(a) So assume $I \overset{!}{\otimes}_I I \xrightarrow{\sim} I$

may this is it.

$$\begin{array}{ccccccc}
 & 0 & & 0 & & & \\
 & \downarrow & & \downarrow & & & \\
\longrightarrow & I\otimes I \otimes I & \longrightarrow & I \otimes I & \longrightarrow & I & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & \| & \\
\longrightarrow & I\otimes I \otimes R & \longrightarrow & I \otimes R & \longrightarrow & I & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & & \\
 & I\otimes I \otimes R/I & \longrightarrow & I \otimes R/I & & & \\
 & \downarrow & & \downarrow & & & \\
 & 0 & & 0 & & &
\end{array}$$

now this is a resolution of I by free right R-modules, so can calculate $I \otimes_R I$ and again get I.

I think I am getting it.

Anyway life is harder than this.

I want to try for a pro object version of this excision proof. So how to proceed? The idea is that ~~all arguments~~ the noetherian arguments ought to generalize in some form. First, Given a module M we can consider the inverse system $I^{\otimes_R n} \otimes_R M$

$$\longrightarrow I \otimes_R I \otimes_R M \longrightarrow I \otimes_R M \longrightarrow M$$

and this inverse system is good. Denote this inverse system $I^g \otimes_R M$. There's certainly a sense in which $I^g \otimes_R I^g \longrightarrow I^g$ is an isomorphism.

(b) Some things to understand.

Consider ~~the case $I \otimes_R I \to I$~~

the case where I is flat (right flat say)

and $I = I^2$. $I\mathcal{J} = I$.

And given an R-module M we have

$$0 \longrightarrow \underbrace{\mathrm{Tor}_1^R(R/I, M)}_{\text{left exact functor of } M} \longrightarrow \underbrace{I \otimes_R M \longrightarrow M}_{\text{exact functors of } M} \longrightarrow \underbrace{M/IM}_{\text{right exact functor of } M} \longrightarrow 0$$

Also have

$$\mathrm{Tor}_n^R(R/I, M) \xrightarrow{\sim} \mathrm{Tor}_{n-1}^R(I, M) = 0$$

$n \geq 2$ since I is flat.

No higher derived functors.

$$0 \longrightarrow I \longrightarrow \tilde{I} \longrightarrow k \longrightarrow 0$$

$$0 \longrightarrow \mathrm{Tor}_1^I(k,k) \longrightarrow I \otimes_I k \longrightarrow \mathbb{R} \xrightarrow{\sim} k \longrightarrow 0$$

$$\mathrm{Tor}_1^I(k,k) = I/I^2 = 0 .$$

~~$\mathrm{Tor}_1(k,k)$~~ ~~Is k flat~~

$$\mathrm{Tor}_1^R(R/I, M) = \mathrm{Ker}\,(I \otimes_R M \longrightarrow M)$$

$$\mathrm{Tor}_1^R(R/I, R/I) = \mathrm{Ker}\,(I/I^2 \longrightarrow R/I) = I/I^2 = 0$$

(c) So if I flat and $I = I^2$, there are no other modules to consider.

So what happens?

I'm considering I flat $I = I^2$.

Then $I \otimes_R I \xrightarrow{\sim} I$

~~$\mathrm{Tor}_1^R(R,k)$~~ $0 \to I \to \tilde{I} \to k \to 0$

So $\mathrm{Tor}_n^R(k,k) = 0 \quad n \geq 2$

$= I/I^2 = 0 \quad n = 1$

$= k \quad n = 0$

Basically we have this ~~Serre~~ Serre-like subcategory of $\tilde{I}$-mod consisting of vector spaces. Is there any homology?

$0 \to I \to \tilde{I} \to k \to 0$

$0 \to \mathrm{Hom}_{\tilde{I}}(k, M) \to M \to \mathrm{Hom}_{\tilde{I}}(I, M) \to \mathrm{Ext}^1_{\tilde{I}}(k, M) \to 0$

One can expect higher Ext's, since I is flat and not projective.

I know that good = good' in this situation.

I know that good = good'.

~~I flat~~ ~~$I \otimes_R M$~~

M flat $\quad I \otimes_R M \hookrightarrow M \quad I \otimes_R M = IM$

(d) What next. Go back over excision to see what you might learn.

Basically we have a proof of excision.

$$\begin{array}{ccccccccc} 0 \leftarrow & C_\lambda(R/I) & \leftarrow & C_\lambda(R) & \leftarrow & I \overset{L}{\otimes}_R & \leftarrow & \Sigma[I \overset{L}{\otimes}_R]^{(2)}_\sigma & \leftarrow \\ & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\ 0 \leftarrow & C_\lambda(0) & \leftarrow & C_\lambda(I) & \leftarrow & I \overset{L}{\otimes}_I & \leftarrow & \Sigma[I \overset{L}{\otimes}_I]^{(2)}_\sigma & \leftarrow \end{array}$$

Here we have the usual confusion, but the basic GFT formula is nonunital; the thing is that ~~we~~ you use $B(R)$ which calculates $\mathrm{Tor}^{\tilde{R}}$ same as Tor^R for unital modules.

So what are the arguments.

First you need $I \overset{L}{\otimes}_R I \longrightarrow I$ quis

and this is the ~~proof that~~ lemma

assuming $M = IM$, M is I-flat $\Longleftrightarrow$ M R-flat

$$X \otimes_I M \xrightarrow{\sim} X \otimes_R M$$

This sort of flatness property reduces the resolutions to

$$\begin{array}{ccccccccc} 0 \leftarrow & C_\lambda(R/I) & \leftarrow & C_\lambda(R) & \leftarrow & I \overset{L}{\otimes}_R & \leftarrow & \Sigma\, I \overset{L}{\otimes}_R & \leftarrow \Sigma^2 \\ & & & C_\lambda(I) & \leftarrow & I \overset{L}{\otimes}_I & \leftarrow & \Sigma\, I \overset{L}{\otimes}_I & \leftarrow \end{array}$$

~~So everything works~~

Finally need $I \overset{L}{\otimes}_I \longrightarrow I \overset{L}{\otimes}_R$ to be quis

$$I \overset{L}{\otimes}_I \Leftarrow I \overset{L}{\otimes}_R I \overset{L}{\otimes}_I \simeq I \overset{L}{\otimes}_I I \overset{L}{\otimes}_R \Rightarrow I \overset{L}{\otimes}_R$$

(e) Now your program is to do the same sort of things. I-adically. So for example we can try to substitute the proobject I^n for I. Now place yourself in the situation where I is flat in R.

Anyway, what else. How do we go on? The problem is how we are going to handle I over itself.

So let's try various things, e.g.

$$I \otimes_I I \xrightarrow{\sim} I \otimes_R I$$

what would be the adic generalization.

$$I^n \otimes_I I^m \longrightarrow I^n \otimes_R I^m$$

OKAY try this

$$\begin{array}{ccc} I \otimes_I I & \longrightarrow & I \otimes_R I \\ \uparrow \exists & \nwarrow & \uparrow \\ I \otimes_I I^2 & \longrightarrow & I \otimes_R I^2 \end{array}$$

$x, y, z \in I$, $r \in R$

$$xr \otimes_I yz = xry \otimes_I z = x \otimes_I ryz$$