$\boxed{b}$ ~~So let us understand something respect~~

Review V Jones

$B \subset A \xrightarrow{\rho} B$ define alg structure on

$A \otimes_B A$ by $(a_1, a_2)(a_3, a_4) = (a_1 \rho(a_2 a_3), a_4)$

$= (a_1, \rho(a_2 a_3) a_4)$.

Let $(x_i, y_i) \in A \otimes_B A$. TFAE

1) $(x_i, y_i) =$ identity element for the element.

3) $a(x_i, y_i) = (x_i, y_i) a \quad \forall a$

$\rho(x_i) y_i = x_i \rho(y_i) = 1$

2) $\rho(a x_i) y_i = a \qquad x_i \rho(y_i a) = a \quad \forall a$

4) $A \otimes_B A \xrightarrow{\sim} \mathrm{Hom}_{B^{op}}(A, A)$

$(x_i, y_i)(a_1, a_2) = (x_i \rho(y_i a_1), a_2)$. So $2) \Rightarrow 1)$

$(x_i, y_i)(a_1, a_2) = (a_1, a_2)$

$1) \Rightarrow \quad x_i \rho(y_i a_1) a_2 = a_1 a_2 \qquad \forall a_2$. Take $a_2 = 1$

get $x_i \rho(y_i a_1) = a_1$

$3) \Rightarrow 2)$ ass $(a x_i, y_i) = (x_i, y_i a)$

$\rho(x_i) y_i = x_i \rho(y_i) = 1$.

Then $\rho(a x_i) y_i = \rho(x_i) y_i a = a$.

Also $x_i \rho(y_i a) = a x_i \rho(y_i) = a$.

$(x_j, y_j)(a x_i, y_i) = (x_j, \rho(y_j a x_i) y_i)$

$\boxed{C}$

$$(ax_i, y_i) = (x_j\rho(y_j a x_i), y_i)$$
$$= (x_j, \rho(y_j a x_i) y_i)$$
$$= (x_j, y_j a).$$

$$(a_1, a_2) \longmapsto (\alpha \longmapsto a_1\rho(a_2\alpha))$$

$$(f(x_i), y_i) \longleftarrow\!\!\shortmid\; f$$

$$(a_1, a_2) \longmapsto (\alpha \longmapsto \underbrace{a_1\rho(a_2\alpha)}_{f(\alpha)})$$

$$(f(x_i), y_i) = (a_1\rho(a_2 x_i), y_i)$$

....

$$\alpha \longmapsto f(x_i)\rho(y_i\alpha) = f(x_i\rho(y_i\alpha)) = f(\alpha).$$

Next you take $A \hookrightarrow A\otimes_B A$, $a \longmapsto (ax_i, y_i)$

$(ax_i, y_i) * 1 = ax_i\rho(y_i) = a.$

Take $\rho : A\otimes_B A \longrightarrow A$. This must be an A-bimodule map, hence determined by a B-central elt of A.

$\boxed{d}$

$$A_0 = B$$
$$A_1 = A$$
$$A_2 = A \otimes_B A$$
$$A_3 = A_2 \otimes_{A_1} A_2$$

~~These~~ Joachim told me about ~~~~ a left action of A on a right Hilbert A-module H. This may not be correct, but there should be a Hilbert version ~~~~ corresponding to the GNS picture.

~~~~ Give example $B \subset A \xrightarrow{\rho} B$

I form $\Gamma = A \oplus A \otimes B \otimes A$ ~~~~, a Γ module is an A module M. tog with a B-module N a ~~direct~~ split injection $N \underset{\iota}{\overset{j}{\leftrightarrows}} M$ $\ni$ $j a \iota = \rho(a)$.

So what is the actual Stinespring stuff.

$\rho : A \longrightarrow B$ completely positive map of C^* algebras. What is GNS originally:

A a C^* alg $\rho : A \longrightarrow \mathbb{C}$ a positive linear fnl i.e. ~~$\rho(a^* a)$~~ $\langle a_1, a_2 \rangle = \rho(a_1^* a_2)$ sesquilinear ≥ 0.

Then get Hilbert space repn. $A \longrightarrow \mathrm{End}(H)$, vector $v \in H$ such that $H = \overline{Av}$, $\rho(a) = \langle v, av \rangle$

~~Stinespring. Given $A \longrightarrow B$ and~~

Idea is that I got from $B \subset A \xrightarrow{\rho} B$ $\rho(1) = 1$, ρ B-bimodule map a GNS alg ~~$A \otimes B$~~ $A \oplus A \otimes B \otimes A$ factoring through $A \otimes_B A$.

Recall now the history.
~~~~

$\boxed{e}$ GNS original dealt with $\rho : A \to \mathbb{C}$ $\rho \geq 0$
gets ~~... $A \to \mathrm{End}(H)$~~
a Hilbert space repn of A.

