

18 March

! Review of module theory - mainly to get the main paper done.

The main point of the paper is to understand Morita equivalences between idempotent rings, to prove Morita invariance of cyclic homology for h-unital rings. What are the main steps? ideas?

$A \xrightarrow{w} B$ homom. determines adjoint functors

$$M(A) \begin{matrix} \xrightarrow{w_!} \\ \xleftarrow{w^*} \end{matrix} M(B)$$

$$M \longmapsto B \otimes_A M$$

$$A \otimes_A N \longleftarrow N$$

$$B^{(2)} \rightarrow B \rightarrow \tilde{B}$$

are B^{op} -nil isos

hence A^{op} -nil is

$$B \otimes_A M \xrightarrow{\sim} B \otimes_A M \xrightarrow{\sim} \tilde{B} \otimes_A M$$

for M finitely A -used.

$$\text{Hom}_A(M, A \otimes_A N) \xrightarrow{\sim}$$

$$\text{Hom}_A(M, N) \xrightarrow{\sim} \text{Hom}_A(M, \text{Hom}_B(B, N))$$

|S adjunction

$$\text{Hom}(w_! M, w_! M') = \text{Hom}(M, w^* w_! M')$$

$$\text{Hom}_B(B \otimes_A M, N)$$

adjunction maps:

$$\alpha : B \otimes_A A \otimes_A M \longrightarrow N$$

$$b \otimes a_1 \otimes a_2 \otimes n \longmapsto b, w(a_1 a_2) n$$

$$\beta : M = A \otimes_A M \longrightarrow A \otimes_A B \otimes_A M$$

$$a_1 a_2 a_3 m \longmapsto a_1 \otimes a_2 \otimes w(a_3) m$$

Thm. ① $w_!$ fully faithful (equivalently $w^* w_! \xrightarrow{\sim} \text{Id}$)

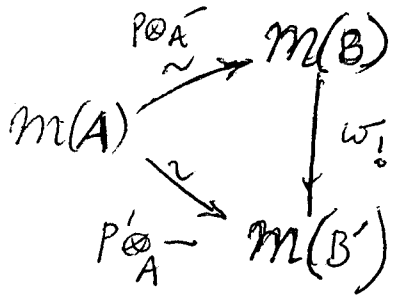
② $\beta : \text{Id} \xrightarrow{\sim} w^* w_!$

$$A^{(2)} \xrightarrow{\sim} A \otimes_A B \otimes_A A^{(2)}$$

③ $A \xrightarrow{w} B$

$A \otimes A^{op}$ nil isom. i.e. $A \text{Ker}(w) A = 0$
and $w(A) B w(A) \subset w(A)$

2. ~~Coordinate systems~~ Coordinate systems - a real time waster
 What is the basic idea?



easy part: suppose given $\begin{pmatrix} 1 & v \\ u & w \end{pmatrix} : \begin{pmatrix} A & Q \\ P & B \end{pmatrix} \rightarrow \begin{pmatrix} A & Q' \\ P' & B' \end{pmatrix}$

$w(b_1 b_2) = w(b_1) w(b_2)$ $v(ag) = a v(g)$ $g p = v(g) u(p)$
 $u(pa) = u(p)$ $v(gb) = v(g) w(b)$ $u(p) v(g) = w(pg)$
 $u(bp) = w(b) u(p)$

Claim: canonical isom.

$$B' \otimes_B P \otimes_A M \xrightarrow{\sim} P' \otimes_A M \quad M \text{ - free}$$

$$Q \otimes_B B^{(2)} \otimes_B N' \xrightarrow{\sim} Q' \otimes_{B'} N'$$

$$P \xrightarrow{u} P'$$

$u(p) = 0$
 B -nil iso. then $v(g) u(p) = 0 \cdot p = 0$

$$Q \xrightarrow{v} Q'$$

is a B^{op} -nil isom.

$$w(b) u(p) = u(bp) \quad \left(\begin{matrix} P \otimes B = 0 \\ \text{gen } B \end{matrix} \right)$$

$$Q \otimes_B B^{(2)} \xrightarrow{\sim} Q \otimes_B B^{(2)}$$

1. Greg only has limited ability to practice football

English comprehension

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28.9.95

3. $Q \xrightarrow{\nu} Q'$ B -nil ism.
 $Q \otimes_B N \xrightarrow{\sim} Q' \otimes_B N$ any firm N .

