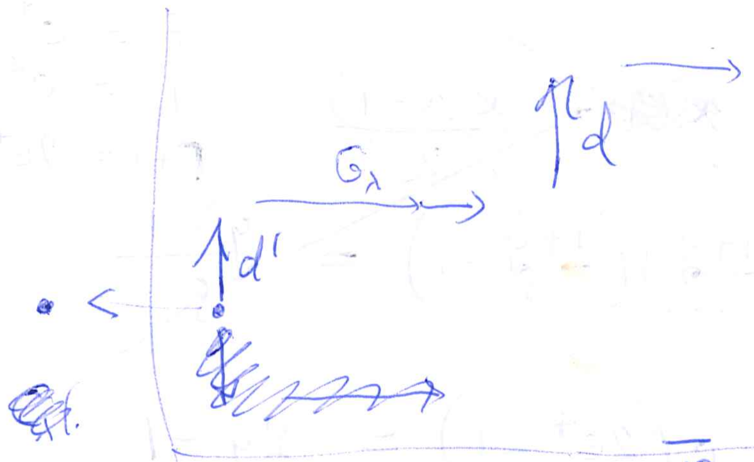
$b'(\xi a \xi) = a\xi - \xi a = -da$

How do you get the S operator of ~~interest~~ degree $+2$ for the d diffl.

Recall that if (M, b, B) is B acyclic, you choose $h \ni [B, h] = 1$ then $S = [b, h]$ is an endom. of degree -2 representing S on M/BM.

F Consider then $(C(A), b', d)$ and choose a contraction for b' say $-d'$. Then $[d, -d'] = -dd' - d'd$ is an operator of degree $+2$ commuting with d, b'.

Let's see if this works.



OK

$$0 \leftarrow C_\lambda(A) \overset{\pi}{\leftarrow} C(A) \overset{1-\lambda}{\longleftrightarrow} C(A) \overset{\bar{N}}{\longleftrightarrow} C_\lambda(A) \leftarrow 0$$

The diagram chasing S operator is

$$S\pi x = \bar{N}_\lambda^{-1}\, d\, G_\lambda\, d'\, P_\lambda x$$

Problem is to find k of degree $+2$ such that $[b', k] = d$ and $[k, d] = 0$

~~$[k, d] = [k, [b', k]]$~~

now $[b', -d'd] = d$

$[b', -dd'] = d$

but $(-d'd)d \qquad d(-d'd)$

$\overset{\shortparallel}{0} \qquad -dd'd = dsd = s^2 d$

G Try something like $d^{[2]}$ inserting 1's

$$k(a_0) = \xi a_0 \xi = (1, a_0, 1)$$

$$b' k(a_0 \xi a_1) = (1, a_0 a_1, 1)$$

$T(A)$ algebra $d_\ell(a) = \xi a$ $d_\ell(\xi) = \xi^2$

$d_\ell(\; a_0 \xi a_1$ WAIT.

Consider the graded algebra $T(A)$. $|A| = 1$.
Let $\xi = 1_A$. Then introduce derivation d_ℓ, d_r of degree $+1$ by $d_\ell(a) = \xi a$ in part $d_\ell(\xi) = \xi^2$
$d_r(a) = -a\xi$ ——— $d_r(\xi) = -\xi^2$.

$$\begin{aligned} d_\ell(a_1 \cdots a_n) &= \xi a_1 a_2 \cdots a_n \\ &\quad - a_1 \xi a_2 \cdots a_n \\ &\quad + \cdots \\ &\quad + (-1)^{n-1} a_1 \xi \cdots a_{n-1} \xi a_n \end{aligned}$$

$\therefore d_\ell = d''$

$$\begin{aligned} d_r(a_1 \cdots a_n) &= -a_1 \xi a_2 \cdots \\ &\quad + a_1 a_2 \xi \cdots \\ &\quad + (-1)^n a_1 \cdots a_{n-1} a_n \xi \end{aligned}$$

$d_r = d'$

$$d_\ell^2(a) = d_\ell(\xi a) = \xi^2 a - \xi(\xi a) = 0.$$

$$d_r^2(a) = d_r(-a\xi) = -(-a\xi)\xi + a(-\xi^2) = 0.$$

$s\alpha = \xi \alpha$

$$[d_\ell, s]\alpha = d_\ell(\xi\alpha) + \xi d_\ell \alpha = \xi^2 \alpha = s^2 \alpha$$

H Thus $(d_\ell - s)^2 = 0$

$$\boxed{d_\ell - s = \delta}.$$

$$[d_r, s](\alpha) = d_r(\xi\alpha) + \xi d_r(\alpha)$$

$$= d_r(\xi)\alpha = -\xi^2\alpha = -s^2\alpha$$

$\therefore (d_r + s)^2 = 0$ $\boxed{d_r + s = d}$

Let's see if we can find operators of degree +2 commuting with d, d' $\hat{=} d_r$

$d = d_r + s$

Look at derivations of degree 2.

$$A \longrightarrow A \otimes A \otimes A$$

three poss. $D_\ell : a \longmapsto \xi^2 a$

$D_m : a \longmapsto -\xi a \xi$

$D_r : a \longmapsto a\xi^2$.

