C. Topics in Cyclic Cohomology and K-theory, Trinity Term 1992.

125 pages of notes. The lecture course is concerned with the analogy between de Rham (co)homology for commutative algebras over **C** and cyclic homology theory for noncommutative algebras, and considers the following topics. Smooth algebraic varieties; noncommutative analogue of smoothness. Quasi fee algebras. Resolutions of b and b'; Hochschild homology. Laplace operator and spectral resolution of Karoubi operator. Connes's B operator. Connes's exact sequence. Connes–Tsygan double complex. Periodic cyclic homology. Negative cyclic homology. Hodge filtration. Models for cyclic theory. The Fedosov product. Harmonic decomposition. Cartan homotopy formula for complex. Connes notion of a connection. Quasi free extensions of algebras. Goodwillie's theorem. Curvature and Yang–Mills connections. The torsion of the connection. The exponential map.

Editor's remark The lecture notes were taken during lectures at the Mathematical Institute on St Giles in Oxford. There have been subsequent corrections, by whitening out writing errors. The pages are numbered, but there is no general numbering system for theorems and definitions. For the most part, the results are in consecutive order, although in one course the lecturer interrupted the flow to present a self-contained lecture on a topic to be developed further in the subsequent lecture course. The note taker did not record dates of lectures, so it is likely that some lectures were missed in the sequence. The courses typicaly start with common material, then branch out into particular topics. Quillen seldom provided any references during lectures, and the lecture presentation seems simpler than some of the material in the papers.

• D. Quillen, Cyclic cohomology and algebra extensions, *K-Theory* **3**, 205–246.

• D. Quillen, Algebra cochains and cyclic cohomology, *Inst. Hautes Etudes Sci. Publ. Math.* **68** (1988), 139–174.

• J. Cuntz and D. Quillen, Cyclic homology and nonsingularity, *J. Amer. Math. Soc.* **8** (1995), 373–442.

Commonly used notation

k a field, usually of characteristic zero, often the complex numbers

A an associative unital algebra over k, possibly noncommutative

$\bar{A} = A/k$ the algebra reduced by the subspace of multiples of the identity

$\Omega^n A = A \otimes (\bar{A} \otimes \ldots \otimes \bar{A})$

$\omega = a_0 da_1 \ldots da_n$ an element of $\Omega^n A$

$\Omega A = \oplus_{n=0}^{\infty} \Omega^n A$ the universal algebra of abstract differential forms

e an idempotent in A

d the formal differential (on bar complex or tensor algebra)

b Hochschild differential

b', B differentials in the sense of Connes's noncommutative differential geometry

λ a cyclic permutation operator

K the Karoubi operator

$\circ$ the Fedosov product

G the Greens function of abstract Hodge theory

N averaging operator

P the projection in abstract Hodge theory

D an abstract Dirac operator

∇ a connection

I an ideal in A

V vector space

M manifold

E vector bundle over manifold

τ a trace

$T(A) = \oplus_{n=0}^{\infty} A^{\otimes n}$ the universal tensor algebra over A

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Topics in Cyclic Cohomology and K-theory

Professor D G Quillen

Analogy between de Rham (co)-homology for commutative algebras over $\mathbb{C}$ and cyclic homology theory for noncommutative algebras

<u>Commutative picture</u>

$$\{\text{finitely generated commutative algebras } A\} = \{\text{affine schemes of finite type over } \mathbb{C}\}$$

$$\downarrow$$

$A \longmapsto Var(A)$

where

$$Var\, A = Hom_{alg}(A, \mathbb{C}) \subseteq \mathbb{C}^n \qquad \{\text{topological spaces}\}.$$

Interesting homology is $H^i(Var A, \mathbb{C})$

differential forms for a commutative algebra

$$\Omega^{\cdot}_A \qquad A \xrightarrow{d} \Omega^1_A \to \Omega^2_A \to$$

(universal super commutative DGA generated by A)

$H^i(\Omega^{\cdot}_A)$

(Deep) <u>de Rham theorem</u> If A is smooth (var A nonsingular and A has trivial radical $\sqrt{0} = 0$)

then $\quad H^i(\Omega_A^\cdot) \xrightarrow{\sim} H^i(\mathrm{Var}\,A, \mathbb{C})$
for the natural map given by generators.

Proof: Uses Hodge theory, resolution of singularities etc.

