18 March

Review of module theory – mainly to get the main paper done.

The main point of the paper is to understand Morita equivalences between idempotent rings, to prove Morita invariance of cyclic homology for h-unital rings. What are the main steps? ideas?

$A \xrightarrow{w} B$ homom. determines adjoint functors

$$\mathcal{M}(A) \underset{w^*}{\overset{w_!}{\rightleftarrows}} \mathcal{M}(B)$$

$$M \longmapsto B \otimes_A M$$

$$A^{(2)} \otimes_A N \longleftarrow N$$

$B^{(2)} \to B \to \tilde{B}$ are B^{op}-nil isos hence A^{op}-nil isos

$B^{(2)} \otimes_A M \xrightarrow{\sim} B \otimes_A M \xrightarrow{\sim} \tilde{B} \otimes_A M$ for M firm A-mod.

$$\mathrm{Hom}_A(M, A^{(2)} \otimes_A N) \xrightarrow{\sim}$$

$$\mathrm{Hom}_A(M, N) \xrightarrow{\sim} \mathrm{Hom}_A(M, \mathrm{Hom}_B(B, N))$$

$\wr\!\!\mid$ adjunction

$$\mathrm{Hom}_B(B \otimes_A M, N).$$

$\mathrm{Hom}(w_! M, w_! M') = \mathrm{Hom}(M, w^* w_! M')$

adjunction map:

$\alpha : B \otimes_A A^{(2)} \otimes_A N \longrightarrow N$ $\qquad b \otimes a_1 \otimes a_2 \otimes n \longmapsto b_1 w(a_1 a_2) n$

$\beta : M = A^{(3)} \otimes_A M \longrightarrow A^{(2)} \otimes_A B \otimes_A M$ $\qquad a_1 a_2 a_3 m \mapsto a_1 \otimes a_2 \otimes w(a_3) \otimes m$

Thm. ① $w_!$ fully faithful (equivalently $w^* w_! \overset{\beta}{\xleftarrow{\sim}} 1$)

② $\beta : 1 \xrightarrow{\sim} w^* w_!$ $\qquad A^{(2)} \xrightarrow{\sim} A^{(2)} \otimes_A B \otimes_A A^{(2)}$

③ $A \xrightarrow{w} B$ $\qquad A \otimes A^{op}$ nil isom. i.e. $A \,\mathrm{Ker}(w) A = 0$

and $w(A) B w(A) \subset w(A)$

2. Coordinate systems – a real time waster

What is the basic idea?

$$\begin{tikzcd} & m(B) \arrow[dd, "w_!"] \\ m(A) \arrow[ur, "P\otimes_A -", "\sim"'] \arrow[dr, "\sim", "P'\otimes_A -"'] & \\ & m(B') \end{tikzcd}$$

easy part: Suppose given $\begin{pmatrix} 1 & v \\ u & w \end{pmatrix} : \begin{pmatrix} A & Q \\ P & B \end{pmatrix} \to \begin{pmatrix} A & Q' \\ P' & B' \end{pmatrix}$

$w(b_1 b_2) = w(b_1)\, w(b_2)$

$u(pa) = u(p)a$

$u(bp) = w(b)\, u(p)$

$v(aq) = a\, v(q)$

$v(qb) = v(q)\, w(b)$

$qp = v(q)u(p)$

$u(p)v(q) = w(pq)$

Claim: canonical isos.

$B' \otimes_B P \otimes_A M \xrightarrow{\sim} P' \otimes_A M$ $\qquad$ M firm

$Q \otimes_B B^{(2)} \otimes_B N' \xrightarrow{\sim} Q' \otimes_{B'} N'$

$P \xrightarrow{u} P'$ $\qquad$ B-mil iso. then $v(q)u(p) = qp = 0$ $\qquad u(p) = 0$

$w(b)u(p) = u(bp)$ $\qquad$ $P_1 Q_1 P = 0$ gen B

$Q \xrightarrow{v} Q'$

is a B^{op}-mil isom.