Stinespring generalization $\rho : A \to \mathcal{B}(H)$
ρ completely positive,
gets a repn of A on a larger Hilbert space ~~....~~ a completion of $A \otimes H$.

$$\langle a_1, a_2 \rangle = \rho(a_1^* a_2)$$

Then $\langle a_1, a a_2 \rangle = \rho(a_1^* a a_2) = \rho((a^* a_1)^* a_2) = \langle a^* a_1, a_2 \rangle$

$$\langle 1, a1 \rangle = \rho(a).$$

$$\langle a_1 \otimes h_1, a_2 \otimes h_2 \rangle \stackrel{\text{defn.}}{=} \langle h_1, \rho(a_1^* a_2) h_2 \rangle$$

Complete positive prob. says that $\forall\ a_1, \dots, a_n \in A$ the ~~..~~ hermitian inner product on H^n given by $(h_i) \longmapsto \langle h_i, \rho(a_i^* a_j) h_j \rangle$ is ≥ 0.

So we get $A \otimes H$ as an A-module with left translation. ~~............~~
This is the left A-module gen. by H. ~~...~~ So there's an obvious $H \xrightarrow{i} A \otimes H$. There is an adjoint j: $\langle 1 \otimes h', a \otimes h \rangle = \langle h', \rho(a) h \rangle$

so $j(a \otimes h) = \rho(a) h$.

Kasparov gen. of Stinespring ~~..~~ concerns a ~~.....~~ representation of A on a ~~.....~~ Hilbert B-module rather ~~that~~ than Hilbert space.

$\boxed{f}$ ~~[illegible]~~ Representation of A on a Hilbert B-module?

Wrong B. So what is a Hilbert B-module

A right B-module H with map

$$H \times H \longrightarrow B$$
$$h_1, h_2 \longmapsto \langle h_1, h_2 \rangle$$

bilinear over $\mathbb{C}$ and

$$\langle h_1 b_1, h_2 b_2 \rangle = b_1^* \langle h_1, h_2 \rangle b_2$$

$$\langle h, h \rangle \text{ is } \geq 0 \text{ in } B \quad (C^* \text{ alg}).$$

e.g. $H = B^n$ with $\langle (h_i), (h_i') \rangle = \sum h_i^* h_i'$

Wrong B. Use C because the role of B is played by $\mathrm{End}_{\mathbb{C}}^{\circ}(H)$

So much for a review. Go back now ~~[illegible]~~ to $S \underset{\text{subalg}}{<} A \xrightarrow[\text{S-bim map}]{\rho} S$ $\quad \rho(1) = 1$.

I know how to handle this via GNS.

Thus if H is an S-module (e.g. ~~[illegible]~~ repn. of S on a Hilbert $\mathbb{C}$-module), then I get an A-module $A \otimes_S H$

S-module H

and $H \underset{i}{\overset{j}{\rightleftarrows}} A \otimes_S H$ $\qquad j(a \otimes h) = \rho(a)h$, $\quad i(h) = 1 \otimes h.$

and an action of $A \otimes_S A$ on $A \otimes_S H$

$$(a_1, a_2) \mapsto \big((a, h) \longmapsto a_1 i j a_2 (a, h)\big)$$
$$= (a_1 \rho(a_2 a), h)$$

So actually in the good case: $A \otimes_S A$ has an identity ~~[illegible]~~: we have what sort of structure?

$$\begin{tikzcd} & (A\otimes_S A\text{-modules}) \arrow[d] \\ (S\text{-modules}) \arrow[ur, "\text{equiv.}"] \arrow[r] & (A\text{-modules}) \end{tikzcd}$$

?

~~[illegible] I want a H~~

There's a Hilbert picture behind GNS.

If $\rho : A \longrightarrow S$ is completely pos.

$S = \mathrm{End}_{C^\circ}(H)$ $\quad$ H right Hilbert C-module

then your GNS algebra has a Hilbert repr.. on $A \otimes_S H$

In general given $A \xrightarrow{\rho} S$ you get $A \oplus A\otimes S \otimes A$ left-acting on $A \otimes S$

$$e = 1\otimes 1 \otimes 1$$

$$(a_1 \otimes s \otimes a_2)(1\otimes 1\otimes 1) = a_1 \otimes s\rho(a_2) \otimes 1$$

~~Thus for~~ Repeat this

Given $A \xrightarrow{\rho} S$ $\qquad \rho(1) = 1$.

you get
$$\begin{tikzcd} A\otimes S \arrow[d, shift left, "j"] \\ S \arrow[u, shift left, "1\otimes_A -"] \end{tikzcd} \quad j = \rho\otimes 1$$

whence Γ acts on $A\otimes S$, so we have $\forall$ S module N an A-module $(A\otimes S)\otimes_S N = A \otimes N$
$$\begin{tikzcd} A\otimes N \arrow[d, shift left] \\ N \arrow[u, shift left] \end{tikzcd}$$

But when we have in addition $S \subset A$ such the ρ is S-bilinear there is more structure

$\boxed{h}$ So how can Hilbert structure arise?

Consider $\rho : A \longrightarrow S \subset \mathrm{End}(H)$.

Normally we take $A \otimes H$ and define inner product $\langle a_1 \otimes h_1, a_2 \otimes h_2 \rangle = \langle h_1, \rho(a_1^* a_2) h_2 \rangle$

Suppose now that ρ is S-bilinear

Certainly $\rho(a_1^* a_2) s h_2 = \rho(a_1^* a_2 s) h_2$

$$\langle h_1, \rho(\underbrace{(a_1 s)^*}_{s^* a_1^*} a_2) h_2 \rangle$$

$$= \langle h_1, s^* \rho(a_1^* a_2) h_2 \rangle$$

$$= \langle s h_1, \rho(a_1^* a_2) h_2 \rangle$$ because we are assuming that S is a $*$ subalg of $\mathrm{End}(H)$ and we use the inner product on H.