When $\underline{A}$ is not smooth

Theorem (Hartshorne on algebraic varieties)
Suppose $A = R/I$ where $\underline{R}$ is smooth
(eg $\underline{R}$ polynomial ring) then

$$H^i\big(\varprojlim_n \Omega_R^\cdot / I^n \Omega_R^\cdot\big) \xrightarrow{\sim} H^i(\mathrm{Var}\,A)$$

Note: $\Omega_A^\cdot$ is used only for $\underline{A}$ smooth to compute the interesting homology

Noncommutative Case

Noncommutative differential forms
A associative unital algebra over $\mathbb{C}$

$$\Omega^n A = A \otimes \underbrace{\bar{A} \otimes \dots \otimes \bar{A}}_{n}$$

where

$$\bar{A} = A/\mathbb{C}$$

$(a_0, \dots, a_n)$ = image of $a_0 \otimes \dots \otimes a_n$ in $\Omega^n A$.

Put $\Omega A = \bigoplus_{n \geq 0} \Omega^n A$

Define $d(a_0, \dots, a_n) = (1, a_0, \dots, a_n)$
Integration by parts suggests
$(a_0, \dots, a_n)(a_{n+1}, \dots, a_k) = \sum_1^n (-1)^{n-i} (a_0, \dots, a_i a_{i+1}, \dots, a_k)$

Proposition: i) This defines a unique DG algebra structure on ΩA satisfying

$$a_0 da_1 \dots da_n = (a_0, a_1, \dots, a_n)$$

ii) Given a DG algebra $\Gamma = \oplus \Gamma^n$ and a homomorphism $u: A \to \Gamma^0$ there is a unique DG algebra homomorphism $u_*: \Omega A \to \Gamma$ extending the $\underline{u}$

$$a_0 da_1 \dots da_n \, a_{n+1} da_{n+2} \dots da_k$$

$$da_n \, a_{n+1} = d(a_n a_{n+1}) - a_n da_{n+1} \quad \text{etc.}$$

Define operators b, B on ΩA as follows:

$$b(\omega\, da) = (-1)^{|\omega|}(\omega a - a\omega) \qquad |\omega| = \text{degree}$$

i.e.

$$b(a_0 da_1 \dots da_{n+1}) = (-1)^n (a_0 da_1 \dots da_n a_{n+1} - a_{n+1} a_0 da_1 \dots da_n)$$

$$= a_0 a_1 da_2 \dots da_{n+1}$$
$$+ \sum_i (-1)^i a_0 da_1 \dots d(a_i a_{i+1}) \dots da_{n+1}$$
$$+ (-1)^{n+1} a_{n+1} a_0 da_1 \dots da_n$$

$= \sum_{i} (-1)^i (a_0, \dots, a_i a_{i+1}, \dots, a_{n+1})$

$\quad + (-1)^{n+1} (a_{n+1} a_0, a_1, \dots, a_n)$

<u>b</u> is the most important operator.

$$B(a_0\, da_1 \dots da_n) = \sum_{i=0}^{n} (-1)^{in}\, da_i \dots da_n\, da_0 \dots da_{i-1}$$

<u>B</u>, <u>b</u> anti-commute $\quad \begin{cases} b^2 = B^2 = 0 \\ bB + Bb = 0 \\ (b+B)^2 = 0 \end{cases}$

$\hat{\Omega}^{ev} A = \prod_{2n} \Omega^{2n} A$

Definition: Periodic cyclic homology of <u>A</u> is

$$HP_i(A) = H_i\left(\hat{\Omega}^{ev} A \overset{b+B}{\underset{}{\rightleftarrows}} \hat{\Omega}^{od} A \right) \quad i \in \mathbb{Z}/2$$

This is the interesting homology.
Need a noncommutative analogue of smoothness.

Definition: <u>A</u> is quasi free if it satisfies the equivalent properties

1) Any homomorphism $A \to R/I$ where I is nilpotent lifts to a hom $A \to R$.
2) $H^n(A, M) = 0$ for all bimodules M over A and $n \geq 2$.
3) $\Omega^1 A$ is a projective bimodule over A.

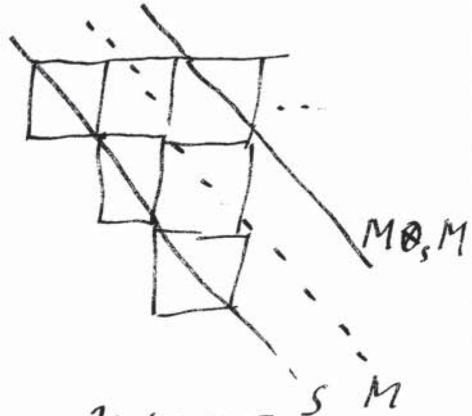
Examples (1) Free algebra $T(V) = \mathbb{C} \oplus V \oplus V^{\otimes 2} \oplus \dots$
analogue of polynomial ring.
(2) Separable algebra $M_n \mathbb{C}$ (non commutative analogue of etale)

<u>A</u> quasi free, <u>S</u> separable $\Rightarrow A \otimes S$ quasifree
eg $M_n A = M_n \otimes A$

If <u>A</u> is quasi free and <u>M</u> is a projective <u>A</u> bimodule

then $T_A M = A \oplus M \oplus (M \otimes_A M) \oplus \dots$
is also quasifree.