$$[d_r, D_m](a) = d_r(-\xi a \xi) + D_m(+a\xi)$$

$$= \xi^2 a \xi + \xi(-a\xi)\xi - \xi a(-\xi^2)$$

$$+ (-\xi a \xi)\xi + a(-\xi^3)$$

$$= \xi^2 a \xi - \xi a \xi^2 - a\xi^3$$

$-\xi^2 = d_r(\xi^2) = d_r(\xi)\xi - \xi d_r(\xi)$, $= 0$

$$[d_r, D_\ell](a) = d_r(\xi^2 a) + D_\ell(+a\xi)$$

$$= \xi^2(-a\xi) + (\xi^2 a)\xi + a(\xi^3) = a\xi^3$$

I $\quad [d_r, D_r](a) = d_r(a\xi^2) + D_r(+a\xi)$

$$= (-a\xi)\xi^2 + (a\xi^2)\xi + a\xi^3 = a\xi^3$$

$$D_\ell - D_r = ad(\xi^2)$$

$$[d_r, D_\ell - D_r] = [d_r, ad(\xi^2)] = ad(d_r(\xi^2)) = 0.$$

$$(s\, ad(\xi))(a) = \xi(\xi a + a\xi) \simeq (D_\ell - D_m)(a)$$

$$[d_r, s\, ad(\xi)] = -s^2 ad(\xi) - s\, ad\, d_r\xi)$$

$$= -s^2 ad(\xi) + s\, ad(\xi^2)$$

$$-\xi^2(\xi a + a\xi) + \xi(\xi^2 a - a\xi^2)$$

$$[D_m, s](\alpha) = D_m(\xi\alpha) - \xi D_m\alpha$$

$$= -\xi^3\alpha$$

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different approach suggested by [L].
The point is to understand the degeneracy structure. $\varepsilon_i(a_0, \dots, a_n) = (-1)^i (a_0, \dots, a_{i-1}, 1, a_i, \dots, a_n)$

$0 \le i \le n+1.$

$$\varepsilon_i(a_1, \dots, a_n) = (-1)^n (a_1, \dots, a_i, 1, a_{i+1}, \dots, a_n)$$

$$0 \le i \le n$$

J

$$\varepsilon_i \varepsilon_j = -\varepsilon_{j+1} \varepsilon_i \qquad 0 \leq i \leq j \leq n$$

$$\varepsilon_i \varepsilon_j (a_1, \ldots, a_n) = \varepsilon_i (-1)^j (a_1, \ldots, a_j, 1, a_{j+1}, \ldots, a_n)$$

$$= (-1)^{i+j} (a_1, \ldots, a_i, 1, a_{i+1}, \ldots, a_j, 1, a_{j+1}, \ldots, a_n)$$

$$\varepsilon_{j+1} \varepsilon_i = (-1)^i \; \varepsilon_{j+1} (a_1, \ldots, a_i, 1, a_{i+1}, \ldots, a_j, 1, a_{j+1}, \ldots, a_n)$$

$$d = \sum_{i=0}^{n} \varepsilon_i \qquad \text{on } A^{\otimes n}$$

$$d^2 = \sum_{a=0}^{n+1} \varepsilon_a \sum_{b=0}^{n} \varepsilon_b$$

$$= \sum_{0 \leq b < a \leq n+1} \varepsilon_a \varepsilon_b + \sum_{0 \leq a \leq b \leq n} \varepsilon_a \varepsilon_b \quad (\varepsilon_a \varepsilon_b = -\varepsilon_{b+1} \varepsilon_a)$$

$$\sum_{0 \leq i < j \leq n+1} \varepsilon_j \varepsilon_i - \sum_{0 \leq i < j \leq n+1} \varepsilon_j \varepsilon_i = 0$$

$$f = \sum_{i=0}^{n} c_i \varepsilon_i$$

$$df + fd = \sum_{a=0}^{n+1} \varepsilon_a \sum_{b=0}^{n} c_b \varepsilon_b + \sum_{a=0}^{n+1} c_a \varepsilon_a \sum_{b=0}^{n} \varepsilon_b$$

$$= \sum_{a=0}^{n+1} \sum_{b=0}^{n} (c_b + c_a) \, \varepsilon_a \varepsilon_b$$

$$= \sum_{0 \leq b < a \leq n+1} (c_b + c_a) \, \varepsilon_a \varepsilon_b - \sum_{0 \leq a \leq b \leq n} (c_b + c_a) \, \varepsilon_{b+1} \varepsilon_a$$

$$K = \sum_{0\le b<a\le n+1} (c_b + c_a)\,\varepsilon_a \varepsilon_b - \sum_{0\le b<a\le n+1} (c_{a-1} + c_b)\,\varepsilon_a \varepsilon_b$$

$$= \sum_{0\le b<a\le n+1} (c_a - c_{a-1})\,\varepsilon_a \varepsilon_b$$

So which one am I working with b', d

Interestingly enough you ~~have~~ can take

$$c_a = 1 \qquad a \geq \alpha$$
$$c_a = 0 \qquad a < \alpha$$

$$[d, f] = \sum_{0\le b<\alpha} \varepsilon_\alpha \varepsilon_b \qquad\qquad d''(-\lambda^{-1} s)$$

So we would like maybe b', d.

