a Kapranov's paper reveals the possibility that I might ~~be able to understand quivers;~~ begin to learn about quivers, quantum groups. There's really a lot to learn. Gelfand's moves. This is the first point to understand, and probably this will give examples of tilting. Exceptional divisors? Bondal. Here you have a hint. Take a quiver. ~~You have~~ It is possible to reverse arrows. Go back to your Kronecker example.

$$\bullet \rightrightarrows \bullet$$

There should be some equivalence with $\bullet \leftleftarrows \bullet$

$$W \longrightarrow H \otimes V \qquad\qquad H^* \otimes W \longrightarrow V.$$

It seems that you don't have a good enough perspective.

What can I say about path algebras? Suppose no loops. You have? Maybe you can understand the derived category equivalence in a simple case. What's the geometry? Answer? Path algebras are tensor algebras of a bimodule. Product of fields R/I_{rad}. What do you know? Module theory.

~~What do you know about~~ $K_0 R$ free abelian group generators ind. projectives. Think of having a sheaf on a poset. Good familiar model which you have studied in the past. Each simple object has a proj. hull.

6 So what to do?

discuss examples of quivers.

$\bullet \longrightarrow \bullet \longrightarrow \bullet \qquad V_0 \xrightarrow{a} V_1 \xrightarrow{b} V_2$

split off $\mathrm{Ker}(a)$, $\mathrm{Coker}(b)$. Then you have V_1 equipped with subspace aV_0, $\mathrm{Ke}(b)$.

Consider $A, B \subset V$ ~~generators~~ splits into

$$\begin{tikzcd}[column sep=small] & V \arrow[d, dash] & \\ & A+B \arrow[dl, dash] \arrow[dr, dash] & \\ A \arrow[dr, dash] & & B \arrow[dl, dash] \\ & A\cap B \arrow[d, dash] & \\ & 0 & \end{tikzcd}$$

$A \cap B \oplus A/A\cap B \oplus B/A\cap B \oplus V/A+B$

which should be independent. So you get 4 indecomposables $V = k$ A, B are k or 0.

The quiver has 6 indecomposables.

$k \longrightarrow 0 \longrightarrow 0$

$0 \longrightarrow 0 \longrightarrow k$

$0 \longrightarrow k = k \qquad A = B = 0$

$k = k \longrightarrow 0 \qquad A = k,\ B = k$

$0 \longrightarrow k \longrightarrow 0 \qquad A = 0,\ B = k$

$k = k = k \qquad A = k,\ B = 0.$

c Try next $V_0 \to V_1 \to V_2 \to V_3$

Somehow there's this reversing all the arrows at a sink or source, which I have yet to understand. Equivalence of indecomposables.

Simple examples: ~~Let us start with gild~~

Suppose we have $i \xrightarrow{a} j$. Can ~~restrict~~ split off the cokernel of a, whence you've replaced

---

Gelfand et al. functors. Take $V_0 \xrightarrow{a} V_1 \xrightarrow{b} V_2$ first. This is an abelian category. To reverse the first arrow, we first split off Ker(a), ~~thus~~ get a injective ?. Other side – split off cokernel of b, get b surj. and pass to $\mathrm{Ker}(b) \subset V_1$. ~~Pass to cokernel~~

Transitions: ~~V_0~~

$$V_0 \xrightarrow{a} V_1 \xrightarrow{b} V_2$$

~~Take Ker b~~

$$V_0 \xrightarrow{a} V_1 \xrightarrow{b} \mathrm{Im}\, b$$

~~Ker b~~

$$V_1 \hookleftarrow \mathrm{Ker}\, b$$

~~$V_0 \to V_1 \quad V_2$~~

Anyway ~~what~~ happens? ~~$V_0 \to V_1$~~

Look carefully at the case. ~~V_0~~ $W \xrightarrow{\varphi} V$

abelian category of these reps. Two simple objects – canonical filtration.

I ~~@~~ idea maybe is this. ~~[illegible]~~ ~~[illegible] process!!~~ better example Take sheaves on the arc simplex. and compare with cosheaves. Sheaves and cosheaves. You take the Grothendieck filtration by support - Cousin complex stuff. I once knew this pretty well, but now ~~[illegible]~~ mix it with hdim ≤ 1.

