42 Does this help?

In any case let M be an A-module, say firm. Then take base change

$$\tilde{C} \otimes_{\tilde{A}} M$$

and make firm over C i.e.

$$C \otimes_A M = \begin{pmatrix} A \\ P \end{pmatrix} \otimes_A (A \quad Q) \otimes_A M$$

$$= \begin{pmatrix} A \\ P \end{pmatrix} \otimes_A M.$$

So it seems OKAY

$$\begin{pmatrix} A & Q \\ P & B \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} a & q \\ p & b \end{pmatrix} \begin{pmatrix} a' & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} aa' & 0 \\ pp' & 0 \end{pmatrix}$$

Patterns Suppose we have $\begin{pmatrix} A & A \\ P & P \end{pmatrix}$ $P \otimes_A A \xrightarrow{\sim} P$

You must give $A \otimes_{\mathbb{Z}} P \longrightarrow A$, i.e. $P \to \mathrm{Hom}_A(A, A)$ map of A^{op}-modules.

Think a little about what you can say. Coordinates $(Q, P, Q\otimes P \twoheadrightarrow A)$.
Actually there might be a simple independence argument here. The point is that once P is fixed you have the functor $P\otimes_A -$ which is suitable for various Q. The different choices of Q might lead to different B all related by excision.

I think I want to work on Morita invariance for Hochschild & cyclic homology. The argument should be very simple for HH.

WRONG $A \overset{L}{\otimes}_A \longleftarrow Q \overset{L}{\otimes}_B P \overset{L}{\otimes}_A = P \overset{L}{\otimes}_A Q \overset{L}{\otimes}_B \longrightarrow B \overset{L}{\otimes}_B$

The interpretation is needed + some lemmas. Go over the lemmas.

Claim that $Q \overset{L}{\otimes}_B P \longrightarrow Q \otimes_B P \twoheadrightarrow A$ has cone whose homology is killed by $QP = A$ on both sides. $F \to P$ a proj res over B.

$$\begin{tikzcd} Q\otimes_B F \arrow[r, "p\cdot"] \arrow[rrr, bend right, "qbp\cdot"'] & F \arrow[r, "b\cdot"] & F \arrow[r, "q\cdot"] & Q\otimes_B F \end{tikzcd}$$

$$q'\otimes f \longmapsto (pq')f \longmapsto b(pq')f \longrightarrow q\otimes b(pq')f$$
$$= (qbp)q'\otimes f$$

So I've forgotten some things.

$$Q\otimes_B F \longrightarrow Q\otimes_B P$$

$$Q\otimes_B F$$

44 So how do I go about ~~the process~~ recalling the steps? ~~You consider~~ I recall some general arguments, namely if X has nil homology then so does $P\otimes_A X$.

$$\begin{tikzcd} P\otimes_A X \arrow[r, "q\cdot"] \arrow[rrr, bend right, "paq\cdot"'] & X \arrow[r, "a\cdot"] & X \arrow[r, "p\cdot"] & P\otimes_A X \end{tikzcd}$$

shows that if $a\cdot H_i X = 0$, then $paq\cdot H_i(P\otimes_A X) = 0$.
So A^n kills $H_i(X) \Rightarrow PA^nQ$ kills $H_i(P\otimes_A X)$.
and $PA^nQ = B$ for some n.

So apply this to $\mathrm{Cone}(E \longrightarrow Q)$ where E is proj resolution of Q. Get $\mathrm{Cone}(P\overset{L}{\otimes}_A Q \longrightarrow P\otimes_A Q)$ has B-nil homology.

What points are important?
What do I need for Morita invariance?

What do I need for Morita invariance?
You ~~know that~~ want

$$P\overset{L}{\otimes}_A Q \overset{L}{\otimes}_B \longrightarrow B\overset{L}{\otimes}_B$$

to be a quis. Method I might use:

$$P\otimes_{-\tau}\mathcal{B}(A)\otimes_{\tau} Q\otimes_{-\tau}\mathcal{B}(B)\otimes_{\tau} \longrightarrow B\otimes_{-\tau}\mathcal{B}(B)\otimes_{\tau}$$

~~two steps~~
~~1) $X\otimes_{-\tau}\mathcal{B}(B)\otimes_{\tau}$ acyclic~~
~~2) $X =$~~

The above map is augmentation for double cx.
Put $X = \mathrm{Cone}(P\otimes_{-\tau}\mathcal{B}(A)\otimes_{\tau} Q \longrightarrow B)$. Claim
1) homology of X killed by B on both sides
2) $X\otimes_{-\tau}\mathcal{B}(B)\otimes_{\tau}$ is acyclic.