$Q \otimes_B B^{(2)} \xrightarrow{\sim} Q' \otimes_B B^{(2)}$

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1. Gregory has invited Dorothy to practice football

English Comprehension

Cynthia Quillen

28.9.95

3. $Q \xrightarrow{v} Q'$ $\quad B^{op}$-nil isom.

$$Q \otimes_B N \xrightarrow{\sim} Q' \otimes_B N \quad \text{any firm } N.$$

$$P' \otimes_A Q \otimes_B N \xrightarrow{\sim} P' \otimes_A Q' \otimes_B N$$

$$P' \otimes_A Q \otimes_B P \otimes_A M \xrightarrow{\sim} B' \otimes_B P \otimes_A M$$

$$\boxed{\begin{array}{l} P' \otimes_A M \xrightarrow{\sim} B' \otimes_B P \otimes_A M \qquad M \text{ A-firm} \\ p' \otimes qpm \longmapsto p' v(q) \otimes p \otimes m \\ b' u(p) \otimes m \longleftarrow b' \otimes p \otimes m \end{array}}$$

and then you have map the other way

~~[illegible]~~ Other way start with

$$u : P \longrightarrow P' \qquad B\text{-nil isom.}$$

$$B^{(2)} \otimes_B P \xrightarrow{\sim} B^{(2)} \otimes_B P'$$

$$Q \otimes_B B^{(2)} \otimes_B P \otimes_A Q' \otimes_{B'} N' \xrightarrow{\sim} Q \otimes_B B^{(2)} \otimes_B P' \otimes_A Q' \otimes_B N'$$

$$\boxed{\begin{array}{l} Q' \otimes_{B'} N' \xrightarrow{\sim} Q \otimes_B B^{(2)} \otimes_B N' \\ q b_1 b_2 p q' \otimes n' \longmapsto q \otimes b_1 \otimes b_2 \otimes u(p) q' n' \\ v(q b_1 b_2) \otimes n' \longleftarrow q \otimes b_1 \otimes b_2 \otimes n' \end{array}} \quad N' \ B'\text{-firm.}$$

$$v(q b_1 b_2) \otimes u(p) q' n'$$

4. What was the problem I worked so hard on? The converse maybe. Transpose.

$$\begin{array}{ccc} \mathcal{M}(A) & \xrightarrow[\sim]{P\otimes_A -} & \mathcal{M}(B) \\ & \searrow^{\sim}_{P'\otimes_A -} & \downarrow w_! \\ & & \mathcal{M}(B') \end{array}$$

and an isom. $B'\otimes_B P$

$w : B \longrightarrow B'$ morphism.

you know it is $B\otimes B^{op}$-mil isom $\therefore w_!$ f.f.

$$B' w(B) B' = B'$$

$$P'Q' w(B) P'Q' \qquad \text{But} \qquad Q'w(B)\, w(B)P'$$

$$v(Q)\, u(P) = QP = A$$

?

$$\therefore P'Q'w(B)P'Q' \supset P'AQ' = P'Q'P'Q' = B^2$$

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why is $u : P \longrightarrow P'$ a B-mil iso

$u(p) = 0 \implies 0 = v(q)u(p) = qp \implies p_1 qp = 0$

$\implies PQ_p = 0$

$B \cdot \mathrm{Ker}(p) = 0$

$$\cancel{(pq)p' = p(qp')} \qquad = \cancel{p\, v(q) u(p')}$$

$$w(pq)p' = u(p)\underbrace{v(q)p'}_{a} = u(pa)$$

$$\therefore\ w(B)P' \subset u(P) \qquad w(B)\ \mathrm{Coker}(u) = 0$$

$w : B \longrightarrow B'$ $\quad B\otimes B^{op}$-mil isom

$w(b) = 0 \implies v(q)\, w(b)\, u(p) = v(qb)u(p) = qbp$

$\implies (p_1 q)\, b\, (pq_1) = 0. \qquad B\,\mathrm{Ker}(w)\, B = 0$

$$w(p_1q_1)\, p'q'\, w(p_2q_2)$$

$$= u(p_1)\underbrace{(v(q_1)p')}_{a}\underbrace{(q'u(p_2))}_{a_2}v(q_2) = u(p_1 a_1)\, v(a_2 q_2) = w(p_1 a_1 a_2 q_2)$$

5. $B' \subseteq B' w(B) B'$ | $B' = P'Q'$

$$\begin{pmatrix} A & Q' \\ P' & B' \end{pmatrix}$$

$Q'P' = A$
$P'Q' = B'$.