God I am slow. quotient

$$W \xrightarrow{a} V$$

Take sheaf on ~~[illegible]~~ Δ_1

$$F_0 \longrightarrow F_{(0,1)} \longleftarrow F_1$$

locally const on strata

I want to understand Gelfand-Ponomarev.

First you need to understand $\bullet \longrightarrow \bullet \longrightarrow \bullet$

which is apparently equivalent to $\bullet \longrightarrow \bullet \longleftarrow \bullet$

and to $\bullet \longleftarrow \bullet \longleftarrow \bullet$. You can change a source to a sink. Why. Take sink

$$\begin{tikzcd} C \bullet \arrow[dr] & \\ B \bullet \arrow[r] & \bullet V \\ A \bullet \arrow[ur] & \end{tikzcd}$$

Can ~~assume that~~ split off the cokernel of the maps coming into V. ~~Anyway~~

$$\begin{tikzcd} \bullet \arrow[dr, bend left, "c"] \arrow[dr, bend right, "b"'] & \\ & \bullet \\ \bullet \arrow[ur, "a"'] & \end{tikzcd}$$

Look at example.

$$\bullet \xrightarrow{a} \bullet \xrightarrow{b} \bullet$$

can split off the kernel of a and the cokernel of b, so can assume a inj, b surj, e $\quad \cdot \xrightarrow{a} \cdot \xrightarrow{b} \cdot \quad$ Any module has a largest submodule ∋ b surjective, quotient is supported at end. ~~Split exact~~ canonical Exact sequence $0 \to X' \to X \to X'' \to 0$ adjoint functors. ~~[illegible]~~ Now what about

$$\cdot \xrightarrow{a} \cdot \xleftarrow{c} \overset{0}{\cdot}$$

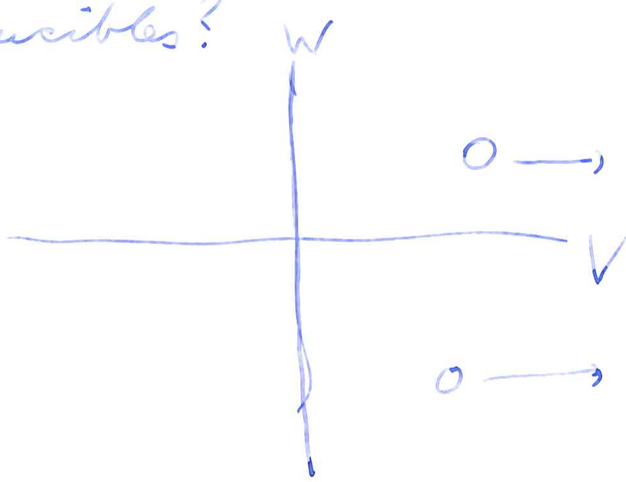
here have largest thing supported at 0, quotient has c injective. Can ask whether you have ~~[illegible]~~ derived equivalence

---

notice that $\cdot \longleftrightarrow \cdot \longleftrightarrow \cdot$ ~~[illegible]~~ amounts ~~reduced~~ to the study of ~~[illegible]~~ correspondences.

$$\begin{tikzcd} \Gamma \arrow[r] \arrow[d] & W \\ V & \end{tikzcd}$$

trivial reduction to the case where $\Gamma \subset V \times W$ and $\Gamma \twoheadrightarrow V$, $\Gamma \twoheadrightarrow W$. Can you see the irreducibles?



$$\begin{tikzcd} & & K \arrow[r, equal] \arrow[d] & K \arrow[d] & \\ 0 \arrow[r] & L \arrow[r] \arrow[d, equal] & \Gamma \arrow[r] \arrow[d] & W \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & L \arrow[r] & V \arrow[r] \arrow[d] & W/K \arrow[d] & \\ & & 0 & 0 & \end{tikzcd}$$

seems that Γ is the fibre product of surjections

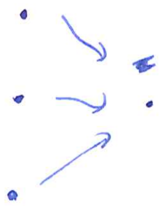
$V \twoheadrightarrow$	$V/L = W/K$	$\twoheadleftarrow W$
k	0	0
0	0	k

indecomposables are then

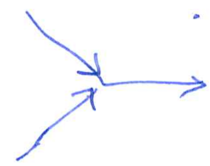
k k k

† In fact I have already handled self corresp, which is much harder. $\Gamma \rightrightarrows V$

Idea for handling derived equivalences comes from duality theory. ~~Duality theory~~.