$$P \otimes_A Q \otimes_B N \xrightarrow{\sim} (P \otimes_A Q') \otimes_B N$$

$$P \otimes_A Q \otimes_B P \otimes_A M \xrightarrow{\sim} B' \otimes_B P \otimes_A M$$

$$P \otimes_A M \xrightarrow{\sim} B' \otimes_B P \otimes_A M \quad M \text{ A-firm}$$

$$\begin{aligned} \nu \otimes g \otimes p \otimes m &\mapsto \nu \otimes (g \otimes p \otimes m) \\ b' \otimes u(p) \otimes m &\longleftarrow b' \otimes p \otimes m \end{aligned}$$

and then you have map the other way
~~Other way~~ Other way start with
 $u: P \rightarrow P'$ B -nil ism.

$$B^{(2)} \otimes_B P \xrightarrow{\sim} B^{(2)} \otimes_B P'$$

$$Q \otimes_B B^{(2)} \otimes_B P \otimes_A Q' \otimes_{B'} N' \xrightarrow{\sim} Q \otimes_B B^{(2)} \otimes_B (P \otimes_A Q' \otimes_{B'} N')$$

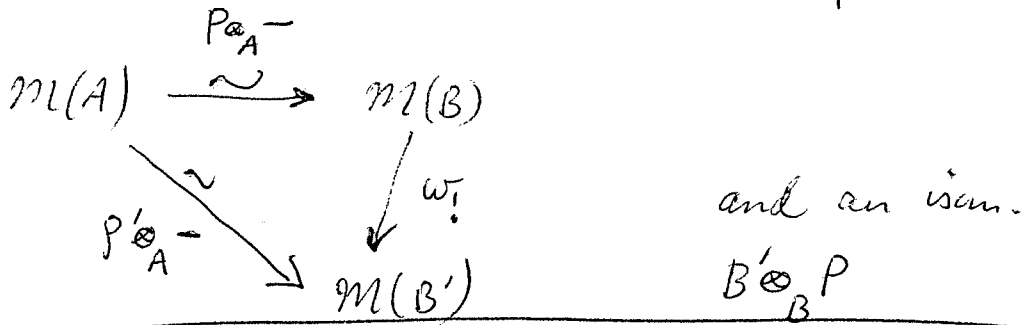
$$Q' \otimes_{B'} N' \xrightarrow{\sim} Q \otimes_B B^{(2)} \otimes_B N'$$

N' B' -firm.

$$\begin{aligned} g \otimes b_1 \otimes b_2 \otimes p \otimes g' \otimes n' &\mapsto g \otimes b_1 \otimes b_2 \otimes u(p) \otimes g' \otimes n' \\ \nu \otimes (g \otimes b_1 \otimes b_2) \otimes n' &\longleftarrow g \otimes b_1 \otimes b_2 \otimes n' \end{aligned}$$

$$\nu \otimes (g \otimes b_1 \otimes b_2) \otimes u(p) \otimes g' \otimes n'$$

4. ~~What~~ What was the problem I worked so hard on?
 The converse maybe. ~~Module~~ Transpose.



$w: B \rightarrow B'$ isom.
 you know it is $B \otimes B'^{\text{op}}$ and isom. $\therefore w$ f.f.

$$\begin{aligned}
 B' w(B) B' &= B' \\
 P' Q' w(B) P' Q' & \quad \text{But } Q' w(B) w(B) P' \\
 & \quad \text{⑤ } v(Q) u(P) = QP = A \\
 \therefore P' Q' w(B) P' Q' & \supset P' A Q' = P' Q' P' Q' = B^2
 \end{aligned}$$

why is $u: P \rightarrow P'$ a B -nil iso
 $u(p) = 0 \Rightarrow 0 = v(g)u(p) = gp \Rightarrow p, gp = 0$
 $\Rightarrow PQ_p = 0$
 ~~$(pg)P' = p(gp')$~~
 ~~$= (p v(g))u(p')$~~
 $B \cdot \text{Ker}(p) = 0$

$$\begin{aligned}
 w(pg)P' &= u(p) \underbrace{v(g)P'}_a = u(pa) \\
 \therefore w(B)P' &\subset u(P) \quad w(B) \text{Coker}(u) = 0
 \end{aligned}$$

$w: B \rightarrow B'$ $B \otimes B'^{\text{op}}$ - nil iso
 $w(b) = 0 \Rightarrow v(g)w(b)u(p) = v(gb)u(p) = gb_p$
 $\Rightarrow (pg) b (pg_1) = 0 \quad B \text{Ker}(w) B = 0$

~~$w(p_1 g_1) P' g_1' w(p_2 g_2)$~~
 $= u(p_1) \underbrace{(v(g_1)P')}_a \underbrace{(g_1' u(p_2))}_{a'} v(g_2) = u(p_1 g_1) v(a_2 g_2) = w(p_1 g_1 g_2 g_2)$

5. $B' \subset B'w(B)B' \quad | \quad B' = P'Q'$

$$\begin{pmatrix} A & Q' \\ P' & B' \end{pmatrix}$$

$$\begin{cases} Q'P' = A \\ P'Q' = B' \end{cases}$$

A, B'
idemp.

