(b)

$$\mathrm{Hom}_{A^{op}}(A\otimes_B A, A) \simeq \mathrm{Hom}_{B^{op}}(A,A)$$

$$\uparrow$$

$$A\otimes_B A$$

$$A\otimes_B A \longrightarrow \mathrm{Hom}_{B^{op}}(A,A) \longrightarrow \mathrm{Hom}_{A^{op}}(A\otimes_B A, A)$$

$$(a_1,a_2) \longmapsto (\alpha \mapsto a_1\rho(a_2\alpha)) \longmapsto \left(\alpha_1\otimes\alpha_2 \longmapsto a_1\rho(a_2\alpha_1)\alpha_2\right)$$

~~and this is indeed the pairing~~

$$((a_1,a_2),(a_3,a_4)) \xrightarrow[\text{in } A_2]{\text{product}} (a_1\rho(a_2a_3), a_4) \xmapsto{\mu=\rho_2} a_1\rho(a_2a_3)a_4.$$

$$\cancel{A_2\otimes_{A_1} A_2 \longrightarrow A_2}$$

First picture has operators

$$T_a = as \qquad T_a^* = \iota_\rho a$$

$$\boxed{T_a^* T_{a'} = \iota_\rho a a' s = \rho(aa')}$$

Second picture has operators

$$T_{(a_1,a_2)} = a_1 s a_2$$

$$T^*_{(a_1,a_2)} = a_1 \iota_\rho a_2$$

$$\boxed{T^*_{(a_1,a_2)} T_{(a_3,a_4)} = a_1 \iota_\rho a_2 a_3 s a_4 = a_1\rho(a_2a_3)a_4}$$

(c) ~~Maybe this is the difference~~
~~So the difference then~~

What are the Toeplitz operators?

$$T_{a_1}\cdots T_{a_p}\, T^*_{a_{p+1}}\cdots T^*_{a_{p+q}}$$

$$= a_1 s\cdots a_p s\, \iota_\rho a_{p+1}\cdots \iota_\rho a_{p+q}$$

$$T_{a_1}T_{a_2}T^*_{a_3}\underbrace{T^*_{a_4}(a_5,a_6)}$$

$$\underbrace{\rho(a_4a_5)(a_6)}$$

$$\rho(a_3\rho(a_4a_5)a_6)(\cdot)$$

$$(a_1, a_2\rho(a_3\rho(a_4a_5)a_6))$$

The other.

$$a_1 s a_2\cdots a_p s a_{p+1}\iota_\rho a_{p+2}\cdots \iota_\rho a_n$$

Notice here that an operator of degree 0 has an odd number of [redacted] a's

$$(a_1 s a_2 \iota_\rho a_3)(a_4 s a_5 \iota_\rho a_6) \qquad (a_1,a_2,a_3)(a_4,a_5,a_6)$$

$$= a_1 s a_2\rho(a_3a_4)a_5\iota_\rho a_6$$

There's a difference between $s\iota_\rho$ in the first case and $s1\iota_\rho$ in the second.

$$(s1\iota_\rho)(a_1,a_2) = s1(\rho(a_1)a_2) \qquad ?$$

$$= 1, \rho(a_1)a_2$$

(d) $$(a_1 S a_2)(a_3, a_4, \dots, a_n)$$
$$= (a_1, a_2 a_3, a_4)$$
$$(a_1 \rho a_2)(a_3, a_4, \dots)$$
$$= (a_1 \rho(a_2 a_3) a_4, \dots)$$

What's the difference between these guys

Generators are left mult by B
and the operators aS $\iota_\rho a$

ie $$(a_1 S \cdot \iota_\rho a_2)(a_3, a_4, \dots) = a_1(1, \rho(a_2 a_3) a_4, \dots)$$
$$= (a_1, \rho(a_2 a_3) a_4, a_5, \dots)$$

relation
$$(a_1 S a_2 \iota_\rho a_3)(a_4, a_5, \dots)$$
$$= a_1 S a_2 \iota_\rho (a_3 a_4, a_5, \dots$$
$$= a_1 S (a_2 \rho(a_3 a_4) a_5, \dots$$
$$= (a_1, a_2 \rho(a_3 a_4) a_5, a_6)$$

There might be a difference at the bottom

Basically it seems we have the same operators with an extra a

$$a_1 S \dots a_p S a_{p+1} \iota_\rho \dots a_{n-1} \iota_\rho \underline{a_n}$$
not defined on B

$a_1 \iota_\rho a_2$ defined to be zero on A

$$a_1 S a_2 \iota_\rho a_3$$

(e) Try again.

In the case of $T_B(A) = B \oplus A \oplus A \otimes_B A \oplus \cdots$

the basis operators are

$$T_a = as \qquad T_a^* = \iota_\rho a$$

Relations $\left(b_1 T_a b_2\right)(\alpha, \cdots)$

$$= \left(b_1 a s b_2\right)(\alpha, \cdots)$$

$$= (b_1 a, b_2 \alpha, \cdots) = (b_1 a b_2, \alpha, \cdots) = T_{b_1 a b_2}(\alpha, \cdots)$$

$$\left(b_2 T_a^* b_1\right)(\alpha, \cdot) = b_2 \iota_\rho (a b_1 \alpha, \cdots)$$