$A \oplus B \twoheadrightarrow V$



Try to understand a sink. X repn. of quiver 0 is the sink. ~~Then have~~ Have canonical exact sequence

$$0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$$

X'' supported at 0. $\quad X''_0 = X_0 \Big/ \sum_{i \xrightarrow{a} 0} \mathrm{Im}(a)$

splitting noncanonically.

$$X'_i = X_i \qquad i \neq 0$$

$$X'_0 = \sum_{i \xrightarrow{a} 0} a(X_i)$$

Work with a source 0.

$$0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$$

$$X'_0 = \bigcap_{0 \xrightarrow{a} \tau(a)} \mathrm{Ker}\left(X_0 \xrightarrow{a} X_{\tau(a)}\right)$$

Have subquiver where you delete 0. But this is not what you are talking about. ~~[illegible]~~.

‡ Try some examples.

$$\bullet \longrightarrow \bullet$$

9 Canonical resolution of $\begin{matrix} V_1 \\ \downarrow \\ V_0 \end{matrix}$?

$$\begin{pmatrix} 0 \\ \downarrow \\ V_1 \end{pmatrix} \longrightarrow \begin{pmatrix} V_1 \\ \downarrow \\ V_0 \oplus V_1 \end{pmatrix} \longrightarrow \begin{pmatrix} V_1 \\ \downarrow \\ V_0 \end{pmatrix} \longrightarrow 0$$

You want some sort of derived cat. equivalence if this is possible.

Functors F. Object T. of proj dim 1.

$\mathbf{R}\mathrm{Hom}(T, -)$. should yield the equivalence

need $\mathrm{Ext}^1(T, T) = 0$. Example

$\mathcal{O} \oplus \mathcal{O}(1)$ $\qquad \mathrm{Ext}^1\begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \mathcal{O}(1), & \mathcal{O}(1) \end{pmatrix} = 0 \qquad H^1(\mathcal{O}, \mathcal{O}(1), \mathcal{O}(-1)) = 0.$

You want T to generate in some ways.

So in the case of dim ≤ 1 chain complexes you have three indecomposables.

P_1	0	k	
Q	k	0	projective.
P_0	k	$= k$	

$$0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow Q \longrightarrow 0$$

$$\begin{matrix} 0 \longrightarrow & \mathrm{Hom}(Q, F) & \longrightarrow & \mathrm{Hom}(P_0, F) & \longrightarrow & \mathrm{Hom}(P_1, F) & \longrightarrow & \mathrm{Ext}^1(Q, F) & \longrightarrow 0 \\ & \parallel & & & & & & & \\ & H_1(F) & & F_1 & & F_0 & & H_0(F) & \end{matrix}$$

h What game? ~~So what are you up to?~~

T is probably $Q \oplus P_0$?

$T = Q \oplus P_1$? No

$k \xrightarrow{0} 0$

$0 \xrightarrow{0} k$

$$H_0(P_1) \neq 0$$
$$\parallel$$

$$\mathrm{Ext}^1(Q \oplus P_1, Q \oplus P_1) = \begin{pmatrix} 0 & \mathrm{Ext}^1(Q, P_1) \\ & \end{pmatrix}$$

No.

$$T = Q \oplus P_0 \qquad \mathrm{Ext}^1 : \begin{pmatrix} Q,Q & Q,P_0 \\ P_0,Q & P_0,P_0 \end{pmatrix} \longmapsto \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Yes.

$$\mathrm{Hom} : \begin{pmatrix} & \\ & \end{pmatrix} \longleftrightarrow \begin{pmatrix} k & 0 \\ k & k \end{pmatrix}$$

So the idea might be to consider

$$R\mathrm{Hom}(Q \oplus P_0, F) =$$

$H^0(F) \supset H^0(F(-1)) = 0$

$H^1(F) \twoheadleftarrow H^1(F(-1)) = 0$

What is the ~~tilting~~ idea?

$$\mathrm{Hom}(T, F) = 0 \quad \text{means} \quad H_1(F) = 0 \Longleftarrow \text{and} \quad F_1 = 0.$$

$$\mathrm{Ext}^1(T, F) = 0 \quad \text{means} \quad H_0(F) = 0.$$

So any F has canonical

$$0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0$$

So what's going on? Splitting.