So the inner product descends to $A \otimes_S H$

which means what?

You haven't got the criteria straight.

~~So what do we have? In the Hilbert picture I am not certain~~

Hilbert picture: First GNS $\rho : A \longrightarrow \mathbb{C}$

GNS situation A $*$ alg, $\rho : A \longrightarrow \mathbb{C}$ linear positive $1 \mapsto 1$

get $H = A$ with $\langle a_1, a_2 \rangle = \rho(a_1^* a_2)$.

$$A \otimes A \longrightarrow \mathrm{End}(A)$$
$$(a_1, a_2) \longmapsto (a \mapsto a_1 \rho(a_2 a))$$
$$= (a \longmapsto a_1 \langle a_2^*, a \rangle)$$

M

$\boxed{i}$

$$\mathbb{C} \subset A \subset \mathrm{End}(A)$$
$$\uparrow \cong$$
$$A \otimes A$$

Start again. A finite diml $*$ alg.

$\rho : A \longrightarrow \mathbb{C}$ linear fnl $\ni$ $\rho(a^*a) \geq 0$

assume $\rho(a^*a) = 0 \implies a = 0$. Then we get an isomorphism

$$A \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{C}}(A, \mathbb{C})$$
$$a_1 \longmapsto (\alpha \longmapsto \rho(a_1 \alpha))$$

of right A-modules hence

$$A \otimes A \xrightarrow{\sim} A \otimes \mathrm{Hom}_{\mathbb{C}}(A, \mathbb{C}) \simeq \mathrm{End}(A)$$

GNS

I take I need zilch.

Have lost picture.

given $A \xrightarrow{\rho} \mathbb{C}$ want central $\sum x_i \otimes y_i \in A \otimes A$ $\ni$ $\rho(x_i) y_i = x_i \rho(y_i) = 1$. In the case of a sep alg have complete classification!! ~~So what~~ ?? Next.

So let's try again to handle things

Compare: Algebraically we have $\mathbb{C} \overset{\rho}{\subset} A$ such that $\rho(a_1 a_2)$ is non degenerate, so gives

$$A \xrightarrow{\sim} A'$$
$$a \longmapsto (\alpha \longmapsto \rho(a\alpha)).$$

So when is this the case. If x_i is a basis of A and y_i is the dual basis defined by

$$\rho(y_j x_i) = \delta_{ji}$$

$\boxed{7}$ If $\rho(y_j x_i) = \delta_{ji}$ and $a = \sum x_i c_i$

Then $\rho(y_j a) = \sum \underbrace{\rho(y_j x_i)}_{\delta_{ji}} c_i = c_j$

so $\boxed{a = \sum x_i \rho(y_i a)}$

Algebraically then we have A a f.d. alg and $\rho \in A'$ such that $\rho(a_1 a_2)$ non-degenerate.
Not necessary that $\rho(1) = 1$.
~~So what else is happening?~~
In the Hilbert case you have $\rho(a^* a) > 0$ for $a \neq 0$.

So I really don't understand what to do.
~~Take $x_1, x_2, \ldots$ There are some key points!!~~
~~In the finite dim'l~~
What other points are there?

$$A \otimes A \xrightarrow{\sim} \mathrm{End}(A)$$

Have A acting by left mult on itself and ~~similarly~~ also on the right. Have A acting
A_ℓ and A_r are always centralizers of each other.

So a curious invariant is $\sum x_i y_i$

Thus if ρ is non degenerate on x_i basis, y_i dual basis $\rho(y_j x_i) = \delta_{ji}$, then we know
$\sum x_i \otimes y_i \in A \otimes A$ is central
so $\sum x_i y_i \in A^?$.

$\boxed{k}$ Assume A separable let
τ be trace on left reg. repn.
Let $\xi_i \otimes \eta_i \in A \otimes A$ be the canonical separability elt.

$\rho(a) = \tau(w a)$.

$$\tau(w \cdot w^{-1} \eta_j \xi_i) = \delta_{ji}$$

so $(\xi_i \otimes w^{-1} \eta_i) = (x_i \otimes y_i)$

Then the ~~general~~ central element is $\xi_i w^{-1} \eta_i$

Let's go back over the construction

$B \subset A \xrightarrow[\text{bimodule}]{\rho} B$ Assume $\exists$ $x_i \otimes y_i \in A \otimes_B A$ $\ni$ central
$x_i \rho(y_i) = \rho(x_i) y_i = 1$.