~~Q'P'Q'~~
 $B' = B'z = P'Q'P'Q'P'Q' = P'AAQ'$

~~P'Q'P'Q'P'Q'~~
 $= P'v(Q)u(P)v(Q)u(P)Q'$

$$= P'v(Q)w(B)u(P)Q'$$

$$\subset B'w(B)B'$$

~~to what?~~ ~~argument~~

argument

$$v: Q \rightarrow Q'$$

B' -nil coin

$$\therefore Q \otimes_B P \otimes_A M \xrightarrow{\sim} Q' \otimes_B P \otimes_A M$$

$$M \xrightarrow{\sim} Q' \otimes_B P \otimes_A M$$

$$g \otimes p \otimes m \mapsto v(g) \otimes p \otimes m$$

$$\boxed{P' \otimes_A M \xrightarrow{\sim} B' \otimes_B P \otimes_A M}$$

M firm

$$p' \otimes g \otimes p \otimes m \mapsto p'v(g) \otimes p \otimes m$$

$$b'u(p) \otimes m$$

$$\longleftarrow b' \otimes p \otimes m$$

~~to what?~~

$$u: P \rightarrow P'$$

B -nil coin

$$B^{(2)} \otimes_B P \otimes_A Q' \xrightarrow{\sim} B^{(2)} \otimes_B P' \otimes_A Q'$$

$$v(g, b_1, b_2) \otimes u(p) \otimes m'$$

$$Q \otimes_B B^{(2)} \otimes_B P \otimes_A Q' \otimes_{B'} N' \xrightarrow{\sim} Q \otimes_B B^{(2)} \otimes_B P' \otimes_A Q' \otimes_{B'} N'$$

$$(g \otimes b_1 \otimes b_2 \otimes p) \otimes g' \otimes u' \mapsto \varepsilon \otimes b_1 \otimes b_2 \otimes u(p) \otimes g' \otimes n'$$

$$Q' \otimes_B N' \xrightarrow{\sim} Q \otimes_B B^{(2)} \otimes_B N'$$

2. I need to recover the outline of Morita equivalence theory. Maybe also the proof of Morita invariance of cyclic homology.

~~main proof goes~~ main proof goes as follows.

$$\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$$

~~main proof~~

$$\begin{pmatrix} A & \text{flat} \\ \Leftrightarrow & P, Q \text{ flat} \end{pmatrix}$$

$$A \overset{L}{\otimes} A = Q \overset{L}{\otimes}_B P \overset{L}{\otimes} A = P \overset{L}{\otimes}_A Q \overset{L}{\otimes}_B = B \overset{L}{\otimes}_B$$

~~To get this to work~~ To get this to work I need E some sort of flat bimod. res. of A.

$$A \overset{L}{\otimes} A = A \overset{\otimes}{\otimes}_A E \overset{\otimes}{\otimes}_A$$

$$= Q \overset{\otimes}{\otimes}_B P \overset{\otimes}{\otimes}_A E \overset{\otimes}{\otimes}_A = \underbrace{P \overset{\otimes}{\otimes}_A E \overset{\otimes}{\otimes}_A Q}_{\text{flat bimod. res. of } B} \overset{\otimes}{\otimes}_B$$

flat bimod. res. of B

$$P \overset{\otimes}{\otimes}_A E \overset{\otimes}{\otimes}_A Q \text{ made of } P \overset{\otimes}{\otimes}_k Q$$

\Leftarrow

$$E \text{ made of } \tilde{A} \overset{\otimes}{\otimes}_k \tilde{A}$$

~~so now consider a coherent sheaf.~~

so main argument. ~~suppose~~ B_1, B_2

How to formulate theorem. Roughly says

that given B_1, B_2 k -flat k -central, ~~then~~ then a Morita equiv. $\mathcal{M}(B_1) \simeq \mathcal{M}(B_2)$ induces

an isom. $HC(B_1) \simeq HC(B_2)$. ~~then~~ You want to put in various remarks, amplifications.

isom. maps yield same HC isom.

for a map homom. the HC isom is ind. by the hom.