F' $\qquad F_1 \twoheadrightarrow B_0$

$F :$ $\qquad F_1 \longrightarrow F_0$

$F'' :$ $\qquad 0 \longrightarrow H_0(F)$

i Anyway so we do get the tilting picture. But I really want to see the derived categ equiv. You need then an explicit thing calculating $R\,\mathrm{Hom}(T, -)$. Clearly should be

$$0 \longrightarrow P_1 \longrightarrow P_0 \oplus P_0 \longrightarrow \overbrace{P_0 \oplus Q}^{T} \longrightarrow 0$$

Except you want the action of $\mathrm{Hom}(T,T)$.

So you want ~~—~~ a functorial injective resolution. How should this go?? Anyway. There should be a standard one —

$$\begin{tikzcd}
0 \arrow[r] & \left(\begin{matrix} F_1 \\ \downarrow d \\ F_0 \end{matrix}\right) \arrow[r] & \left(\begin{matrix} F_1 \oplus F_0 \\ \downarrow d+1 \\ F_0 \end{matrix}\right) \arrow[r] & {}
\end{tikzcd}$$

$$F_1 \xrightarrow{\binom{1}{d}} F_1 \oplus F_0$$

$$\begin{tikzcd}
F_1 \arrow[r, "\binom{1}{0}"] \arrow[d, "d"'] & F_1 \oplus F_0 \arrow[r, "(0\ 1)"] \arrow[d, "(d\ 1)"] & F_0 \arrow[d, "0"] \\
F_0 \arrow[r, equal] & F_0 & 0
\end{tikzcd}$$

have no H_0 hence acyclic for $\mathrm{Ext}^1(Q, -)$.

<u>This is the</u> standard resolution YES. Apply then $\mathrm{Hom}(T, G)$

$= \mathrm{Hom}(Q \oplus P_0, G) = H_1(G) \oplus G_1$ and you get for $R\,\mathrm{Hom}(T,F)$ the complex.

$$H_1(F) \oplus F_1 \longrightarrow F_0 \oplus F_0 \qquad ?$$

j This is confusing. I need $\mathrm{Hom}(T, G)$ as ~~something~~ a kind of complex.

$$\mathrm{Hom}_1(Q \oplus P_0, G) = H_1(G) \oplus G_1$$

This should be the complex $H_1(G) \subset G_1$

so $R\mathrm{Hom}(T, F)$ seems to be

$$\begin{array}{ccccc}
H_1(F) & \left(F_1 \right. & \xrightarrow{-d} & \left. F_0 \right) & H_0(F) \\
\cap & \downarrow \binom{1}{-d} & & \cap & \\
F_1 & \left. F_1 \oplus F_0 \right. & \xrightarrow[(0\ \ 1)]{} & F_0 & 0
\end{array}$$

So this is rather confusing.

usually $\begin{pmatrix} H_1(F) \\ \cap \\ F_1 \end{pmatrix}$ sits inside $\begin{pmatrix} F_1 \\ \downarrow \\ F_0 \end{pmatrix}$ with quot. ?

7/13
review this again. I'm studying the quiver $\bullet \to \bullet$ whose module are complexes $E_1 \to E_0$. We have this tilting corresp. (actually two).

Can split off ~~ker~~ H_0 cokernel then has d surj.

$H_1(E) = \mathrm{Hom}(Q, E)$ $\quad$ $H_0(E) = \mathrm{Ext}^1(Q, E)$

$Q : \quad k \longrightarrow 0$ $\qquad$ $0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow Q \longrightarrow 0$

$P_0 : \quad k = k$

$P_1 : \quad 0 \longrightarrow k$ $\qquad$ $H_1(E) \to E_1 \to E_0$

$k \quad T = Q \oplus P_0$

$$\mathrm{Hom}(T, E) = H_1(E) \oplus E_1 \qquad = 0 \Leftrightarrow \quad E_1 = 0.$$

Start again ~~to~~ organize.

Look first at indecomposable calculation.

$$0 \to H_1(F) \to F_1 \to F_0 \to H_0(F) \to 0$$

split off $H_1(F)$ and $H_0(F)$; get three types

Q	k	0	reps	$F \mapsto H_1(F)$
P_1	0	k	proj reps	$F \mapsto F_0$
P_0	$k = k$		proj "	$F \mapsto F_1$

If we treat in analogy with sheaves on $\mathbb{P}^1$, then get canonical extn.

$$0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0 \quad ?$$

Aim is what? Gelfand-P picture for the ind.