Assume A separable. Then

$$(A \otimes_B A)^{\natural} \xrightarrow{\sim} (A \otimes_B A)_{\natural} = A$$

Take $B = \mathbb{C}$. Have canonical elt $\xi_i \otimes \eta_i$

~~Can mult~~ consider

The point is to

First $B = \mathbb{C}$. Then central elts $\xi_i w \otimes \eta_i = \xi_i \otimes w \eta_i$
have invariant w. as $\eta_i \xi_i = 1$.
Corresp ρ is $a \mapsto \tau(w^{-1} a)$
To repeat the process I need

$$A \subset \mathrm{Hom}_{B^{op}}(A, A) \longrightarrow A$$

$\boxed{11}$ To repeat process I need to be able to map $A \otimes_B A \xrightarrow{\sim}$

$$\frac{\cancel{8}a^2}{\cancel{8}a^4}$$

$$\frac{aa}{aaaa} \qquad \frac{\cancel{2}\cdot\cancel{2}}{\cancel{2}\cdot\cancel{2}\cdot 2\cdot 2}$$

$$\| $$

$$\frac{1}{aa} = \frac{1}{a^2} = a^{-2}$$

$$\frac{\cancel{a}\cancel{a}\cancel{a}aa}{\cancel{a}\cancel{a}\cancel{a}}$$

$$\frac{12a^5b^4}{3a^3b^2} = 4a^{+2}b^{+2}$$

$$\begin{tikzcd} A\otimes_B A \arrow[r] \arrow[d, dashed] & \mathrm{Hom}_{B^{op}}(A,A) \\ A & \end{tikzcd}$$

need this bimodule map.
need an element in the commutant of B.

What possibilities? I take 1

In which case

$$\big((a_1,a_2)\otimes(a_3,a_4)\big)\cdot\big((a_5,a_6)\otimes(a_7,a_8)\big)$$

$$= (a_1,a_2)\,\mu\big((a_3\,\rho(a_4a_5),\,a_6)\big) \otimes (a_7,a_8)$$

$\rho(1)=1$:

$$\cancel{= (a_1,\ a_2a_3\,\rho(a_4a_5)\,a_6) \otimes_A (a_7,a_8)}$$

$$\cancel{= a_1,\ a_2a_3\,\rho(a_4a_5)}$$

$$= (a_1,a_2)\ a_3\,\rho(a_4a_5)\,a_6 \otimes (a_7,a_8)$$

$$= (a_1,a_2a_3)\,\rho(a_4a_5) \otimes (a_6a_7,\ a_8)$$

Leads to $(a_1,a_2,a_4)\cdot(a_5,a_6,a_8)$

$$= (a_1,\ a_2\,\rho(a_4a_5)\,a_6,\ a_8)$$

I could have put $\mu(a_1,a_2) = a_1\,\chi\,a_2$

$\chi^{\in A}$ centralized by B.

M

Review: Assume A sep., $\xi_i \otimes \eta_i \in A \otimes A$ is canon sep. elt. Take $\rho(a) = \tau(\omega^{-1}a)$ ω inv.

$$x_i \otimes y_i = \xi_i \omega \otimes \eta_i = \xi_i \otimes \omega \eta_i$$

Obvious invariant is $x_i y_i = \xi_i \omega \eta_i$

Now $a \longmapsto \xi_i a \eta_i$ is a projection of A onto its center $A^\natural$, probably the canonical proj. So $\rho(1) = 1$ means $\tau(\omega^{-1}) = 1$.

Situation: Suppose $B \subset A$ both separable. We know that

$$(A \otimes_B A)^\natural \xrightarrow{\sim} (A \otimes_B A)_\natural = A \otimes_B$$
$$\parallel$$
$$(A \otimes^B A)^A \xrightarrow{\sim} (A \otimes^B A)_\natural = ((A \otimes A)_\natural)^B = A^B.$$

exact

Thus we have a description of central elements of $A \otimes_B A$ in terms of elements in the commutant B' of B in A. There should be a canonical element corresponding to $1 \in B'$. Now I need the corresponding ρ. ~~which means which leads to a great deal~~

$$\begin{tikzcd} \mathrm{Hom}_{B^{op}}(A, B) \arrow[r, two heads] & \mathrm{Hom}_B(B, B) \\ B \otimes_B A \arrow[u, "\wr"] & B \arrow[u, equal] \end{tikzcd}$$

Apparently $A \xrightarrow{\sim} \mathrm{Hom}_{B^{op}}(A, B) \xrightarrow{\text{res}} \mathrm{Hom}_{B^{op}}(B, B) = B$

$a \longmapsto (\alpha \longmapsto \rho(a\alpha))$

But doesn't have ρ yet.

$\mathrm{Hom}_{B^{op}}(A, B) = B \otimes^B \mathrm{Hom}(A, \mathbb{C}) \simeq B \otimes_B \mathrm{Hom}(A, \mathbb{C}) = \mathrm{Hom}(A, \mathbb{C})$

$\boxed{11}$ This looks close to working.

$\mathrm{Hom}_{B^{op}}(A, B)$

Problem: Assume $B \subset A$ both separable algs. Then we have a canonical relative separability element in $A \otimes_B A$ which is the image of the canonical sep. element in $A \otimes A$. The question is whether there is a corresponding $\rho : A \longrightarrow B$.