A, B' idemp.

$$B' = B'^3 = P'Q'P'Q'P'Q' = P'AAQ'$$
$$= P' v(Q) u(P) v(Q) u(P) Q'$$
$$= P' v(Q) w(B) u(P) Q'$$
$$\subset B' w(B) B'.$$

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argument $v : Q \to Q'$ B^{op}-nil coein

$$\therefore\ Q \otimes_B P \otimes_A M \xrightarrow{\sim} Q' \otimes_B P \otimes_A M$$
$$\|$$
$$M \xrightarrow{\sim} Q' \otimes_B P \otimes_A M$$
$$q\, p\, m \longmapsto v(q) \otimes p \otimes m$$

$$\boxed{P' \otimes_A M \xrightarrow{\sim} B' \otimes_B P \otimes_A M}$$

$\underline{M \text{ firm}}$

$$p' \otimes q p m \longmapsto p' v(q) \otimes p \otimes m$$

$$b' u(p) \otimes m \longleftarrow b' \otimes p \otimes m$$

$u : P \to P'$ B-nil coein

$$B^{(2)} \otimes_B P \otimes_A Q' \xrightarrow{\sim} B^{(2)} \otimes_B P' \otimes_A Q'$$

$v(q, b_1 b_2) \otimes u(p) q' n'$

$$Q \otimes_B B^{(2)} \otimes_B P \otimes_A Q' \otimes_{B'} N' \xrightarrow{\sim} Q \otimes_B B^{(2)} \otimes_B P' \otimes_A Q' \otimes_{B'} N'$$
$$(q \otimes b_1 \otimes b_2 \otimes p) \otimes q' \otimes n' \longmapsto q \otimes b_1 \otimes b_2 \otimes u(p) q' n'$$
$$Q' \otimes_{B'} N' \xrightarrow{\sim} Q \otimes_B B^{(2)} \otimes_B N'$$

6. I need to recover the outline of Morita equivalence theory. Maybe also the proof of Morita invariance of cyclic homology.

~~[illegible]~~ main proof goes ~~as~~ as follows.

$$\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$$ ~~[illegible]~~ $\Big($ A l+r flat $\iff {}_BP, Q_B$ flat

$$A\overset{L}{\otimes}_A = Q\overset{L}{\otimes}_B P\overset{L}{\otimes}_A = P\overset{L}{\otimes}_A Q\overset{L}{\otimes}_B = B\overset{L}{\otimes}_B$$

~~[illegible]~~ To get this to work I need E some sort of flat bimod res. of A.

$$A\overset{L}{\otimes}_A = A\otimes_A E\otimes_A$$

$$= Q\otimes_B P\otimes_A E\otimes_A = \underbrace{P\otimes_A E\otimes_A Q}_{\text{flat B bimod res. of } B}\otimes_B$$

$P\otimes_A E\otimes_A Q$ made of $P\otimes_k Q$ $\Leftarrow$ E made of $\tilde{A}\otimes_k\tilde{A}$

~~So now consider a coherent sheaf.~~

So main argument. ~~Suppose B_1, B_2~~

How to formulate theorem. Roughly says that given B_1, B_2 k-flat h-unital, ~~and~~ then a Morita equiv. $\mathcal{M}(B_1)\simeq\mathcal{M}(B_2)$ induces an isom $HC(B_1)\simeq HC(B_2)$. ~~[illegible]~~ You want to put in various remark, amplifications.

isom meqs yield same HC isom.

for a meq homom. the HC isom is ind. by the hom.

