e) Let's try to recall about topological Markov ~~[illegible]~~ chains:

~~[illegible]~~ Construction:

Given a finite sets X ~~and a~~ and a correspondence on it, i.e. a finite set Y and two arrows $Y \overset{s}{\underset{t}{\rightrightarrows}} X$. Then we form the infinite fibre product

$$\cdots \xrightarrow{t} X \xleftarrow{s} Y \xrightarrow{t} X \xleftarrow{s} Y \xrightarrow{t} X \xleftarrow{s} Y \cdots$$

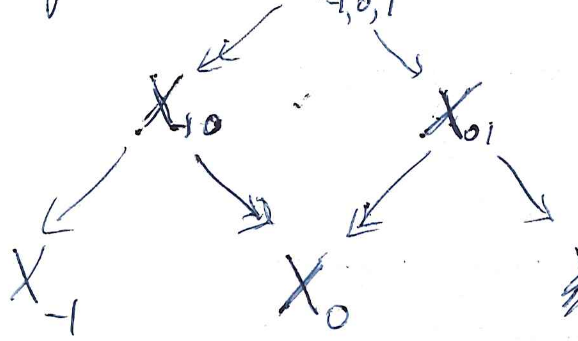
to simplify suppose $(s,t) : Y \to X \times X$ is injective i.e. Y is a relation on X. Then we are looking at sequences $(x_n) \in X^{\mathbb{Z}}$ such that $\forall\, n$ $(x_n, x_{n+1}) \in Y$. Let's call this space $\Omega \subset X^{\mathbb{Z}}$. Its a compact totally disconnected with $\mathbb{Z}$ action. ~~I want to~~

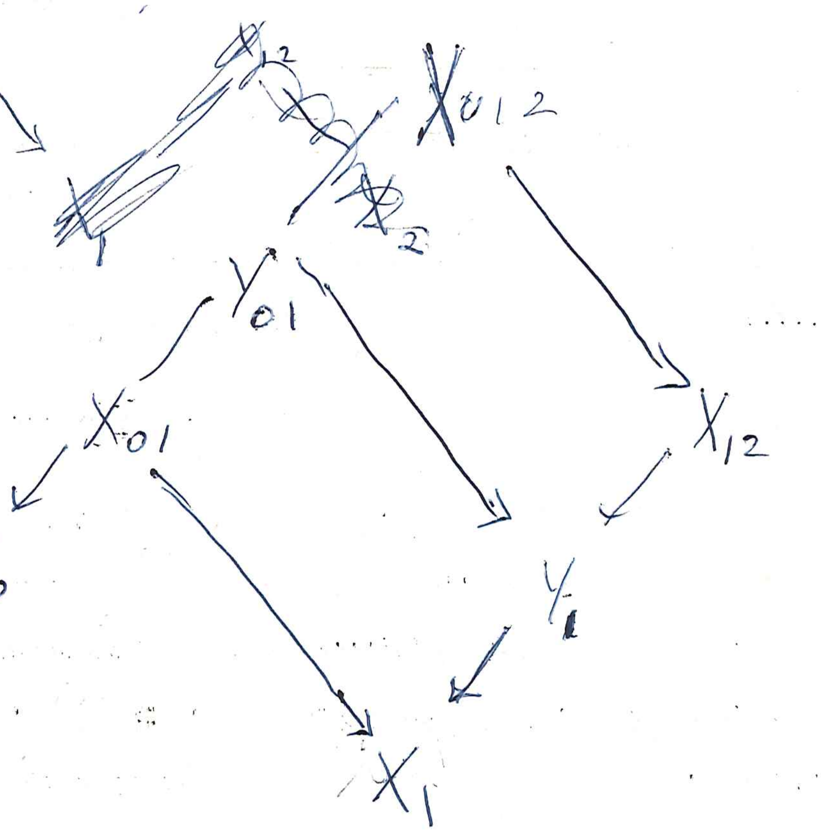
You want to assume that $\Omega \xrightarrow{pro} X$ is surjective. Given any x_0 you need to be able to connect it to an x_1 and an x_{-1}. This amounts to the surjectivity of s,t.