Delicate points: HC is a functor on rings ^{defined the category}

But the natural situation is a 2-cat. so to add ~~objects~~ B you assoc. $\mathcal{M}(B)$ a cat.

o each B_1, B_2 you get a groupoid of equivalences $\mathcal{M}(B_1) \simeq \mathcal{M}(B_2)$ and for each B_1, B_2, B_3 have

~~main outline~~

7. composition of equivalences. ~~Also~~ This gives you a 2-groupoid. You also have ~~a~~ a 1-cat whose objects are rings B and maps are meghom. And a weak (sort of) functor from the 1-cat to the 2-groupoid.

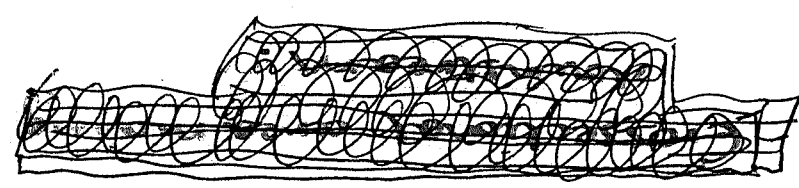
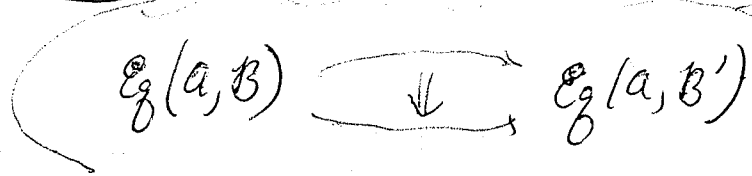
Now how do I want to understand this situation? ~~If~~ You fix A , and look at dual pairs $(P, Q, \phi: Q \otimes P \rightarrow A)$. These form a category pure and simple, ~~and~~ and there's a functor $(P, Q) \mapsto P \otimes_A Q$ to rings and meghomas. ~~Anyway what else~~

Work in 2-cat $\underline{\text{Cat}}$. Fix A , consider

$B \mapsto \underline{\text{Hom}}(A, B)$. Can I form a

2-cat over $\underline{\text{Cat}}$ with fibre over B equal to $\underline{\text{Hom}}(A, B)$. Object would be a pair B ,

2-morphisms go between 1-morphisms with same endpoints.



8. Guess. Objects of the Total 2-cat should be pairs $(B, \Phi: A \rightrightarrows B)$. Then give

(B, Φ) (B', Φ') you want a groupoid

a maps. $\underline{\text{Hom}}((B, \Phi), (B', \Phi'))$ has

objects $B \xrightarrow{F} B'$ where such that $F\Phi = \Phi'$

or together with an isom. $F\Phi \xrightarrow{\sim} \Phi'$.

It's the latter I'm used to ~~and there is no improvement~~

Example: $(B, F: m(A) \xrightarrow{\sim} m(B))$. these pairs form a category, maps are $(B \xrightarrow{w} B', \theta: w_! F \xrightarrow{\sim} F')$

The category theory remains as confusing as ever. You've defined a category, but you still haven't ~~decided~~ decided whether there is a 2-cat structure of interest.

Idea: Consider collection of cats $m(B)$ equiv. to $m(A)$ as a full 2-subcat of $\text{Cats} + \text{Equiv}$.

~~Consider the collection of cats. This should be a~~

Fix an A and ~~everything~~ everything should be equivalent to ~~the~~ a Picard cat.

So when I do cat theory properly I should get a Picard cat = 2 groupoid with one object

What can I say about self-equivalences of $m(A)$ = category of invertible ^{equiv} bimodules.

9. discussion today. ^{important.} ~~the~~ idea that a 2-groupoid with a single object is a strict Picard category. ~~the~~

~~the~~ Starting point - you have rings B and a cat $\mathcal{M}(B)$ for each B , you ~~consider~~ get a 2-groupoid with ~~the~~ objects B by assoc. to each (B_1, B_2) the gpoid $\text{Equiv}(\mathcal{M}(B_1), \mathcal{M}(B_2))$. ~~more~~
Suppose fix A , then for each B look at $\text{Equiv}(\mathcal{M}(A), \mathcal{M}(B))$ and for each pair (B_1, B_2) look at $\text{Equiv}(\mathcal{M}(A), \mathcal{M}(B_1)) \times \text{Equiv}(\mathcal{M}(B_1), \mathcal{M}(B_2))$.