Delicate points: HC is a functor on the defined category rings. But the natural situation is a 2-cat. So to each ~~objects~~ B you assoc. $\mathcal{M}(B)$ a cat. To each B_1, B_2 you get a groupoid of equivalences $\mathcal{M}(B_1)\simeq\mathcal{M}(B_2)$ and for each B_1, B_2, B_3 have

7. composition of equivalences. This gives you a 2-groupoid. You also have a 1-cat whose objects are rings B and maps are meghom. And a (weak (sort of) functor from the 1-cat to the 2-groupoid.

Now how do I want to understand this situation? You fix A and look at dual pairs $(P, Q, \phi : Q \otimes P \twoheadrightarrow A)$. These form a category pure and simple, and there's a functor $(P,Q) \mapsto P \otimes_A Q$ ~~to~~ rings and meghoms. ~~Anyway what else~~

Work in 2-cat $\underset{\sim}{Cat}$. Fix $\mathcal{A}$, consider

$\mathcal{B} \mapsto \underset{\sim}{Han}(\mathcal{A}, \mathcal{B})$. Can I form a

2-cat over $\underset{\sim}{Cat}$ with fibre over $\mathcal{B}$ equal to $\underset{\sim}{Han}(\mathcal{A}, \mathcal{B})$. Object would be a pair $\mathcal{B}$,

2-morphisms go between 1-morphisms with same ndpoints

$$\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Uparrow"] & \bullet \end{tikzcd}$$

$$\begin{tikzcd} Eq(\mathcal{A}, \mathcal{B}) \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & Eq(\mathcal{A}, \mathcal{B}') \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{B} \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \mathcal{B}' \end{tikzcd}$$

8. Guess. Objects of the Total 2-cat should be pairs $(B, \Phi : a \xrightarrow{\sim} B)$. Then give (B, Φ) (B', Φ') you want a groupoid a maps. $\underline{\mathrm{Hom}}((B,\Phi),(B',\Phi'))$ has

objects $B \xrightarrow{F} B'$ either such that $F\Phi = \Phi'$ or together with an isom. $F\Phi \xrightarrow{\sim} \Phi'$.

It's the latter I'm used to ~~and there is no improvement~~

Example: $(B, F : m(A) \xrightarrow{\sim} m(B))$ these pairs form a category, maps are $(B \xrightarrow{W} B', \theta : w_! F \xrightarrow{\sim} F')$

The category theory remains as confusing as ever. You've defined a category, but you still haven't ~~[illegible]~~ decided whether there is a 2-cat structure of interest.

Idea: Consider collection of cats $m(B)$ equiv. to $m(A)$ as a full 2-subcat of Cats + Equiv.

~~Consider the collection of cats. This should be~~

Fix an A and everything should be equivalent to a Picard cat.

So when I do cat theory properly I should get a Picard cat = 2 groupoid with one object

What can I say about self-equivalences of $m(A)$ = category of invertible bimodules.

9. discussion today. important idea that a 2-groupoid with a single object is a strict Picard category.

Starting point - you have rings B and a cat $\mathcal{M}(B)$ for each B, you ~~consider~~ get a 2-gpoid with objects B by assoc. to each (B_1, B_2) the gpoid $\mathrm{Equiv}(\mathcal{M}(B_1), \mathcal{M}(B_2))$.

Suppose fix A, then for each B look at $\mathrm{Equiv}(\mathcal{M}(A), \mathcal{M}(B))$ and for each pair (B_1, B_2) look at $\mathrm{Equiv}(\mathcal{M}(A), \mathcal{M}(B_1)) \times \mathrm{Equiv}(\mathcal{M}(B_1), \mathcal{M}(B_2))$.

It seems clear that I get a 2-groupoid over the previous one. It should be true that this 2-gpoid is equiv to a 1-gpoid.

So let's use $\mathcal{M}$ contexts, or ~~just~~ the bimodules.