Is there a canonical $\rho \in \mathrm{Hom}_{B^\circ}(A, B)^B$

$$\mathrm{Hom}_{B^\circ}(A, B) = \left(B \otimes \mathrm{Hom}_{\mathbb{C}}(A, \mathbb{C})\right)^B$$

where B acts internally, i.e. to the right of B and the ~~left~~ right on A, hence left on $\mathrm{Hom}(A, \mathbb{C})$. Since B separable for any left B module M: $(B \otimes M)^B \xrightarrow{\sim} B \otimes_B M = M$

Thus it appears that

$$\left(B \otimes \mathrm{Hom}_{\mathbb{C}}(A, \mathbb{C})\right)^B \simeq \mathrm{Hom}(A, \mathbb{C})$$

has a canonical element.

Problem: If A separable, τ canonical trace, $x_i \otimes y_i$ the canon sep elt, then $\tau(1) = \dim A$ and $\mu : A \otimes A \to A$ takes the identity $x_i \otimes y_i$ to $x_i y_i = 1$. What about the next step?

$$(A \otimes A) \otimes_A (A \otimes A) \longrightarrow A \otimes A$$

$$(a_1, a_2) \otimes (1, a_3) \longmapsto (a_1\, \tau(a_2), a_3)$$

$$(x_i, 1, y_i) \longmapsto \tau(1)(x_i, y_i)$$

$\boxed{0}$

Let's begin with $B \subset A$ both separable. Then the canonical sep. elt in $A \otimes A$ yields a relative one in $A \otimes_B A$ and I want to find a corresponding ρ. Group alg case. Suppose $B = \mathbb{C}[H] < A = \mathbb{C}[G]$. Want B bimodule map $\mathbb{C}[G] \longrightarrow \mathbb{C}[H]$. Obvious map seems to kill $G - H$. So take ρ to do this. Take canonical central elt.

$$\frac{1}{|G|} \sum_{g \in G} g \otimes g^{-1}$$

$$\cancel{\frac{1}{|G|} \sum_{g \in G} g \otimes \rho(g^{-1}) = \frac{1}{|G|} \sum_{h} h \otimes h^{-1}}$$

$$\frac{1}{|G|} \sum_{g \in G} g \rho(g^{-1}) = \frac{1}{|G|} \sum_{h \in H} 1 = \frac{|H|}{|G|}$$

So you ask for a bimodule map.

So what's the game. Consider $H < G$. Then $\mathbb{C}[G]$ is free over $\mathbb{C}[H]$ both left and right.

~~Pick $\rho = 1_H$ characteristic fn. of H.~~

Pick $\rho(g) = \begin{cases} 0 & g \notin H \\ g & g \in H. \end{cases}$

Pick coset reps. $X \xrightarrow{\sim} G/H$

$$\frac{1}{[G:H]} \sum_{x \in X} x \, \frac{1}{|H|} \sum_{h \in H} (h \otimes h^{-1}) \, x^{-1}$$

$$\downarrow$$

$$\frac{1}{[G:H]} \sum_{x \in X} x \otimes x^{-1} \quad \in A \otimes_B A$$

P

What is ρ?

$$\frac{1}{[G:H]} \sum_x x\,\rho(x^{-1}) = \frac{1}{[G:H]}$$

Is $A \otimes_B A$ a group alg? NO.

e.g. $A = \mathbb{C}$ then $A \otimes A = \mathrm{End}(A)$ is simple.

~~canonical~~

canonical sep. elt. $x_i \otimes y_i$

is symmetric $x_i \otimes y_i = y_i \otimes x_i$

Thus ~~stan~~

~~Standard element~~

Review arg. $0 \to \Omega^1 A \to A \otimes A \to A \to 0$

splits $\Leftrightarrow$ $\exists$ separability elt: $\sum x_i \otimes y_i \in A \otimes A$ i.e. central and $x_i y_i = 1$

$\Leftrightarrow$ every derivation is inner.

$\exists$ sep. element $\Rightarrow$ every module projective so A semi simple

But also suppose x_i and y_i ind.

$a x_i f(y_i) = x_i f(y_i a)$

$\therefore \sum \mathbb{C} x_i \subset A$ left ideal

get faithful f.d. repn. Apply Wedderburn to get $A = \prod M_{n_i}\mathbb{C}$. then get nondegeneracy of trace as left reg. repn. From this get symmetric sep. elt.

~~Furthermore.~~ Then given A bimodule M.

$$M^{\natural} \xrightarrow{\sim} M_{\natural}$$

$\sum x_i m y_i \leftarrow\!\!\shortmid m$ well-defined

$A \otimes^{\natural} A \xrightarrow{\sim} A \otimes_{\natural} A \xrightarrow{\sim} A$

$$(A \otimes A)^{\natural} \xrightarrow{\sim} (A \otimes A)_{\natural} \xrightarrow{\sim} A$$

$\quad \cup\!\!\!\shortmid$

$x_i \otimes y_i \qquad y_i x_i = 1$

get! $x_i \otimes y_i$ inside central $x_i y_i = 1$.

also outside central as $a x_i \otimes y_i$, $x_i \otimes y_i a$

So we win.

