§0 $A \to B$ homo + Mag

$$
\begin{tikzcd}
A \arrow[r] \arrow[d] & \begin{pmatrix} A & Q \\ P & B \end{pmatrix} \arrow[d] & B \arrow[l, hook'] \arrow[dl, "\text{lower right}"] \\
B \arrow[r, "\text{upper}", "\text{left corner}"'] & M_2 B &
\end{tikzcd}
\qquad
\begin{tikzcd}
K_1 A \arrow[r, two heads] \arrow[d] & K_1 C \arrow[d] & K_1 B \arrow[l, two heads] \arrow[dl, "\sim"'] \\
K_1 B \arrow[r, "\sim"] & K_1(M_2 B) &
\end{tikzcd}
$$

same map

$\Longrightarrow \quad K_1 A \twoheadrightarrow K_1 B.$

$$
\begin{tikzcd}
A \arrow[r] & \begin{pmatrix} A \\ P \end{pmatrix} \otimes_A (A \ \ Q') = C' \arrow[d] & B' \arrow[l] \arrow[d] \\
 & \begin{pmatrix} A \\ P \end{pmatrix} \otimes_A (A \ \ Q) = C & B \arrow[l]
\end{tikzcd}
$$

$$
\begin{tikzcd}
K_1 A \arrow[r, "\sim"] \arrow[dr] & K_1 C' \arrow[d, "\sim"] & K_1 B' \arrow[l] \\
 & K_1 C &
\end{tikzcd}
$$

PROFESSOR D.G. QUILLEN
MATHEMATICAL INSTITUTE
OXFORD

{π 10/9/95 - 1444 Try Morita invariance for K_2. First review the steps for K_1.

The key point is that if $A \xrightarrow{\phi} B$ is a ring homom. which induces a Morita equivalence, then $K_1 A \twoheadrightarrow K_1 B$. Why? I'm assuming $A = A^2$, $B = B^2$. The rest should be Vaserstein's identity. I can suppose $A \overset{\phi}{\hookrightarrow} B$, because $A \twoheadrightarrow \phi(A)$ will be a nilp extn, so $GL(A) \twoheadrightarrow GL(\phi A)$. Then I have the Morita context

$$\begin{pmatrix} A & AB \\ BA & B \end{pmatrix}$$

& the rest is Vaserstein.

~~Another step is surjectivity.~~

Injectivity step

$P \otimes_A Q = B$ is B flat.

$\Updownarrow$

$$C = \begin{pmatrix} A & Q \\ P & B \end{pmatrix}$$

assume either Q is A-flat or P is A^{op} flat. $\Longleftrightarrow$ $P \otimes_A Q = B$ is right flat

Q A-flat $\Longrightarrow$ $Q = \varinjlim F_\alpha$ $\qquad F_\alpha \simeq A^{n_\alpha}$

$\qquad\qquad\qquad\quad \| $

$\qquad\qquad\qquad AQ = \varinjlim AF_\alpha \qquad F_\alpha^* A$

$$P \longrightarrow \mathrm{Hom}_A(Q, A) \longrightarrow \mathrm{Hom}_A(F_\alpha, A)$$

$$C = \begin{pmatrix} A & Q \\ P & B \end{pmatrix} = \varinjlim_\alpha \underbrace{\begin{pmatrix} A & AF_\alpha \\ P & P \otimes_A AF_\alpha \end{pmatrix}}_{C_\alpha}$$

$$K_1 C = \varinjlim_\alpha K_1 C_\alpha$$

need C_α idempotent

OK. as $\hat{C}_\alpha = \begin{pmatrix} A \\ P \end{pmatrix} \otimes_A (A\ Q)$

Anyway we have

$$A \longrightarrow C_\alpha \longrightarrow M_n(A)$$

which gives injectivity.

Actually I'm try to prove Morita invariance of $K_1 A$ for firm rings. So the above gives

$K_* A \hookrightarrow K_* C$ if either P_A, ${}_A Q$ are flat.

So for K_1 you get $K_1 A \xrightarrow{\sim} K_1 C \xleftarrow{\sim} K_1 B$ when A, B flat on one side.

What method? ~~You want to choose~~ Suppose you want to get from A to B

(A, A, μ)

Suppose A, B M.eq + firm. Choose $0 \to K \to A' \xrightarrow{f} A \to 0$ with A' A-flat firm. Then we know that $K_1 A' \xrightarrow{\sim} K_1 A$.

$$
\begin{array}{ccccc}
K_1 A' & \longrightarrow & K_1 C' & \xleftarrow{\sim} & K_1 B \\
\downarrow \wr & & \downarrow \wr & & \| \\
K_1 A & \longrightarrow & K_1 C & \longleftarrow & K_1 B
\end{array}
$$

Start again. Restrict to firm ~~stuff~~ first.

$$C = \begin{pmatrix} A & Q \\ P & B \end{pmatrix} = \begin{pmatrix} A \\ P \end{pmatrix} \otimes_A (A \quad Q) = \begin{pmatrix} Q \\ B \end{pmatrix} \otimes_B (P \quad B)$$

comp. firm.

suppose ${}_A Q$ flat (equiv. B is left flat).

Then $Q = \varinjlim F_i = \varinjlim_i AF_i$

$C = \varinjlim C_i \qquad C_i = \begin{pmatrix} A \\ P \end{pmatrix} \otimes_A (A \quad AF_i)$

$\qquad \to \begin{pmatrix} A \\ F_i^* A \end{pmatrix} \otimes_A (A \quad AF_i) \cong M_{n+1}(A)$.

<u>Prop.</u> C completely firm, B left or right flat $\implies K_* A \hookrightarrow K_* C$. iso for K_1.

Then given C comp. firm, choose $A' \twoheadrightarrow A$ get

$$
\begin{array}{ccccc}
K_* A' & & K_1 C' & \xleftarrow{\sim} & K_1 B \\
\downarrow \wr & & \downarrow \wr & & \| \\
K_1 A & & K_1 C & \longleftarrow & K_1 B
\end{array}
$$