Anyway this thing $\mathrm{Equiv}(\mathcal{M}(A), \mathcal{M}(A))$ somehow becomes the Pic. groupoid

Equiv

10. Let's go back to Meg theory.
various cats + 2-cats.
Start with ~~maps~~ maps $B \mapsto \mathcal{M}(B)$.
To each B get cat $\mathcal{M}(B)$

To each meghom $B \xrightarrow{w} B'$ get $w_! : \mathcal{M}(B) \xrightarrow{\sim} \mathcal{M}(B')$
transitive up to canon isom, so you get a
cofibred cat over rings and Meghoms. Objects
are pairs (B, M), maps $(B, M) \longrightarrow (B', M')$ given by
$w: B \to B'$ + $B' \otimes_B M \xrightarrow{\sim} M'$. Cocart sections?

2-gpoid obj. B, 1-arrow equiv $\mathcal{M}(B_1) \to \mathcal{M}(B_2)$,
2-arrow is isom. of functors.
component of
This 2-gpoid should be "equiv" to 2 gpoid with
-object.

~~Let's~~ try to put into words what you might
be looking for. Fund. 2 gpoid of cats and equiv.
Pick component of this + get Picard cat. ~~Somehow~~
~~you start with the fibred cat of~~

You have base category of rings + meghom and
a cofibred cat over it such that transitions are equiv.
What does topology say? You have a quasi-fibration
fibres are cats but not gpoids. So it's
like a fibre bundle w structural group, but in the
homotopy sense. Principal bundle ~~with~~ fibre
over B should be the $\boxed{\text{1-groupoid}}$ of equiv $\mathcal{M}(A) \xrightarrow{\sim} \mathcal{M}(B)$

So we form the principal bundle, ~~objects~~
cofibred cat over rings + meghom with
fibre $\mathrm{Equiv}(\mathcal{M}(A), \mathcal{M}(B))$

11. Megtheory again. I am still ~~trying~~ to fit the cat stuff in some order. We start with the map $B \mapsto \mathcal{M}(B)$ associating a cat to a ring. Then we get some sort of structure on the ~~collec~~ family of rings, namely, for each pair of rings B_1, B_2 a gpoid $Eq(\mathcal{M}(B_1), \mathcal{M}(B_2))$ composition etc. This is a 2-gpoid. ~~Next~~ Fix a component. Up to equivalence this comp. is equiv to full subcategory with one object A. So you have a strict Picard cat. <u>Question</u>: Is it always like usual Pic. cats? Is k-inv. stable, or nonzero if stable. This question arises maybe in Cat.

Anyway this is what happens when you try to work with rings up to Morita equivalence. You end up with a 2 groupoid, disjoint union of Picard categories. Nothing more you can say it seems.

The next step is to bring in meghoms. You have a based cat $\mathcal{B}$ of B and meghoms, and naturally a cofibred cat assoc. to $B \mapsto \mathcal{M}(B)$. The extn functors are equivs, so we have the analogue of quasi-fibring. At the moment I have a cof. cat over a base cat, something like a fibre bundle. There should be an assoc. principal bundle for some top. group. The top group should be the Picard cat. So what.

2. Anyway think a little bit about Meg
Anyway ~~[illegible]~~ recall earlier problem about 2-categories. You begin with rings A idempotent, each A has $M(A)$ attached, for each pair A, B ~~you~~ you have $\mathcal{E}(A,B) = \mathrm{Equiv}(M(A), M(B))$, a groupoid, each A, B, C you have compos. fun

$$\mathcal{E}(A,B) \times \mathcal{E}(B,C) \longrightarrow \mathcal{E}(A,C)$$

associative strictly, quasi-inverses exist. ~~[illegible]~~ Certainly you have a 2-category, ~~something~~ you might call a 2-groupoid. ~~Anyway~~ Now you ~~pass~~ restrict to a component. This should be equivalent to ~~looking at~~ restricting to a single ring. Then you have a Picard category $\mathcal{E}(A,A)$ strictly associative, but with quasi-inverses. ~~So so~~ All this stuff so far is essentially pure category theory. ~~The next thing you need to do~~ I think you have ~~missed~~ forgotten the idea of ~~doing~~ trying to understand a component by constructing a fibre type category. What happens when we do this ith rings?? So we fix A and then end up ith assigning to each B the ~~category of~~ groupoid quiv$(M(A), M(B))$. An object should now be a ing B together with $F: M(A) \xrightarrow{\sim} M(B)$. Then given $3, F)$ (B', F') we have