45 I think I just checked 1).
Go over it again. You have

$$\begin{array}{ccc} \tilde{A} \otimes_{-\tau} B(A) \otimes_{\tau} Q & \longrightarrow & \tilde{A} \otimes_A Q \\ \| & & \| \\ Y = \mathrm{Cone}\,\big(F & \xrightarrow[\text{proj } A\text{-res.}]{} & \tilde{Q}\,\big) \end{array}$$

Then $\underbrace{}_{Y \text{ acyclic} \Rightarrow}$ $P \otimes_A Y$ has homology killed by B on left

$$\begin{array}{ccc} \| & & \\ \mathrm{Cone}\,\big(P \otimes_A F & \longrightarrow & P \otimes_A Q\big) \\ \| & & \| \\ X = \mathrm{Cone}\,\big(P \otimes_{-\tau} B(A) \otimes_{\tau} Q & \longrightarrow & B\big) \end{array} \qquad \therefore B\, H_*(X) = 0$$

---

other side should be same.

$$\mathrm{Cone}\,\big(P \otimes_{-\tau} B(A) \otimes_{\tau} \tilde{A} \longrightarrow P \otimes_A \tilde{A} = P\big) \quad \text{acyclic.}$$

$$\Rightarrow X = \mathrm{Cone}\,\big(P \otimes_{-\tau} B(A) \otimes_{\tau} Q \longrightarrow P \otimes_A Q = B\big) \quad \text{has } H_*(X)\, B = 0$$

This proves 1). Now need 2).
Here you use the Postnikov filtration of X.
So this reduces you to case of an B-bimodule
N such that $BN = NB = 0$. i.e. an abel. gp.

$$N \otimes_{-\tau} B(B) \otimes_{\tau} = N \otimes_{\mathbb{Z}} B(B)$$

Confused about difference between B and $\bar{B}$.
How does this work?
Your X is $\mathrm{Cone}\,\big(P \overset{L}{\otimes}_A Q \to P \otimes_A Q\big)$

$$\begin{array}{ccc} \| & & \\ P \otimes_{-\tau} B(A) \otimes_{\tau} Q & \longrightarrow & P \otimes_A \tilde{A} \otimes_A Q \\ & & \| \\ & & P \otimes_A Q = B \end{array}$$

$$X = P \otimes_{-\tau} \bar{B}(A) \otimes_{\tau} Q$$

~~[illegible]~~ You want to prove that

$$P \otimes^L_A Q \otimes^L_B \longrightarrow B \otimes^L_B \quad \text{quis}$$

$$P \otimes B(A) \otimes Q \otimes B(B) \qquad ??$$

I guess the first question is whether

$$B \cdot M = M \cdot B = 0 \overset{?}{\Longrightarrow} M \overset{L}{\underset{B}{\otimes}} = 0.$$

The answer is NO. $H_0(\mathbb{Z} \otimes^L_B) = \mathbb{Z} \otimes_B = \mathbb{Z}$.

So it seems I made a mistake in my diary. Perhaps the refined argument

$$A \otimes^L_A = A \otimes^L_A A \otimes^L_A \longleftarrow Q \otimes^L_B P \otimes^L_A A \otimes^L_A \longleftarrow Q \otimes^L_B B \otimes^L_B P \otimes^L_A A \otimes^L_A$$

still works. Probable.

What about your latest slick version. namely chose $E \to \tilde{A}$ flat A-bimod res.

$F \to \tilde{B}$ — B —

Then ~~$\mathbb{Z} \otimes_A F \otimes_B \longrightarrow E \otimes_A$~~

$$(*) \quad P \otimes_A E \otimes_A Q \otimes_B F \otimes_B \longrightarrow B \otimes_B F \otimes_B$$

Since F flat over $B \otimes_{\mathbb{Z}} B^{op}$, $- \otimes_B F \otimes_B$ is exact so $(*)$ should be a quis, provided $P \otimes_A E \otimes_A Q \to P \otimes_A \tilde{A} \otimes_A Q = B$ is a quis. You did this assuming P_A, ${}_AQ$ flat.

So where are we?? Basically happy about Hochschild but unsure about cyclic hom.

Look at the cyclic business. What sort of definition do you want to use? Standard defn is via mixed complex $(\bar{C}(\tilde{A}), b, B)$, i.e. cyclic k-module $[n] \mapsto A^{\otimes n+1}$.

$$\bar{C}(\tilde{A})_n = \tilde{A} \otimes A^{\otimes n}$$
$$= A^{\otimes n+1} \oplus A^{\otimes n}$$

But I'm proposing to work with things like $A \overset{L}{\otimes}_A$, $A \overset{L}{\otimes}_A A \overset{L}{\otimes}_A$, etc.