Next. Suppose you have Ω compact tot. disc. ~~and a par~~ with $\mathbb{Z}$ action, and a partition of Ω i.e. a cont. surj $\Omega \twoheadrightarrow X$ finite ~~quotient~~ space. Then get a map $\Omega \longrightarrow X^{\mathbb{Z}}$. ~~One of the first assumptions to~~ One assumes $\exists$ a finite quotient such that $\Omega \hookrightarrow X^{\mathbb{Z}}$, ~~and then you~~ ~~this~~ is a finite type assumption, then you want

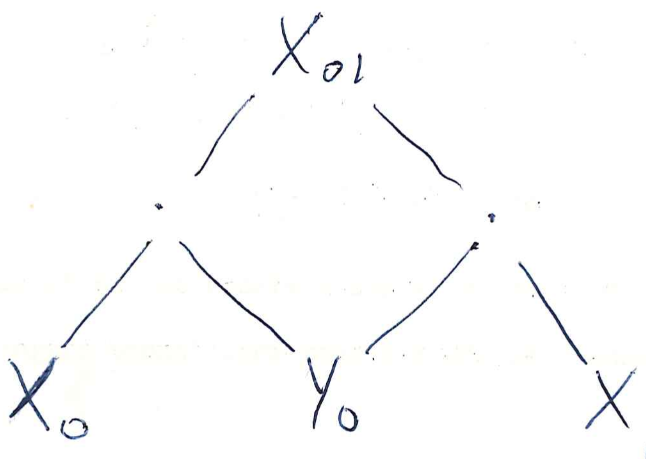
f) something else, which makes it ~~[illegible]~~ Markov. ~~[illegible]~~ What is the equivalence relation Sullivan told me about?

The question is when ~~[illegible]~~ are two partitions $X_{-1,0,1}$ close or related.





Xerox Cindy's book



n) ~~So we~~ $A^2 = A$ assume

If $AM = M$ $\cancel{A \otimes_A A}$ then $A \otimes_A M$ is good

In general $A \otimes_A A \otimes_A M$ is good.

Let's go on to the endomorphisms of the forgetfull functor. Claim they are $\mathrm{Hom}_{A^{op}}(A, A)$.

Let $R = \mathrm{Hom}_{A^{op}}(A, A)$ ring. Also have a homom. $A \xrightarrow{l} R$ image is an ideal?

$$l(a)a' = aa'.$$

Then $x\, l(a)\,(a') = x(aa') = x(a)a'$

Try $l_a(a') = aa'$ $\qquad x \in R$

$$(x l_a)(a') = x(l_a(a')) = x(aa') = x(a)a'$$
$$= l_{x(a)}(a').$$
$$(l_a x)(a') = l_a(x(a')) = a\,x(a')$$

The image is thus a left ideal in R.

Now if $A \otimes_A M \xrightarrow{\sim} M$, then R acts on M; there is a unique R module structure on M extending the A-module structure.

Consider $\mathrm{Modg}(A) \xrightarrow{F} \mathrm{Ab}$

$$M \longmapsto M$$

0) Since $A \otimes_A M \xrightarrow{\sim} M$ functorial isom.

Some interesting points are the fact that the forgetful functor ~~from~~

$$\mathrm{Modg}(A) \subset \mathrm{Mod}(A)$$

is not exact. It's right exact, the higher left derived functors have values in the thick subcategory of null A-modules.

What are these functors?

In the main example $A = Ae \otimes_B eA$, $B = eAe$

$$\mathrm{Modg}(A) \simeq \mathrm{Mod}(B)$$

The forgetful functor is $N \mapsto Ae \otimes_B N$

the derived functors are $\mathrm{Tor}_n^B(Ae, N)$.

Is it clear these are null modules.

$$Ae \otimes_B P.$$

consider multiplication by $a_1 e a_2$.

Actually an A-module X is null iff $eX = 0$. since $eX \subset AeAX \subset AeX$. ~~Clear. Everything is clear.~~ So we get many examples because I've classified A $A = Ae \otimes_B eA$. Describe by

$$A = \begin{pmatrix} B & W_1 \\ V_1 & V_1 \otimes_B W_1 \end{pmatrix}$$

B, ${}_BW_1$, V_{1B} unital. $W_1 \otimes V_1 \longrightarrow B$ arb. bin. map

Ae in particular is $B \oplus V_1$ where V_1 can be arbitrary.

p) ~~Analogy with sheaf theory~~

covariant version of sheaf theory.

But first what are the endos of the forgetful functor. Is it representable?

$$\cancel{\mathrm{Hom}} \quad N \longmapsto Ae \otimes_B N \sim \mathrm{Hom}_B(B, N) \ ?$$

Let $R = \mathrm{Hom}_{B_r}(Ae, Ae)$

$F(N) = Ae \otimes_B N$

$S = \mathrm{Hom}_{\mathrm{Fn}_{add}(\mathrm{Mod}(B),\, Ab)}(F, F)$

Then have a homom $S \longrightarrow R$.

namely ~~S → Hom~~ given $T \in S$. ~~Ca~~

This means $T_N : Ae \otimes_B N \circlearrowleft$ $\forall N$

comp. with $\forall N \longrightarrow N'$ In particular

take $T_B : Ae \otimes_B B \circlearrowleft$ all $B \circlearrowleft^{B}$ left B module maps i.e. right mult.

