[9 Idea. $R = k[x,y]$

$$0 \longrightarrow R \longrightarrow \underset{\substack{\text{field}\\ \text{of frac}}}{K} \longrightarrow \bigoplus_{ht(p)=1} E(A/p) \longrightarrow \bigoplus E(k) \longrightarrow 0$$

complex of injective, hence firm, modules.

so if you have an exact functor

$$\mathrm{mod}(R) \longrightarrow \mathrm{mod}(R)/\text{firm}$$

then this functor kills R, therefore all projectives, etc. But it will be like finite groups. So you don't have an abelian ~~quotient~~ category but maybe a triangulated ~~quotient~~ out.

$D(\mathrm{mod}(R))$ in here have ~~complexes~~ apparently complementary triangulated subcategories.

~~modules hence~~

complexes whose homology is killed by some power of I, better is torsion.

complexes which are firm $R/I \overset{L}{\otimes}_R M \simeq 0$

or cofirm $R\mathrm{Hom}_R(R/I, M) = 0$.

These are the same in the regular local case because of the Koszul ~~complexes~~ resolution.

Consider the inverse system of Koszul complexes

$K(x^n, y^n)$

Recall $K(r)$: $\underset{1}{R} \xrightarrow{r} \underset{0}{R}$

[r So we get an inverse system

$$K(r^n) \qquad \begin{tikzcd} R \arrow[r, "r^n"] & R \\ R \arrow[u, "r"] \arrow[r, "r^{n+1}"] & R \arrow[u, equal] \end{tikzcd}$$
$$\updownarrow$$
$$K(r^{n+1})$$

so $$K(x^n, y^n) = K(x^n) \otimes_R K(y^n)$$

$$\varinjlim_n \mathrm{Hom}_R(K(x^n,y^n), M)$$

$$= \underbrace{\left(\varinjlim_n \mathrm{Hom}_R(K(x^n,y^n), R)\right)}_{\text{complex } \cancel{\text{of form}}} \otimes_R M$$

$$0 \longrightarrow R \longrightarrow R_x \oplus R_y \longrightarrow R_{xy} \longrightarrow 0$$

it's a resolution of the injective hull $E(k)$ of k I think, because $H^i_{\mathfrak{m}}(R) = \begin{cases} E(k) & i = 2 \\ 0 & i \neq 2. \end{cases}$

$$0 \to R \to \underbrace{R_x \oplus R_y \longrightarrow R_{xy}}_{\text{flat firm.}} \longrightarrow E(k) \longrightarrow 0$$

Still ask about

$$\mathrm{Mod}(R)/I\text{-tors} \longrightarrow \mathrm{Fun}(I\text{-firm}, \mathcal{A}b)$$

$$0 \to j_* j^* M \longrightarrow M_x \oplus M_y \longrightarrow M_{xy}$$

~~[illegible]~~ seems to give fully faithful.

[5 So in the noetherian case it appears that the localization functor can be recovered.

~~The~~ it seems to be a kind of faithful-flatness argument.

R commutative I ideal

$I = \sum_j{}' R a_j$ $\qquad$ $\mathrm{Spec}(R) - \mathrm{Spec}(R/I) = \bigcup_j \mathrm{Spec}(R_{a_j})$

~~Check: M is I-torsion.~~

$$R \longrightarrow \bigoplus R_{a_j} \quad \text{not defined}$$

never the less I can examine the maps

$$M \longrightarrow M_{a_j}$$

I know that M not torsion iff $\exists$ sequence (a_n) $\ni$ $F((a_n)) \otimes_R M \neq \phi$.

What I can do is this?

I can consider $\{M \mid \forall_j \ M_{a_j} = R_{a_j} \otimes_R M = 0\}$.

This is a Serre subcategory closed under $\oplus$'s ~~contains~~ Does it contain R/I? Yes because $a_j \in I$ so $(R/I)_{a_j} = 0$. Then it contains all I-torsion Modules. Conversely consider an M in this subcategory say cyclic $M = Rm$. Then $\forall j$ $\exists n_j$ such that $a_j^{n_j} m = 0$. Does this mean

Assume m not an I-torsion element. Then $Im \neq 0$ so $\exists j$ with $a_j m \neq 0$.

[t So where are we???? Anyway.

R comm. noeth.

$$0 \longrightarrow j_* j^* M \longrightarrow \prod_j M_{a_j} \rightrightarrows \prod_{j \neq k} M_{a_j a_k}$$

In the case where ~~R~~ I is fg proj as left R-module, I know things work — there's a kind of Cuntz-Krieger algebra $\mathcal{O}$ such that I cofirm modules $=$ $\mathcal{O}$-modules and I^{op}-firm $=$ $\mathcal{O}^{op}$-modules.

In general ~~one has~~ I can hope for some kind of faithfully flat descent.