It seems clear that I get a 2-groupoid over the previous one. It should be true that this 2-groupoid ~~is~~ is equiv to a 1-groupoid.

So let's use \mathcal{M} contexts, or just the binodes.

Anyway ~~nothing like a groupoid~~ this thing somehow becomes the ~~great~~ $\text{Equiv}(\mathcal{M}(A), \mathcal{M}(A))$ Pic. groupoid

Equiv

10. Let's go back to Meq theory.

various cats + 2-cats.

Start with ~~maps~~ $B \mapsto \mathcal{M}(B)$.

To each B get cat $\mathcal{M}(B)$

To each megham $B \xrightarrow{w} B'$ get $w_! : \mathcal{M}(B) \xrightarrow{\sim} \mathcal{M}(B')$

transitive up to canon isom, so you get a
ofibred cat. over rings and meghoms. Objects
are pairs (B, \mathcal{M}) , maps $(B, \mathcal{M}) \rightarrow (B', \mathcal{M}')$ given by
 $w: B \rightarrow B' + B' \otimes_B \mathcal{M} \xrightarrow{\sim} \mathcal{M}'$. Cocart ~~sections?~~

2-gpoid ~~obj.~~ B , 1-arrows equiv $\mathcal{M}(B_1) \rightarrow \mathcal{M}(B_2)$,

2-arrow is ~~isom.~~ isom. of functors.

~~1 component of~~
This 2-gpoid should be "equiv" to 2 gpoid with
-objd.

~~Let's~~ Try to put into words what you might
be looking for. Fund. 2 gpoid of cats and equiv.
1-ick component of this + ~~get~~ get Picard cat. somehow
~~you should start with the fibred cat of~~

You have base category of rings + meghom and
a cofibred cat ~~of~~ over it such that transitions are equiv.
What does topology say? You have a quasi-fibration
pres are ~~cats~~ cats ~~but~~ but not gpoids. So it's
like a fibre bundle w structural group, but in the
anotopy senses. Principal bundles ~~with~~ fibre
over B ~~should~~ should be the groupoid of equiv $\mathcal{M}(A) \xrightarrow{\sim} \mathcal{M}(B)$

So we form the ~~groupoid~~ principal bundle, ~~objects~~
cofibred cat ~~of~~ over ~~rings~~ rings + meghom with
fibre $\text{Equiv}(\mathcal{M}(A), \mathcal{M}(B))$

11. Megtheory again. I am still trying to fit the cat stuff in some order. We start with the map $B \mapsto M(B)$ associating a cat to a ring. Then we get some sort of structure on the ~~cat~~ family of rings, namely, for each pair of rings B_1, B_2 a 2-goid ~~eg~~ $(M(B_1), M(B_2))$ composition etc. This is a 2-goid. ~~Fix a component.~~ Fix a component. Up to equivalence ~~this comp.~~ this comp. is equiv to full subcategory with one object A . So you have ~~a strict~~ a strict Picard cat. Question: Is it always like usual Pic. cats? ~~Is k-inv. stable, or menzero~~ if stable. This question arises maybe in Cat.

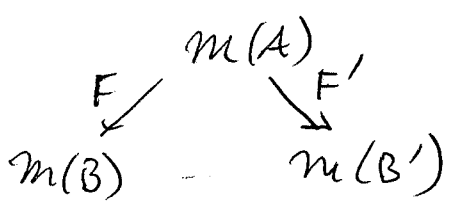
Anyway this is what happens when you try to ~~work~~ work with rings up to Morita equivalence. You end up with a 2 groupoid, ~~and a full subcategory~~ disjoint union of Picard categories. ~~Nothing~~ Nothing more you can say it seems.

The next step is to bring in meghoms! You have a based cat B of B and meghoms, and naturally a ~~co~~ cofibred cat assoc. to $B \mapsto M(B)$. The ~~extra~~ extra functors are equivs, so we have the analogue of quasi-fibring. ~~At the moment I have a of~~ ~~cat over a base cat, something like a fibre bundle.~~ At the moment I have a of cat over a base cat, something like a fibre bundle. There should be an assoc. principal bundle for some top group. The ^{top} group should be the Picard cat. ~~So what.~~ So what.