This is a homom. $S \longrightarrow R$.

on other hand have homom. $R \longrightarrow S$.

Composite $R \to S \to R$ is identity

Other way take T.

$$\begin{tikzcd}
Ae \otimes_B B \arrow[r, "T_B"] \arrow[d, "1 \otimes (\cdot n)"] & Ae \otimes_B B \arrow[d, "1 \otimes (\cdot n)"] \\
Ae \otimes_B N \arrow[r, "T_N"] & Ae \otimes_B N
\end{tikzcd}$$

Commutes for all $n \in N$.

Does this imply $T_N = \tilde{T}_B \otimes 1_N$. clear

8)

$$\begin{array}{ccc}
Ae & \overset{\tilde{T}_B}{\longleftarrow} & Ae \\
\big\uparrow \cong & & \\
Ae \otimes_B B & \overset{T_B}{\longleftarrow} & Ae \otimes_B B \\
\big\downarrow 1\otimes(\cdot n) & & \\
Ae \otimes_B N & \overset{T_N}{\longleftarrow} & Ae \otimes_B N
\end{array}$$

~~$T_N(ae\otimes n) \otimes$~~

~~$ae \otimes n$~~

$$\begin{array}{ccc}
ae & \longmapsto & \tilde{T}_B(ae) \\
\updownarrow & & \\
ae\otimes 1 & \longmapsto & T_B(ae\otimes 1) = \tilde{T}_B(ae)\otimes 1 \\
\big\downarrow & & \big\downarrow \\
ae\otimes n & & T_B(ae\otimes n) = \tilde{T}_B(ae)\otimes 1
\end{array}$$

Thus in this example the endos of the forgetful functor are

$$\mathrm{Hom}_{B_r}(Ae, Ae) = \mathrm{Hom}_{A_r}(A, A).$$

because $A = Ae \otimes_B eA$ is the good right A module corresp to Ae. Remember $Ae = B \oplus \underbrace{e^{\perp}Ae}_{V_{1B}}$

So what comes next?

$Ae.$

$$\mathrm{Hom}_{A_r}(Ae\otimes_B eA,\ Ae\otimes_B eA) = \mathrm{Hom}_{B_r}(Ae,\ \mathrm{Hom}_{A_r}(eA,\ Ae\otimes_B eA))$$

n) So now I understand I think the endos of the ~~forgett~~ forgetfull functor. The problem is however it's not very interesting. How do multipliers come into play? In general you have

$$\mathrm{Hom}_{A_r}(A,A) = \mathrm{Hom}_{B_r}(Ae, Ae)$$ possible left multiplications

$$\mathrm{Hom}_{A_\ell}(A,A) = \mathrm{Hom}_{B}(eA, eA)$$ possible right mult.

To get a multiplier you take a pair satisfying some compatibility.

$$\cancel{(\mu \times a) \times \mu}$$

$$\boxed{(a_1 \times \mu)\, a_2 = a_1(\mu \times a_2)}$$

condition for all pairs.

$$(ea_1 \times \mu)\, a_2 e = ea_1(\mu \times a_2 e)$$

This has to do with the basic pairing $eA \otimes_A Ae \xrightarrow{\sim} B$.

viewpoint: We have examined the analogue of the forgetful functor from sheaves to presheaves. This is the inclusion of good module in modules.

$B \oplus W \quad B$

V

~~Now what I want is the analogue~~

This is like looking at the functors $\Gamma(U, F)$ for each open set U. What is $\Gamma(X,F)$?

5) morphism of topoi ?

pair (f^*, f_*) of adjoint functors where

f^* ~~[illegible]~~ & commutes with finite $\varprojlim$'s.

left adjoint which should be right exact.

$$\begin{array}{ccc} \mathrm{Modg}(A) & \xrightarrow{f} & \mathrm{Modg}(B) \\ & \xleftarrow{\text{right adjoint, subtler, exact.}} & \\ \cap & & \cap \\ \mathrm{Mod}(A) & \xrightarrow[\text{obvious functor}]{} & \mathrm{Mod}(B) \end{array}$$

Obvious functor should be

$$M \longmapsto {}_B Y \otimes_A M$$

we want $B \otimes_B Y = Y \otimes_A A$ ~~so~~ that

good A modules go to good B ~~g~~ modules.

What I really would like is cosheaves example, say ~~[illegible]~~ arising from ~~sheaves with~~ ~~[illegible]~~ sections with compact support. In fact I would like a nice example!

Something to do with partitions of unity? Other questions. Properties of abelian category of good modules. Should have enough projectives

$$\mathrm{Hom}_A(M, N) = \mathrm{Hom}_{\text{good}}(M, \tilde{A} \otimes_A N)$$