~~So~~ Next suppose $B \subset A$ separable

Then there ~~[illegible]~~ should be a canonical separating element in $A \otimes^B A \xrightarrow{\sim} A \otimes_B A$. Namely take the one for A and average for B. Is there a corresponding ρ? ~~Is there a corres~~

So let us try again to make some progress.

$$\mathrm{Hom}_{\mathbb{C}}(A,B) = B \otimes \mathrm{Hom}_{\mathbb{C}}(A,\mathbb{C})$$

$$\mathrm{Hom}_{B^{op}}(A,B) = B \otimes^B \mathrm{Hom}_{\mathbb{C}}(A,\mathbb{C})$$

$$\xrightarrow{\sim} B \otimes_B \mathrm{Hom}_{\mathbb{C}}(A,\mathbb{C}) \xleftarrow{\sim} \mathrm{Hom}(A,\mathbb{C})$$

There seems to be a canonical way to take a linear functional f on A and get a right B-mod. map $A \longrightarrow B$. This uses the canon. elt $\xi_i \otimes \eta_i \in B \otimes B$

$$a \longmapsto f(a\xi_i)\eta_i$$

So $\rho(a) = \tau(a\xi_i)\eta_i = \eta_i \tau(\xi_i a)$

$$\rho(x_j)y_j = \tau(x_j\xi_i)\eta_i y_j = \tau(x_j)\eta_i(\xi_i y_j)$$

$$= \tau(x_j)y_j = 1.$$

Suppose $B \subset A$ are separable algs.

Let $\xi_j \otimes \eta_j \in B \otimes B$

$x_i \otimes y_i \in A \otimes A$

be the canonical separab. elts. τ the canonical trace on A so that $\tau(x_i y_j) = \delta_{ij}$.

Let
$$\rho(a) = \xi_j \tau(\eta_j a) \qquad \rho : A \to B$$
$$= \tau(a\eta_j)\xi_j$$
because $\tau(a\eta_j) = \tau(\eta_j a)$ is a scalar.

Then
$$\rho(ba) = \xi_j \tau(\eta_j ba)$$
$$= b\xi_j \tau(\eta_j a) = b\rho(a)$$
$$\rho(ab) = \tau(ab\eta_j)\xi_j = \tau(a\eta_j)\xi_j b = \rho(a)b$$

Also
$$\rho(x_i)y_i = \tau(x_i\eta_j)\xi_j y_i$$
$$= \tau(x_i)\xi_j \eta_j y_i = \tau(x_i)y_i = 1$$
$$x_i \rho(y_i) = \cancel{x_i \tau(y_i \eta_j)\xi_j}$$
$$= x_i \xi_j \tau(\eta_j y_i)$$
$$= x_i \eta_j \xi_j \tau(y_i) = x_i \tau(y_i) = 1.$$

In the group case
$$\rho(g) = \frac{1}{|H|}\sum_{h \in H} \underbrace{\tau(gh^{-1})}_{\begin{cases} 0 & g \neq h \\ |G| & g = h \end{cases}} h$$
$$= \frac{|G|}{|H|}\begin{cases} 0 & g \notin H \\ g & g \in H. \end{cases}$$
$$\frac{1}{|G|}\sum_g g\rho(g^{-1}) = \frac{1}{|G|}\frac{|G|}{|H|}\sum_{g \in H} gg^{-1} = 1.$$

$\boxed{5}$ $B = \mathbb{C}$. $\rho(a_1 a_2)$ non deg.

$$A \xrightarrow{\sim} \mathrm{Hom}(A, \mathbb{C}) \quad \text{isom of right modules}$$
$$a \longmapsto (\alpha \mapsto \rho(a\alpha))$$

$$A \otimes A \longrightarrow \mathrm{Hom}(A, A)$$
$$(a_1, a_2) \longmapsto (\alpha \longmapsto a_1 \rho(a_2 \alpha))$$

(x_i, y_i) ~~$a = x_i \rho(y_i a)$~~

If ρ nondegenerate, then so are

$\alpha \mapsto \rho(w\alpha)$ and $\rho(\alpha w)$ for w invertible

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Since $\mathrm{Hom}(A, \mathbb{C})$ is right module $\simeq A$

$A^\times$ acts to the right on generators.

$$A \xrightarrow{w\cdot} A \xrightarrow{1 \mapsto \rho} A^*$$
$$1 \longmapsto w \longmapsto \rho \cdot w = (\alpha \longmapsto \rho(w\alpha))$$

Review & right up.

$B \subset A$ sep. τ canonical trace on A.

$x_j' \otimes y_j' \in B \otimes B$ canon sep elt.

$(x_i, y_i) \in A \otimes A$ canon sep elt.

$\tau(x_i y_j) = \delta_{ij}$

$\rho(a) = x_j' \tau(y_j' a)$
$= \tau(a y_j') x_j'$
$= \tau(a x_j') y_j'$

$a = x_i \tau(y_i a) = \tau(a x_i) y_i$

why?

$a = x_i c_i \implies \tau(y_i a) = c_i$

$\implies a = x_i \tau(y_i a)$

$\rho(ab) = \tau(a b x_j') y_j'$
$= \underbrace{\tau(a x_j') y_j'}_{\rho(a)} b$

$a = c_i y_i \implies \tau(a x_i) = c_i$

$\implies \tau(a x_i) y_i = a$.

$\rho(ba) = x_j' \tau(y_j' ba) = b \underbrace{x_j' \tau(y_j' a)}_{\rho(a)}$ also symmetric.