Let's go back to Q, P flat on both sides.

$$A \otimes_A E \otimes_A \longleftarrow Q \otimes_B F \otimes_B P \otimes_A E \otimes_A$$
$$\parallel$$
$$B \otimes_B F \otimes_B \longleftarrow P \otimes_A E \otimes_A Q \otimes_B F \otimes_B$$

Can you iterate? Obvious approach is to consider the cyclic objects

$$[A \otimes_A E \otimes_A]^{(n+1)} \longleftarrow [Q \otimes_B F \otimes_A P \otimes_A E \otimes_A]^{(n+1)}$$
$$4$$
$$[\qquad\qquad]^{(n+1)} \leftarrow \{\qquad\qquad\}$$

What you need for it to work is to be able to ~~substitute~~ $Q \otimes F \otimes P \mapsto A$ in the appropriate circle

$\otimes\ Q \otimes F\ \otimes$
$E \qquad\qquad P$
$\otimes \qquad\qquad \otimes$
$P \qquad\qquad E$
$\otimes\ F \otimes Q\ \otimes$

~~need~~ then

$$- \otimes_A E \otimes_A Q \otimes_B F \otimes_B P \otimes_A E \otimes_A$$

to be exact.

$$- \otimes_{\tilde{A}^e} (E \otimes_{\mathbb{Z}} E) \otimes_{\tilde{A}^e} (Q \otimes_{\mathbb{Z}} P) \otimes_{\tilde{B}^e} F$$

If you put $\tilde{B}^e$ in for F, then you get

$$- \otimes_{\tilde{A}^e} (E \otimes_{\mathbb{Z}} E) \otimes_{\tilde{A}^e} (Q \otimes_{\mathbb{Z}} P) = P \otimes_A E \otimes_A - \otimes_A E \otimes_A Q$$

 Put in $\tilde{A}^e$ for E to get

$$P \otimes_A \tilde{A} \otimes_Z \tilde{A} \otimes_A - \otimes_A \tilde{A} \otimes_Z \tilde{A} \otimes_A Q$$

$$= P \otimes_Z - \otimes_Z Q = (P \otimes_A \tilde{A}) \otimes_Z - \otimes_Z (\tilde{A} \otimes_A Q)$$

$$- \otimes_A \tilde{A}^e \otimes_A Q \otimes_B \tilde{B}^e \otimes_B P \otimes_A \tilde{A}^e \otimes_A$$

$$- \otimes_Z Q \otimes_Z P \otimes_Z$$

$$\cancel{- \otimes_A \tilde{A} \otimes_A Q \otimes_Z P \otimes_A \tilde{A} \otimes_A}$$

$$\cancel{P \otimes_A \tilde{A} \otimes_A - \otimes_A \tilde{A} \otimes_A Q = P\,?}$$

Go back to

$$- \otimes_A E \otimes_A Q \otimes_B F \otimes_B P \otimes_A E \otimes_A$$

Put $\tilde{B} \otimes_Z \tilde{B}$ for F. get

$$- \otimes_A E \otimes_A Q \otimes_B \tilde{B} \otimes_Z \tilde{B} \otimes_B P \otimes_A E \otimes_A$$

$$= P \otimes_A E \otimes_A - \otimes_A E \otimes_A Q$$

Put $\tilde{A} \otimes_Z \tilde{A}$ in for E

$$P \otimes_A \tilde{A} \otimes_Z \tilde{A} \otimes_A - \otimes_A \tilde{A} \otimes_Z \tilde{A} \otimes_A Q = P \otimes_Z - \otimes_Z Q$$

So the conclusion seems to be that I can't go any further without assuming P, Q flat over groundring.

49 Review yesterday

I reviewed the details of M inv. for HH, decided that

1) long form: $A\overset{L}{\otimes}_A = A\overset{L}{\otimes}_A A\overset{L}{\otimes}_A = Q\overset{L}{\otimes}_B P\overset{L}{\otimes}_A A\overset{L}{\otimes}_A = \cdots$ necessary

2) the left + right flat version works for HH but does not seem to generalize to HC.

See what we need to handle cyclic homology. ~~The~~ You have to iterate your proof for HH. This means considering

$$\left[Q\overset{L}{\otimes}_B B\overset{L}{\otimes}_B P\overset{L}{\otimes}_A A\overset{L}{\otimes}_A\right]^{(n+1)}$$

The proof uses in this expression we ~~can stop~~ have

$$A\overset{L}{\otimes}_A Q\overset{L}{\otimes}_B B\overset{L}{\otimes}_B P\overset{L}{\otimes}_A A \xrightarrow[\text{qu}]{} A\overset{L}{\otimes}_A A \xrightarrow[\text{fu}]{\cancel{\text{a gues}}} A$$

Your proof uses in the end ~~a~~ the fact that HC is given by the pre cyclic object

$$n \longmapsto \left[A\overset{L}{\otimes}_A\right]^{(n+1)} = \left[A\otimes_{\mathbb{Z}} \mathcal{B}(A)\otimes_{\mathbb{Z}}\right]^{(n+1)}$$

Standard model for $\mathcal{B}(A)$ is

$$\xrightarrow{b'} A^{\otimes 2} \xrightarrow{b'} A \to \mathbb{Z}$$

~~In this~~ there is unital + nonunital stuff to be checked here.