It gives you two abelian cats.

(1) complexes $F_1 \longrightarrow F_0$

(2) complexes $E^1 \longleftarrow E^0$

canonical extension. split noncan

$$0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0$$

$$\begin{pmatrix} H_1(F) \\ \\ 0 \end{pmatrix} \qquad \begin{pmatrix} F_1 \\ \uparrow \\ F_0 \end{pmatrix} \qquad \begin{pmatrix} B_0 \\ \downarrow \\ F_0 \end{pmatrix}$$

$$0 \Leftarrow E'' \Leftarrow E \longleftarrow E' \longleftarrow 0$$

$$\begin{pmatrix} H^1(E) \\ \uparrow \\ 0 \end{pmatrix} \longleftarrow \begin{pmatrix} E^1 \\ \uparrow \\ E^0 \end{pmatrix} \Longleftarrow \begin{pmatrix} B^0 \\ \uparrow \\ E^0 \end{pmatrix}$$

equiv

$B_0 \hookrightarrow F_0$

$F_0/B_0 \twoheadleftarrow F_0$

l So what do ~~I~~ I find? Anyway, nothing!!

So for the GP stuff it's clear. But there's some DG ~~...~~ angle

~~...~~ Take $T = Q + P_0$ $\begin{matrix} k \to 0 \\ k = k \end{matrix}$

$$\mathrm{Hom}(T, F) = H_1(F) \oplus F_1$$

~~quotients~~ of $\oplus$ sums of T ~~have~~ surj differential leads to

$$\begin{array}{ccc} F^+ & F & F^- \\ \begin{pmatrix} F_1 \\ \downarrow \\ B_0 \end{pmatrix} & \begin{pmatrix} F_1 \\ \downarrow \\ F_0 \end{pmatrix} & \begin{pmatrix} 0 \\ \\ H_0(F) \end{pmatrix} \end{array}$$

these are T free.

So you want to understand why there's a DG equiv.

$$R\mathrm{Hom}(T, F) \qquad T = \begin{pmatrix} k \xrightarrow{0} 0 \\ \oplus \\ k = k \end{pmatrix}$$

$$0 \longrightarrow P_1 \longrightarrow P_0 \oplus P_0 \longrightarrow P_0 \oplus Q \longrightarrow 0$$

$\exists$ functorial <u>injective</u> resolution

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_1 & \xrightarrow{\binom{1}{0}} & F_1 \oplus F_0 & \xrightarrow{(0\ 1)} & F_0 & \longrightarrow & 0 \\ & & \downarrow d & & \downarrow (d\ 1) & & \downarrow & & \\ 0 & \longrightarrow & F_0 & = & F_0 & \longrightarrow & 0 & & \end{array}$$

~~$H_1(F)$~~ $\mathrm{Hom}(T, F)$ is $F_1 + H_1(F)$ as a complex.

so it ought to be $H_1(F) \hookrightarrow F_1$

$$\begin{array}{ccccccc} H_1(F) & \longrightarrow & F_1 & \xrightarrow{d} & F_0 & \cdots\to & H_0(F) \\ \downarrow & & \downarrow \binom{1}{-d} & & \| & & \downarrow \\ F_1 & & F_1 \oplus F_0 & \xrightarrow{(0\ -1)} & F_0 & & 0 \end{array}$$

m So the picture is clearer, namely, if T is a ~~tilting~~ module then $R\mathrm{Hom}(T,-)$ will go from ~~our modules to~~ the derived cat of modules to $\mathrm{End}(T)$-modules. But the hypothesis that T has p dim ≤ 1, says that for any module M $R\mathrm{Hom}(T,M)$ has two coh. gps. $\mathrm{Hom}(T,M)$ which is related to the largest ~~subobject of~~ submodule of M which is a quotient of copies of T.

At least in this example

Let's try to generalize to some other ~~examp~~ quivers.

First go over the example again

$F: (F_1 \to F_0)$

canonical filtration with quotients

$$H_1(F) \to 0 \quad , \quad F_1/Z_1 \xrightarrow{\sim} B_0 \quad , \quad 0 \to H_0(F)$$

ind are $\quad k \to 0 \qquad k \xrightarrow{\sim} k \qquad 0 \to k = P_1$

$Q \qquad P_0 = $ proj reps $\qquad$ proj rep.