$\boxed{t}$ Calculate 7:20

$$\rho(x_i)y_i = x_j'\,\tau(y_j'x_i)\,y_i = x_j'\,\underbrace{\tau(x_i)y_i}_{1}\,y_j'$$

$$= x_j'y_j' = 1.$$

$$\rho(x_i)y_i = \tau(x_i x_j')\,y_j'\,y_i = \tau(x_i)\,\underbrace{y_j' x_j'}_{1}\,y_i = 1.$$

Example. $B = \mathbb{C}[H] \subset \mathbb{C}[G]$

$$x_j' \otimes y_j' = \frac{1}{|H|}\sum_{h\in H} h^{-1}\otimes h \qquad \tau'(h) = \begin{cases} |H| & h = 1 \\ 0 & h \neq 1 \end{cases}$$

$$x_i \otimes y_i = \frac{1}{|G|}\sum_{g\in G} g^{-1}\otimes g \qquad \tau(g) = \begin{cases} |G| & g = 1 \\ 0 & g \neq 1 \end{cases}$$

$$\rho(g) = \frac{1}{|H|}\sum_{h\in H} h\,\tau(h^{-1}g)$$

$$= \begin{cases} 0 & \text{if } g \notin H \\ \frac{|G|}{|H|}\,g & \text{if } g \in H. \end{cases}$$

$$\rho(x_i)y_i = \frac{1}{|G|}\frac{|G|}{|H|}\sum_{g\in H} \overbrace{g\,g^{-1}}^{1} = 1.$$

In this example $\rho(1) = (G:H)\,1$

Serious question. Given $B \subset A$ both sep + the canonical ρ do we know anything about $\rho(1) = \tau(x_i')y_i'$?

However $\mathbb{C} \subset A \subset A\otimes A$

Question: Suppose A sep. and consider $A\otimes A$ prod via τ and identity the can

$\boxed{u}$ Suppose you give a $x_i \otimes y_i \in (A \otimes_B A)^?$ such that $\exists\, \rho : A \to B$ B-bimodule map

$\ni \quad \rho(x_i) y_i = x_i \rho(y_i) = 1.$

~~We know then that~~ ρ might not be uniquely determined

March 15

Suppose $\mathbb{C} \subset A$, A separable, and we ~~assume~~ take $\tau : A \to \mathbb{C}$ to be the canonical trace whence

$$A \otimes A \xrightarrow{\sim} \operatorname{Hom}_{\mathbb{C}}(A, A)$$
$$a_1, a_2 \longmapsto (\alpha \mapsto a_1 \tau(a_2 \alpha))$$

The identity operator on A is then given by the canonical sep. element $(x_i, y_i) \in A \otimes A$. x_i basis y_i dual basis: $\tau(y_j x_i) = \delta_{ij}$.

Let $\tau(a)$ = trace of left mult by a.

Assume non deg. Let x_i be basis for A y_i the dual basis $\tau(y_i x_j) = \delta_{ij}$.

Then $a x_j = x_i a_{ij}$ where $a_{ij} = \tau(y_i a x_j)$

and $y_i a = a_{ij} y_j$ so

$$a x_j \otimes y_j = x_i a_{ij} \otimes y_j = x_i \otimes a_{ij} y_j = x_i \otimes y_i a$$

so $x_i \otimes y_i$ is central. Finally

$$\tau(a) = a_{ii} = \tau(y_i a x_i) = \tau(a x_i y_i)$$
$$\tau(a(1 - x_i y_i)) = 0 \quad \forall a \implies x_i y_i = 1.$$

~~anyway~~ $\quad (a_1 \tau(a_2 x_i), y_i)$

$$f(x_i)\tau(y_i \alpha) = f(x_i \tau(y_i \alpha)) = f(\alpha).$$

$\boxed{V}$ ~~[illegible]~~ Repeat:

Start with $\mathbb{C} = B \subset A$ sep, $\tau : A \to \mathbb{C}$ canon. trace whence

$$A \otimes A \xrightarrow{\sim} \mathrm{Hom}(A,A)$$
$$(a_1, a_2) \longmapsto (\alpha \mapsto a_1 \tau(a_2 \alpha))$$
$$(f(x_i), y_i) \qquad f$$

~~Question~~: On $\mathrm{Hom}(A,A)$ have the canonical trace τ' as separable alg which is $\dim A$ times matrix trace tr. ~~Also have Average wrt the subalg A~~

$$\mathrm{tr}(f) = \tau(y_i f(x_i))$$
$$\mathrm{tr}((a_1, a_2)) = \tau(y_i \, a_1 \tau(a_2 x_i))$$
$$= \tau(a_1 \tau(a_2 x_i) y_i)$$
$$= \tau(a_1 a_2)$$

Now have this linear ~~form~~ map

$$g : A \otimes A \xrightarrow{\tau\mu} \mathbb{C} \longrightarrow A$$

To make into a bimodule map

$$\cancel{\rho(\xi) = g(x_i \xi y_i) \qquad x_i g(\xi y_i)}$$
$$\cancel{\rho(a\xi) = g(x_i a \xi y_i) \qquad x_i g(}$$

$$\rho(\xi) = x_i \, \tau\mu(y_i \xi) = \tau\mu(\xi y_i) x_i$$

$$\rho(a_1 \otimes a_2) = \tau\mu(a_1 \otimes a_2 y_i) x_i$$
$$= \tau(a_1 a_2 y_i) x_i = a_1 a_2 .$$

So the matrix trace on $\mathrm{Hom}(A,A)$ yields the product μ on A.