So start at the beginning with a definition of HC(A). Need a definition of HC(A)! Answer is the homology of the precyclic object $n \mapsto A^{\otimes n+1}$. Known this is given by the standard Connes-Tsygan bicomplex. So now you have to get this to agree with what you need which is

$$n \longmapsto \left[A\overset{L}{\otimes}_A\right]^{(n+1)}.$$

Maybe first treat the unital case. $A = R$. Here you have an idea which ~~says that~~ ~~you can do~~ involves $\left[(R\otimes R)\otimes_R\right]^{(n+1)}$ cyclic module

So where are we? ~~We have a~~

I want to start with $Q \otimes P \xrightarrow{\ell} A$ and somehow embed it in something nonsingular. Start ~~with~~

$$P \longrightarrow \mathrm{Hom}_A(Q, A)$$

Take $Q \otimes P \to A$ and add $Q \otimes P_0 \to A$

First take the unital case. Say $Q = A^n$. Then $\mathrm{Hom}_A(Q, A) = A^n_r$.

Some ~~where~~ time today I have decided to study Morita equivalences. You want somehow a category. Something like fin gen. proj modules. More precisely start with f.g. projective modules P and associate $\mathrm{Aut}(P)$. Then you want to form a direct limit, i.e. if $P' \hookrightarrow P''$ you want $\mathrm{Aut}(P') \longrightarrow \mathrm{Aut}(P'')$. Now you ~~can~~ ~~Aut(P)~~ you have $\mathrm{End}(P)$ $\mathrm{End}(P')$

$$\mathcal{M}(A) = \mathcal{M}(\mathrm{End}(P)) = \mathcal{M}(\mathrm{End}(P'))$$

Specifically you have $\begin{pmatrix} A & Q \\ Q^* & \mathrm{End}_A(Q)^{op} \end{pmatrix}$ $\begin{pmatrix} A & Q' \\ Q'^* & \mathrm{End}_A(Q')^{op} \end{pmatrix}$

But to get the sort of map $\mathrm{Aut}(Q') \longrightarrow \mathrm{Aut}(Q)$ you want, you need to have

$$Q' \otimes Q'^* \subset Q \otimes Q^*$$

compatible with the pairing. Thus you need to give a complement: $Q = Q' \oplus Q''$.

9/8 Idea I have now is to try to ~~stabl~~ stabilize in roughly the same way as when constructing GL. ~~So with~~

idea: Instead of a group homom. $G \to H$ consider a $G \times H$ set.

68 The examples I have are inclusions $A \subset B$ such that $ABA = A$, $BAB = B$.

$$\begin{pmatrix} A & AB \\ BA & B \end{pmatrix}$$

and surjections

$$\cancel{\begin{pmatrix} A & A/KA \\ A/AK & A/K \end{pmatrix}} \qquad AKA = 0$$

$$\begin{pmatrix} A & A/AK \\ A/KA & A/K \end{pmatrix} \qquad \begin{pmatrix} 0 & AK \\ KA & K \end{pmatrix}\begin{pmatrix} A & A \\ A & A \end{pmatrix} \subset \begin{pmatrix} 0 & 0 \\ KA & KA \end{pmatrix}$$

There's a compatibility to be checked, namely, to see that the Morita invariance isomorphism $A \overset{L}{\otimes}_A \simeq B \overset{L}{\otimes}_B$ in these cases is actually given by the homomorphism $A \to B$. at least in degree zero.

$$A\otimes_A = Q\otimes_B P\otimes_A = P\otimes_A Q\otimes_B = B\otimes_B$$

$$A = AB\cdot BA \qquad a = a_1b_1b_2a_2$$

$$a_1b_1b_2a_2 \longleftarrow a_1b_1\otimes b_2a_2 \Longleftrightarrow b_2a_2\otimes a_1b_1 \longmapsto b_2a_2a_1b_1 = a_1b_1b_2a_2$$

~~O.K.,~~ there's no problem defining

$HH(A) \longrightarrow HH(B)$ assoc. to $A \to B$?

~~No~~ $(\tilde{A}, A) \longrightarrow (\tilde{B}, B)$.

<u>Question</u>: Given $M(k)$, $k = k^2$ consider all coords $V\otimes U \twoheadrightarrow k$ and to each associate the group $(U\otimes_k V)_{ab}$. The question is whether this system of groups can be organized into a filtering category?