$$\begin{tikzcd} & M(A) \arrow[dl, "F"'] \arrow[dr, "F'"] & \\ M(B) & & M(B') \end{tikzcd}$$

hen consider pairs $G: M(B) \xrightarrow{\sim} M(B')$, $\xi: GF \xrightarrow{\sim} F'$.
o these form a cat? $G_1 \underset{\sim}{\xrightarrow{u}} G_2$ such that

~~G_1~~

3. I think all this is clear. But why are you interested? So far you have handled the category stuff. ~~How many minutes?~~ ~~So the point is clear.~~

The picture you want involves more than just meg's between rings. You have meg homos. You have a basic cat consisting of idempotent rings and meg homos. You are looking at something like going from a category to a co-groupoid realizing its homotopy type. ~~The Grothendieck picture is clear:~~ How to put this into words? You have for each B a cat $M(B)$, in fact, a functor from rings + meghoms to categories and equivalences. ~~So the picture is~~ ~~The result is~~ ~~but The picture is~~ You want to view this as a quasi-fibre bundle where the fibres are equiv to $M(A)$. The "structure gp" of the fibration should be the Picard groupoid $Eq(M(A), M(A))$. What results???

You have this cat of rings and Meghoms. What are you going to do with it. You can pick a component and ask about its homotopy type. I think what happens is that after you form the "principal fibre bundle" over the cat of rings + meghoms with fibre $Eq(M(A), M(B))$ over B, you get a contractible category. So your cat meghoms should be heq to B of the Picard category of self-equivalences of $M(A)$.

14.
$$\begin{tikzcd} G_1F \arrow[r, "uF"] \arrow[d, "\xi_1"', "\sim"] & G_2F \arrow[dl, "\xi_2", "\sim"'] \\ F' & \end{tikzcd}$$

In principle these pairs G, ξ form a groupoid, but in practice because F is an equivalence, u is determined by $\xi_1\ \xi_2$.

Translate into bimodules.

$\mathrm{Equiv}(\mathcal{M}(A), \mathcal{M}(B))$ [redacted] a groupoid of firms (B,A)-bimodules which are invertible.

$$\begin{tikzcd} & A \arrow[dl, "P"'] \arrow[dr, "P'"] & \\ B \arrow[rr, "R"] & & B' \end{tikzcd}$$

pairs R, ξ such that $\xi : R \otimes_B P \xrightarrow{\sim} P'$

maps $(R_1, \xi_1) \longrightarrow (R_2, \xi_2)$ should be an isom

$u : R_1 \longrightarrow R_2$ s.t.

$$\begin{tikzcd} R_1 \otimes_B P \arrow[dr, "\xi_1"] \arrow[d, "u\otimes 1"'] & \\ R_2 \otimes_B P \arrow[r, "\xi_2"'] & P' \end{tikzcd}$$

~~$\mathbb{Q}[\sigma]_+$-module~~
~~$\mathbb{Q}[\sigma]_-$-module~~

Comm.

But this because P is an equiv. u is completely determined.

$$\begin{tikzcd} R_1 \arrow[d, "u"'] \arrow[r, equal] & R_1 \otimes_B P \otimes_A Q \arrow[d, "u\otimes 1\otimes 1"] \arrow[dr, "\xi_1\otimes 1"] & \\ R_2 \arrow[r, equal] & R_2 \otimes_B P \otimes_A Q \arrow[r, "\xi_2 \otimes 1"'] & P' \otimes_A Q \end{tikzcd}$$

see end of file DISC in directory AO