Let's go over where we are. We have

$$T_B(A) = B \oplus A \oplus A \otimes_B A \oplus \cdots$$

operators $b \in B$ $\quad T_a = as$ left mult by a

$$T_a^* = \iota_\rho a$$

contraction by $\alpha \longmapsto \rho(a\alpha)$ $\qquad \mathrm{Hom}_{B^{op}}(A, B)$

$$A \longrightarrow B$$

So the algebra we get is

$$T_B(A) \otimes_B T_B(A)$$

$$(a_1, \cdots, a_p) \; (a_1', \cdots, a_q') \mapsto a_1 s \cdots a_p s \, \iota_\rho a_1' \cdots \iota_\rho a_q'$$

~~Left mult~~.

$$T_A(A \otimes_B A) = A \oplus A \otimes_B A \oplus A \otimes_B A \otimes_B A$$

operators a

$$T_{(a_1, a_2)} = a_1 s a_2 \qquad T^*_{(a_1, a_2)} = a_1 \iota_\rho a_2$$

(f) ~~[illegible]~~

$$T_B(A\otimes_B A) \otimes_A T_B(A\otimes_B A)$$

$$\cancel{(a_0 s \cdots a_{p-1} s a_p)} \quad \cancel{(a_0' \iota_\rho \cdots \iota_\rho a_q')}$$

$$(a_0, \dots, a_p) \otimes_A (a_0', \dots, a_q') \longmapsto a_0 s \cdots s a_{p+1} a_0' \iota_\rho \cdots \iota_\rho a_q'$$

Now these algebras are really quite similar. We might concentrate on a correspondence between these operators.

If we ignore B, then

$$T_B(A)/B = T_A(A\otimes_B A)$$

so the two are related

$$a \longmapsto a x_i s \iota_\rho y_i$$

$$\cancel{a x_i s \iota_\rho y_i(\alpha) = a x_i(1, \rho(y_i\alpha))}$$

$$(\alpha) \xrightarrow{\iota_\rho y_i} \rho(y_i\alpha)(\cdot) \xrightarrow{s} (\rho(y_i\alpha)) \xrightarrow{a x_i} (a x_i \rho(y_i\alpha)) = a\alpha.$$

$$a x_i s \iota_\rho y_i(\alpha_1, \alpha_2)$$
$$= a x_i s(\rho(y_i\alpha_1)\alpha_2)$$
$$= (a x_i, \rho(y_i\alpha_1)\alpha_2) = (a\alpha_1, \alpha_2).$$

(g) Try again. What's the relation

~~$a x_i s \iota_\rho y_i$~~

Is $1 = x_i s \iota_\rho y_i$ on $T_B(A)$?

$$(a_1) \xrightarrow{y_i} (y_i a_1) \xrightarrow{\iota_\rho} \rho(y_i a_1) \xrightarrow{s} (\rho(y_i a_1))$$

$$\xrightarrow{x_i} (x_i \rho(y_i a_1)) = (a_1).$$

Thus $x_i s \iota_\rho y_i$ is the projection onto degree ≥ 0.

$$a_1 s a_2 s = a_1 s (x_i s \iota_\rho y_i) a_2 s \quad ?$$

$$T_{a_1} T_{a_2} = T_{a_1} T_{x_i} T_{y_i}^* T_{a_2}$$

You ask whether $T_{x_i} T_{y_i}^*$ is the identity.
It isn't but

$$T_{x_i} T_{y_i}^* T_a = T_{x_i} \rho(y_i a) = T_{x_i \rho(y_i a)} = T_a.$$

$$\therefore \left(T_{x_i} T_{y_i}^*\right)^2 = T_{x_i} T_{y_i}^* \qquad \text{projector.}$$

$$T_{x_i} T_{x_j} T_{y_j}^* \underbrace{T_{y_i}^* \; T_{a_1}}_{\rho(y_i a_1)} T_{a_2} = T_{x_i} T_{x_j} T_{y_j}^* T_{\rho(y_i a_1) a_2}$$

$$= T_{x_i} T_{\rho(y_i a_1) a_2}$$

$$= T_{x_i \rho(y_i a_1)} T_{a_2} = T_{a_1} T_{a_2}.$$

(h) ~~At the point is that if I wish to~~

Take $\quad a_1 s a_2 \iota_\rho a_3 = T_{a_1} a_2 T^*_{a_3}$.

This makes sense on $T_A(A \otimes_B A) = \underbrace{T_B(A)/B}_{\text{positive degrees}}$

So how am I going to handle this?

$$T_{a_1} T_{a_2 x_i} T^*_{y_i} T^*_{a_3}$$

On the other hand

$$T_{a_1} T^*_{a_2} = T_{a_1} 1 T^*_{a_2}$$

So what have I learned?

I've learned to recognize A_{2n} as the alg spanned by the operators $T_{a_1} \cdots T_{a_n} T^*_{a'_1} \cdots T^*_{a'_n}$.

Furthermore

a itself is not defined on $T_B(A)$.

So what do you want?

Begin again. have formed Toeplitz algebra

$$T_B(A) \otimes_B T_B(A^*)$$

$$T_{a_1} \cdots T_{a_p} T^*_{a'_1} \cdots T^*_{a'_q}$$

and represented this on $\quad T_B(A)$

$$a_1 s \cdots a_p s \, \iota_\rho a'_1 \cdots \iota_\rho a'_q$$

Also have Toeplitz op represented on $T_A(A \otimes_B A)$

$$T_A(A \otimes_B A) \otimes_A T_A(A \otimes_B A)$$

$$T_{a_0} \cdots T_{a_{p-1}} a_p T^*_{a_{p+1}} \cdots T^*_{a_{p+q}}$$

$$a_0 s \cdots a_{p-1} s a_p \iota_\rho a_{p+1} \cdots \iota_\rho a_{p+q}$$