[M] I have to write up various things.
But first maybe we try to find a representation of the increasing ~~system~~ family A_n.

First: Assume $B \subset A$ both sep.
$\tau = \tau_A$ canon. trace. ~~Then we get a $\rho : A \to B$~~
$\tau' =$ canon. trace on B. $(x_j', y_j') \in B \otimes B$
canon. sep elt $(x_i, y_i) \in A \otimes A$ also

Define $\rho : A \longrightarrow B$

$$\rho(a) = x_j' \, \tau(y_j' a) = \tau(a y_j') x_j' = \tau(a x_j') y_j'$$

Then $(x_j' \otimes y_j') b = b (x_j' \otimes y_j') \Rightarrow \rho(ba) = b\rho(a)$

$x_j' \otimes b y_j' = x_j' b \otimes y_j' \Rightarrow \rho(ab) = \rho(a) b.$

Also

$$\rho(x_i) y_i = x_j' \tau(y_j' x_i) y_i = x_j' y_j' = 1$$

$$x_i \rho(y_i) = \underbrace{x_i \tau(y_i x_j')}\, y_j' = x_j' y_j' = 1.$$

So we can proceed.

The ~~[illegible]~~ conjecture I think is that if I form the family $B \subset A \subset A \otimes_B A$

$$A_0 \subset A_1 \subset A_2$$

starting from this ρ, then up to nonzero scalars we have the standard choices, i.e. $\mu : A \otimes_B A \longrightarrow A$ should coincide (up to scalars) with the canonical ~~choice~~ ρ obtained from the separability of $A \otimes_B A$ and A.

$\boxed{\times}$ OKAY we ~~...~~ concentrate upon the simplest case namely $B = \mathbb{C}$, A separable, and use the canonical elts. We have

$$A \longrightarrow A \otimes A \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{C}}(A, A) = A_2$$

Now $\tau_{A_2} = \dim(A)\, tr$ where tr is the trace of an operator on A. Thus the $\rho : A_2 \to A$ associated to $\tau_{A_2} : A_2 \longrightarrow \mathbb{C}$ is

$$\rho(T) = x_i \,\tau_{A_2}(y_i T) = \dim(A)\ x_i\, tr(y_i T)$$

If $T(\alpha) = a_1 \tau_A(a_2 \alpha)$, then

$$tr(y_i T) = \tau_A(a_2 y_i a_1)$$

so

$$\rho\big((a_1, a_2)\big) = \dim(A)\ x_i\, \tau_A(y_i a_1 a_2)$$
$$= \dim(A)\, a_1 a_2 . = \dim(A)\, \mu((a_1, a_2)) .$$

Now the identity elt in $A_3 = A_2 \otimes_{A_1} A_2$ using μ is $(x_i, 1, y_i)$

Start at beginning.

$\mathbb{C} \subset A$ use $E_1 = \frac{1}{\dim A} \tau_A$ so that $E_1(1) = 1$.

The corresp. identity in $A_2 = A \otimes A$ is $\dim(A)(x_i, y_i)$ since

$$(x_i, y_i)(x_j, y_j) = (x_i\, E_1(y_i x_j), y_j)$$
$$= \frac{1}{\dim(A)}(x_i\, \tau(y_i x_j), y_j) = \frac{1}{\dim(A)}(x_i, y_i)$$

Now ~~...~~ $\mu\big(\dim(A)(x_i, y_i)\big) = \dim(A)\, x_i y_i$
$= \dim(A).$

$\boxed{y}$ Thus I must use $\frac{1}{\dim A}\mu$ for E_2.

I admit to being confused about the scalars.

Let's start again. Take

$$B \subset A \xrightarrow{\rho} B$$

$\left\{ \begin{array}{l} x_i \otimes y_i \in A \otimes_B A \\ \text{central} \\ \rho(x_i) y_i = x_i \rho(y_i) = 1. \end{array} \right.$

This is the general situation. ~~So what~~ ~~Start with~~. So you have

$$\begin{array}{lll} A_0 = B & & 1_B \\ A_1 = A & \rho, & 1_A \\ A_2 = A \otimes_B A, & \mu, & (x_i, y_i) \\ A_3 = A \otimes_B A \otimes_B A, & 1 \otimes \rho \otimes 1, & (x_i, 1, y_i) \\ A_4 = & & (x_i, x_j, y_i, y_j) \end{array}$$

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example: group alg of an abelian group.
A_2 something like a Weyl or Clifford algebra.

$$RA = \Omega^{ev} A = A + \Omega^2 A +$$

$$A^{\otimes 3} \cap IA = \Omega^2 A$$

$$\rho(A) = A$$

$$\rho(A)^2 = A \oplus d\Omega^1 A$$

$$\rho(A)^3 = A \oplus \Omega^2 A$$

$$\rho(A)^4 = \underbrace{A \oplus \Omega^2 A \oplus d\Omega^3 A}$$

have K on this side.