2. Anyway think a little bit about ~~the~~ ~~Meg~~
 Anyway ~~the~~ recall earlier problem.
 about 2-categories. You begin with rings A
 idempotent, each A has $M(A)$ attached, for each
 pair A, B you have $E(A, B) = \text{Equiv}(M(A), M(B))$,
 a groupoid, each A, B, C you have compos. fun
 $E(A, B) \times E(B, C) \rightarrow E(A, C)$ ~~is~~

associative strictly, quasi-inverses exist. ~~That~~
~~is~~ Certainly you have a 2-category, something
 you might call a 2-groupoid. ~~Anyway~~
 Now you ~~pass~~ ^{restrict} to a component. This should be
 equivalent to ~~looking at~~ restricting to a single
 ring. Then you have a Picard category $E(A, A)$ strictly
 associative, but with quasi-inverses. ~~So~~ all
 this stuff so far is essentially pure category theory.

~~The~~ ~~thing~~ ~~you~~ ~~want~~ ~~to~~ ~~do~~ I think you
 have ~~missed~~ forgotten the idea of ~~being~~ trying to
 understand a component by constructing a fibre
 type category. What happens when we do this
 with rings?? So we fix A and then end up
 with assigning to each B the ~~category~~ ~~groupoid~~
 $\text{Equiv}(M(A), M(B))$. An object should now be a
 ring B together with $F: M(A) \xrightarrow{\sim} M(B)$. Then given
 (B, F) (B', F') we have ~~is~~



you consider pairs $G: M(B) \xrightarrow{\sim} M(B')$, $\xi: GF \xrightarrow{\sim} F'$
 do these form a cat? $G_1 \xrightarrow{u} G_2$ such that
~~is~~

3. I think all this is clear. But why are you interested? So far you have handled the category stuff. ~~How many minutes?~~ So the point is clear.

The picture you want involves more than just ~~meq's~~ meq's between rings. ~~meq's~~ You have meq homos. You have a basic cat consisting of ~~self-invertible rings and~~ ~~meq homos.~~ ~~meq's~~ You are looking at something like going from a category to a groupoid realizing its homotopy type. ~~The Brothendick~~ How ~~picture is clear~~ to put this into words? You have for each B a cat $M(B)$, in fact, a functor from rings + meq-homs to categories and equivalences.

~~But the picture is~~ ~~The result is~~ You want to view this as a fibre bundle where the fibres are equiv $M(A)$. The "structure gp"

the fibration should be ~~with~~ ~~the~~ the Picard groupoid $\mathcal{E}_g(M(A), M(A))$. What results???

You have this cat of rings and Meq-homs. What are you going to do with it. You can pick a component and ask about its homotopy type. I think what happens is that after you form the ^{principal} "fibre bundle" over the cat of rings + meq-homs with fibre $\mathcal{E}_g(M(A), M(B))$ over B , you get a contractible category. So your cat ~~of~~ meq-homs should be heq^B of the Picard category of self-equivalences of $M(A)$.

$$17. \quad G_1 F \xrightarrow{u_F} G_2 F$$

$$\xi_1 \downarrow \cong \quad \xi_2 \downarrow \cong$$

$$F' \quad F'$$

In principle these pairs G, ξ form a groupoid, but in practice because F is an equivalence, u is determined by ξ_1, ξ_2 .

Translate into bimodules.

$\text{Equiv}(M(A), M(B))$ a groupoid of ~~finite~~ (B, A) -bimodules which are invertible.

$$\begin{array}{ccc} & A & \\ P \swarrow & & \searrow P' \\ B & \xrightarrow{R} & B' \end{array}$$

pairs R, ξ such that

$$\xi: R \otimes_B P \xrightarrow{\cong} P'$$

maps $(R_1, \xi_1) \rightarrow (R_2, \xi_2)$ should be an isom

$$u: R_1 \rightarrow R_2 \quad \text{isom.}$$

$$\begin{array}{ccc} R_1 \otimes_B P & \xrightarrow{\xi_1} & P' \\ u \otimes 1 \downarrow & & \uparrow \\ R_2 \otimes_B P & \xrightarrow{\xi_2} & P' \end{array}$$

~~$[A, B]$ -module~~
 ~~$[B, A]$ -module~~

Comm.

But this because P is an equiv. u is completely determined.

$$\begin{array}{ccc} R_1 \cong R_1 \otimes_B P \otimes_A Q & \xrightarrow{\xi_1 \otimes 1} & P' \otimes_A Q \\ \downarrow u & \downarrow u \otimes 1 & \\ R_2 \cong R_2 \otimes_B P \otimes_A Q & \xrightarrow{\xi_2 \otimes 1} & P' \otimes_A Q \end{array}$$

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