$F \mapsto F_1 \qquad F \mapsto F_0$

$$0 \to P_1 \to P_0 \to Q \to 0 \qquad \therefore \text{ proj dim } 1.$$

can. filtration

$$(F_1 \to B_0) \hookrightarrow (F_1 \to F_0) \twoheadrightarrow (0 \to H_0(F))$$

surj. arrow

$T = P_0 \oplus Q \qquad$ ~~quotients~~ of $\oplus T$ are $(F_1 \twoheadrightarrow F_0)$.

So what? $\mathrm{Hom}(T,F)$ sees only this surj side of F. One of the problems you have is to recognize the meaning of this $\mathrm{End}(T)$ module.

Let's try another example, say 

n So again you have

$$\begin{pmatrix} F_1 & & \\ & \searrow & B_0 \\ F_2 & \nearrow & \end{pmatrix} \to \begin{pmatrix} F_1 & & \\ & & F_0 \\ F_2 & & \end{pmatrix} \quad \begin{pmatrix} 0 & & \\ & \searrow & H_0 \\ 0 & \nearrow & \end{pmatrix}$$

Let's spend a few more minutes on Gelfand-P

Last time I looked at the case of the quiver

$$\begin{matrix} A & & \\ & \searrow & V \\ B & \nearrow & \end{matrix}$$

~~correspond~~ roughly correspondences

Idea here is to split off the sum $\bar{A} + \bar{B} \subset V$.

$$\begin{pmatrix} F_1 & & \\ & \searrow & B_0 \\ F_2 & \nearrow & \end{pmatrix} \hookrightarrow \begin{pmatrix} F_1 & & \\ & & F_0 \\ F_2 & & \end{pmatrix} \twoheadrightarrow \begin{pmatrix} 0 & & \\ & & H_0 \\ 0 & & \end{pmatrix}$$

basic projectives are

$$\begin{matrix} k & & \\ & \searrow\!\!\! = & k \\ 0 & \nearrow & \end{matrix} \qquad \begin{matrix} 0 & & \\ & \searrow & k \\ k & \nearrow\!\!\! = & \end{matrix} \qquad \begin{matrix} 0 & & \\ & \searrow & k \\ 0 & \nearrow & \end{matrix}$$

$$P_1 \qquad\qquad P_2 \qquad\qquad P_0$$

Am interested in subcat with $B_0(F) = F_0$.

quotients of copies of $Q = \begin{pmatrix} k & & \\ & \searrow & k \\ k & \nearrow & \end{pmatrix}$ and we have

$$0 \longrightarrow P_0 \longrightarrow P_1 \oplus P_2 \longrightarrow Q \longrightarrow 0$$

$$0 \to \mathrm{Hom}(Q, F) \longrightarrow F_1 \oplus F_2 \longrightarrow F_0 \longrightarrow \mathrm{Ext}^1(Q,F) \to 0$$

$$T = P_1 \oplus P_2 \oplus Q$$

0 Back to Gelfand. Take a general quiver with a sink, call it 0. ~~Then~~ Each vertex x gives a P_x projective ~~rep~~. module rep. fibre at x.

$$\mathrm{Hom}(P_x, F) = F(x)$$

have arrows $P_0 \to P_x$ for each $x \xrightarrow{a} 0$ and x is never 0. ~~So we have~~ Define Q by

$$\cancel{0 \to} P_0 \longrightarrow \bigoplus_{x,\, a:x\to 0} P_x \longrightarrow Q \longrightarrow 0$$

Is this injective? $P_x(y)$ free abelian group on paths starting at x ending at y. ~~Free abelian gp~~. So if $\exists$ an $a : x \to 0$, then any path $0 \to \cdot$?

$$P_0 \longrightarrow P_x \qquad \text{injective since } P_x(0) \text{ has at least } a.$$

$\cancel{0 \to \mathrm{Hom}(P_x, P_y)}$ $\quad T = \bigoplus_{x \neq 0} P_x \oplus Q$

~~$0 \to \mathrm{Hom}(P_x \ldots \mathrm{Hom}(T, T) \to$~~

$$\mathrm{Hom}(T, F) = \bigoplus_{x \neq 0} F(x) \oplus \mathrm{Ker}\left\{ \bigoplus_{a:x\to 0} F(x) \to F(0) \right\}$$