You want a degree 2 operator k ∋

$$[b', k] = d \qquad\qquad [d, k] = 0.$$

idea is that $k = [d, f] \Rightarrow [d, k] = 0$

$$[b', [d, f]] = -[d, \underbrace{[b', f]}_{dd'}] = d \ ?$$

~~Take $f = dd'$~~

~~$k = [b', f] = [b', dd'] = -d[b', d'] = -d(-1) = d$~~

~~Then~~ Doing nothing.

L $[b,d'] \stackrel{?}{=} 0$ $[b',d'] = -1$

$[c,d'] \stackrel{?}{=} 1$?

$$d'(a_0,\dots,a_n) = \sum_{i=1}^{n+1} (-1)^i (a_0,\dots,a_{i-1},1,a_i,\dots,a_n)$$

$$d'c(a_0,\dots,a_n) = (-1)^n d'(a_n a_0, a_1,\dots,a_{n-1})$$

$$= (-1)^n \sum_{i=1}^{n} (-1)^i (a_n a_0,\dots,a_{i-1},1,a_i,\dots,a_{n-1})$$

$$c d'(a_0,\dots,a_n) = \sum_{i=1}^{n} (-1)^i (-1)^{n+1} (a_n a_0,\dots,a_{i-1},1,a_i,\dots,a_{n-1})$$

$$+ (-1)^{n+1}(-1)^{n+1}(a_0,\dots,a_n)$$

So consider $[b,d'] = 0$

we know d' acyclic $[b',d'] = -1$ | $[c,d'] = 1$

The image of d' contained in degen. subcomplex

Contraction homotopy is

$$[b, (1-\kappa)^{-1} d d'] = (1-\kappa)^{-1}(1-\kappa) d' = d'$$

$$[b, -d'(1-\kappa)^{-1} d] = d'(1-\kappa)^{-1}[b,d] = d'$$

but neither of these k's commutes with d'

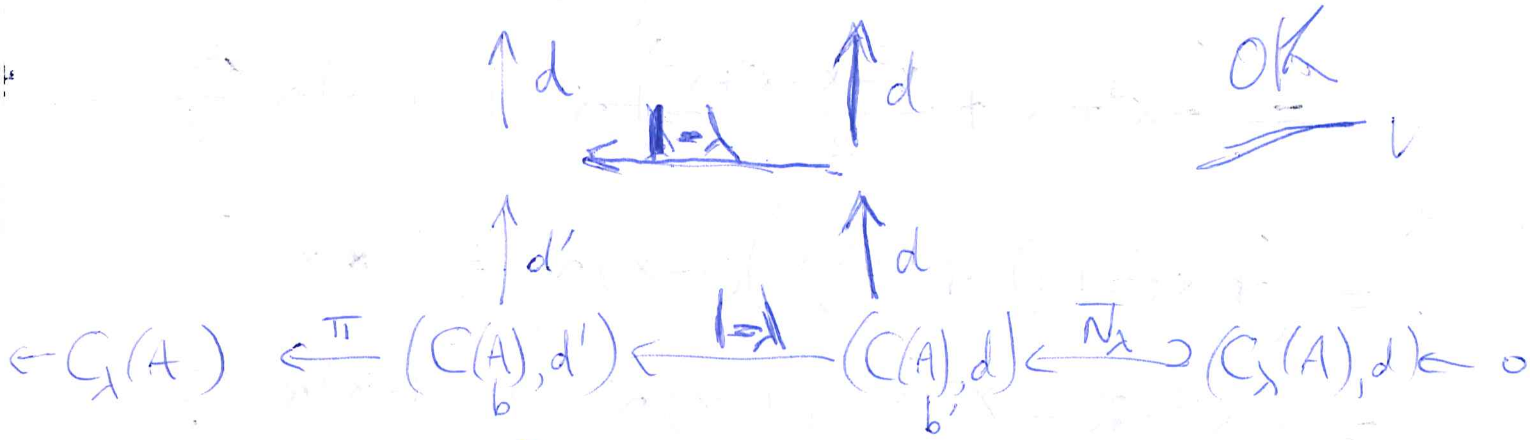
$[b', -d'd] = d$

$[b', dd'] = d$

So both (C, b, d') and (C, b', d) are such that $B = [b,h]$, but we have not arranged that $[B,h] = 0$

M Maybe the best thing is to stick to the cyclic complex and try to understand the S operator.

$sd' = ds$



operator is $\tilde{N}_\lambda^{-1} \, d \, G_\lambda \, d' P_\lambda$

The question is whether this operator has any nice relation to b on $C_\lambda(A)$.

~~The next question is whether~~

The next point

~~Better~~ invariant bilinear form is a linear functional on $(a \otimes_a a \otimes_a)_\sigma$. If I take a to have the zero mult. sames as $S^2_k V$ so this is not a useful point.