$\boxed{r}$ A homotopy should be an odd cochain $\varphi \in (\Omega\tilde{A})^*$ such that $\varphi(b+B) = \varphi$ except in degree zero. [redacted]

[redacted] So we end up taking the cocycle

$$\sum_{n \geq 0} (-1)^n \rho \frac{\omega^n}{n!}$$

and writing ~~it~~ as a coboundary. But this cocycle corresponds to the homomorphism $RA \longrightarrow \mathbb{C}$ induced by $\rho$.

The hard part will be to lift $E$ into $\Omega\tilde{A}$

---

Review it.

$$0 \longrightarrow \Lambda \hat{\otimes} B(\mathbb{C}) \longrightarrow E \longrightarrow \Omega A \longrightarrow 0$$

Maybe you can actually E into $\Omega\tilde{A}$. As $\Lambda$-module $E$ should be a direct sum of these two, so it should be a question of lifting $\Omega A \to \Omega\tilde{A}$ compatibly with $B$. Also you want to lift $\Lambda \tilde{\otimes} B(\mathbb{C})$ into $\Omega\tilde{A}$. That's probably very simple. So what approach.

$$R(\tilde{A}) \longrightarrow \mathbb{C} \times RA$$

I remember this being hard. First case

$$R(\tilde{\mathbb{C}}) \longrightarrow \mathbb{C} \times \mathbb{C}$$

$\tilde{\mathbb{C}} = \mathbb{C}[e]$ where $e = 1$ in $\mathbb{C}$.

$\rho(e) = x$ $\quad x \to 0$ aug $\quad x \to 1$ $\tilde{\mathbb{C}} \to \mathbb{C}$

5] There is a unique lifting of $e$ ~~to~~ an idempotent made ~~out~~ ~~of~~ $\rho(e) = x$. Anyway after lifting the idempotent $e$ you need to lift $RA$ into the centralizer of $\hat{e}$. The question is whether there is a reasonable way to do this? You can take

$$\hat{e}R(\tilde{A})\hat{e} \longrightarrow RA$$

In order to construct a lifting you need only lift $A$ linearly so that $e$ goes to $\hat{e}$

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr, "\rho"] & \\ R(\tilde{A}) \arrow[rr] \arrow[d] & & RA \arrow[d, equal] \\ \hat{e}R(\tilde{A})\hat{e} \arrow[rr] & & RA \end{tikzcd}$$

$$\begin{tikzcd} & W^{\rho(e)} \arrow[d] \\ A \arrow[ur, "\rho\ \text{linear}"] \arrow[r] & \mathbb{C} \times RA \\ a \arrow[r, mapsto] & (0, \rho a) \end{tikzcd}$$

$\rho(e)$ centralizes $\hat{e}$

---

To be checked carefully

$$R(\tilde{A}) \longrightarrow \underbrace{R\mathbb{C} \times RA}_{\widetilde{RA}} = \mathbb{C} \times RA$$

A hom. $\widetilde{RA} \longrightarrow R$ same as an idempotent $e \in R$ and a linear map $\rho : A \longrightarrow eRe$ such that $\rho(1_A) = e$

The claim is that there's a quasi-canonical process for lifting:

t

$$\begin{tikzcd}
R(\tilde{A}) \arrow[r] \arrow[d] & S \arrow[d, "\text{nilpotent}"] \\
\widetilde{RA} \arrow[r] \arrow[ur, dashed, "\text{comm.}"'] & R
\end{tikzcd}$$

How does it work? How does it go?

We have the following data: an idempotent $e \in R$ and a $\rho : A \longrightarrow eRe$ such that $\rho(1_A) = e$

Better can suppose given

$$\begin{tikzcd}
R(\tilde{\mathbb{C}}) \arrow[rr] \arrow[d, phantom, "\cap" rotate=0] & & S \arrow[d, "\text{nilp.}"] \\
R(\tilde{A}) \arrow[rr] \arrow[urr] & & \widetilde{RA} = \mathbb{C} \times RA \ni e
\end{tikzcd}$$

$e$ is the identity of $RA$

Start again. Suppose $S$ is a nilpotent extension of $\mathbb{C} \times RA$ and suppose that we are given $l : \mathbb{C} \times A \to S$ lifting of the subspace $\mathbb{C} \times A \subset \mathbb{C} \times RA$. such that $(1,0)$ goes to the identity of $S$.

$$\begin{tikzcd}
 & S \arrow[d, "\pi"] \\
A \arrow[r] \arrow[ur, "l"] & \mathbb{C} \times RA \\
a \arrow[r, mapsto] & (0, \hat{\rho} a)
\end{tikzcd}$$

$e = 1_A$. Then $\pi l(e) = (0, \hat{\rho} e)$. Unique poly in $l(e)$ which is idempotent in $S$, call it $\hat{e} \in S$.

So $\hat{e}$ has to go to $(0,1) \in \mathbb{C} \times RA$.

Now consider

$$a \longmapsto \hat{e}\,l(a)\,\hat{e} \in \hat{e}S\hat{e} \subset S$$

$$\pi\left(\hat{e}\,l(a)\,\hat{e}\right) = (0,1)(0,\hat{\rho}a)(0,1)$$
$$= (0,\hat{\rho}a).$$

so $\quad \hat{e}\,l(1_A)\,\hat{e} = \hat{e}\,l(1_A)$

So what we find is that we can cut $S$ down to $\hat{e}S\hat{e}$ and also $l$ down to $a \mapsto \hat{e}\,l(a)\,\hat{e}$ and we have

$$\begin{tikzcd} & \hat{e}S\hat{e} \arrow[d] \\ A \arrow[ur, "\hat{e}l\hat{e}"] \arrow[r, "\hat{\rho}"'] & RA \end{tikzcd}$$

The problem is that $\hat{e}\,l(1_A)\,\hat{e}$ ~~is~~ not necessarily equal to $\hat{e}$. This seems to happen already for $A = \mathbb{C}$. $\quad l(1)=x$

$$\begin{tikzcd} & S \arrow[d] \\ \mathbb{C} \arrow[ur, "l"] \arrow[r] & \mathbb{C} \times \mathbb{C} \end{tikzcd}$$

$$1 \longmapsto (0,1)$$