So let's continue with the noetherian case. Observe that if I finitely presented, then the localization has to be

$$j_* j^* M = \varinjlim \operatorname{Hom}_R(I^{\otimes_R n}, M)$$

~~so for M flat we have~~ Thus localization $j_* j^*$, hence j_* ~~j_*~~ commutes with filtered inductive limits. Also

~~$j_* j^* M$~~ M flat $\Longrightarrow$ $j_* j^* M = \mathcal{O} \otimes_R M$

where $j_* j^* R = \mathcal{O}$.

But in the $k[x,y]$ example flat $\Rightarrow$ cofirm

$\therefore$ $\mathcal{O} = R$.

But this example indicates that

$$0 \longrightarrow j_* j^* M \longrightarrow \prod_j M_{a_j} \rightrightarrows \prod_{j,k} M_{a_j a_k}$$

[u is apparently a better formula.

At the moment I am hoping to show, assuming I finite presented left R-module, that the functor

$$\cancel{}\ \mathrm{mod}(R)/I\text{-tors} \longrightarrow \mathrm{Fun}(I^{\perp}\text{ firm}, Ab)$$

is fully faithful.

$$\mathrm{Hom}_{\mathrm{mod}(R)/I\text{-tors}}(j^*M, j^*N) = \mathrm{Hom}_R(M, \underbrace{j_*j^*N})$$

formula for this

in terms of $\cancel{\otimes}$ $F \otimes_R N$ where F flat + firm

e.g. you would like

$$0 \longrightarrow j_*(j^*N) \longrightarrow F_0 \otimes_R N \longrightarrow F_1 \otimes_R N$$

e.g. $j_*j^*N = \mathcal{O} \otimes_R N$ where I fg proj ~~left~~ R-mod.

In the commutative ~~[illegible]~~ case this is OKAY?

So we have this example

$$0 \longrightarrow j_*j^*M \longrightarrow M_x \oplus M_y \longrightarrow M_{xy} \longrightarrow R^1 j_* j^*M \longrightarrow 0$$

~~Anyway~~

$$0 \longrightarrow R \longrightarrow \overset{\text{firm}}{K} \xrightarrow{f} \overset{\text{firm}}{\bigoplus_{|p|=1} E(R/p)} \longrightarrow \overset{\text{firm}}{E(k)} \longrightarrow 0$$

any firm M mapping to R has image 0

$$\mathrm{Ext}^1(F_{\text{firm}}, R)$$

[V Let's try to. ~~prove the theorem.~~ describe $F : I\text{-firm} \longrightarrow Ab.$ which come from modules. Define M by

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, phantom, "\parallel"] & F(R_x) \oplus F(R_y) \arrow[r] & F(R_{xy}) \\
0 \arrow[r] & M \arrow[r] & R_x \otimes_R M \oplus R_y \otimes_R M \arrow[r] & R_{xy} \otimes_R M
\end{tikzcd}$$

assume F commutes with filtered lims.

Then

$$\begin{tikzcd}
0 \arrow[r] & M_x \arrow[r] & F(R_x)_x \oplus F(R_y)_x \arrow[r] \arrow[d, phantom, "\parallel"] & F(R_{xy})_x \arrow[d, phantom, "\parallel"] \\
& & F(R_x) \oplus F(R_{xy}) & F(R_{xy})
\end{tikzcd}$$

$\therefore \quad M_x \xrightarrow{\sim} F(R_x) \qquad M_y \xrightarrow{\sim} F(R_y).$

Notice that for any M $\qquad M_x$ is firm.

$$M_x = R_x \otimes_R M$$

$$\begin{tikzcd}
\therefore \quad I \otimes_R M_x \arrow[dr, "\simeq"'] & = & I \otimes_R R_x \otimes_R M \arrow[dl, "\wr"] \\
& R_x \otimes_R M = M_x &
\end{tikzcd}$$

[W So what? The problem is once I ~~take~~ find a module M together with consistent ~~the~~ maps

$$M_x \longrightarrow F(R_x) \quad , \quad M_y \longrightarrow F(R_y)$$

Then how do I define

$$X \otimes_R M \longrightarrow F(X) \quad ?$$

Can I say something about firm flat modules being generated by certain types?

So if P is firm flat. Then P is cofirm,

so $0 \longrightarrow P \longrightarrow P_x \oplus P_y \longrightarrow P_{xy} \longrightarrow 0$

~~Now $F(P)$ is cofirm.~~

If P firm flat

How well can I understand firm flat modules?

flat $\Longrightarrow$ cofirm.

Important things. ~~[illegible]~~

Given F define M by

$$0 \longrightarrow M \longrightarrow F(R_x) \oplus F(R_y) \longrightarrow F(R_{xy})$$

Can you construct a map?

$$X \otimes_R M \longrightarrow F(X) \qquad \text{for } X \text{ firm}$$

~~[illegible]~~ $X \otimes_R M \longrightarrow X \otimes_R F(R_x) \oplus X \otimes_R F(R_y) \longrightarrow X \otimes_R F(R_{xy})$

point is that $F(R_a) = \varinjlim \left(F(R) \xrightarrow{a\cdot} F(R) \xrightarrow{a\cdot} F(R) \xrightarrow{a\cdot} \cdots \right)$ intuitively.