Triangular matrix algebra
- diagonal algebra is separable.



Definition: $X(A)$: $\quad A \overset{b}{\underset{B = d}{\rightleftarrows}} \Omega^1 A / [\Omega^1 A, A]$

This a quotient of $\hat{\Omega}^{ev} A \overset{b+B}{\underset{b+B}{\rightleftarrows}} \hat{\Omega}^{odd} B$

Proposition: (Analogue of the de Rham Theorem)
If <u>A</u> is a quasi free then
$H_i(X(A)) = HP_i(A)$

Main Result If $\underline{A} = R/I$ where $\underline{R}$ is quasifree, then
$$HP_i A = H_i(\varprojlim_n X(R/I^n)).$$

Remark $\Omega^0_A \to \Omega^1_A \to \Omega^2_A \to$

$F^p \Omega_A \qquad 0 \to \Omega^p_A \to \Omega^{p+1} A \to$

Let $F^n \Omega A \qquad 0 \oplus b\Omega^{n+1} A \oplus \Omega^{n+1} A \oplus \Omega^{n+2} A$

$$HC_n(A) = H_{n+2\mathbb{Z}}(\Omega A / F^n \Omega A,\ b+B)$$

Noncommutative differential forms

$\underline{A}$ associative unital algebra over $\mathbb{C}$

$\bar{A} = A/\mathbb{C}$

Let $\Omega^n A = A \otimes \bar{A}^{\otimes n}$ $(n \geq 0)$ $\Omega^n A = 0$ $(n < 0)$

write $(a_0, \ldots, a_n)$ for the image of $a_0 \otimes \ldots \otimes a_n$ in $\Omega^n A$.

Denote $\Omega^0 A = A$ $\qquad (a) = a$

Define product and $\underline{d}$ on $\Omega A = \oplus_{\mathbb{Z}} \Omega^n A$ by

1) $d(a_0, \ldots, a_n) = (1, a_0, \ldots, a_n)$

2) $(a_0, \ldots, a_n)(a_{n+1}, \ldots, a_k) = \sum_{i=0}^{n} (-1)^{n-i} (a_0, \ldots, a_i a_{i+1}, \ldots, a_k)$

Proposition: 1) These formulas define a DG algebra on ΩA, which is unique such that
$$a_0 da_1 \ldots da_n = (a_0, \ldots, a_n)$$
2) (Universal property of Ω).

Given a differential graded algebra $\Gamma = \oplus_n \Gamma_n$ and an algebra homomorphism there is a unique homomorphism of DG algebras $\Omega A \to \Gamma$ extending $\underline{u}$. $\quad (u: A \to \Gamma)$

Proof: (sketch) The uniqueness part of (1)

In any DG algebra containing $\underline{A}$ is an even degree subalgebra we have
$$d(a_0 da_1 \ldots da_n) = da_0 da_1 \ldots da_n$$
$$(a_0 da_1 \ldots da_n)(a_{n+1} da_{n+2} \ldots da_k)$$
$$= \sum (-1)^n a_0 a_1 da_2 \ldots da_k + \sum_{i=1}^{n} (-1)^{n} a_0 da_1 \ldots d(a_i a_{i+1}) \ldots da_k$$

Existence: Need to check that the given multiplication is associative to ensure we do have a DG algebra. Can be done by calculation.

Alternatively we construct ΩA as a representation. Consider ΩA as a complex with differential $\underline{d}$ defined by (1). Let $\mathcal{E} = \oplus_{n \in \mathbb{Z}} \mathcal{E}^n$ be the algebra of linear operators $\mathcal{E}^n = \mathrm{Hom}$ of degree $\underline{n}$ on ΩA.

Then $\mathcal{E}$ is a DGA
$$dw = [d, w] = d \cdot w - (-1)^{|w|} w \cdot d.$$

Let $l : A \to \mathcal{E}^0$ by $l(a)(a_0, \dots, a_n) = (aa_0, \dots, a_n)$ be the homomorphism given by left multiplication.

Define

$$l_* : \Omega A \to \mathcal{E}$$
$$l_*(a_0, \dots, a_n) = la_0 [d, la_1] \dots [d, la_n].$$

Then

$$l_*(\Omega) = \text{the DG subalgebra of } \mathcal{E} \text{ generated by } lA$$

Let $\omega : \mathcal{E} \to \Omega A$ be $w \mapsto w(1)$

$$la_0 [d, la_1] \dots [d, la_n](1) = (a_0, \dots, a_n)$$

Have l_* is injective

$\therefore$ l_* is an isomorphism of ΩA with the DGA subalgebra of $\mathcal{E}$ generated by lA. This defines a DG algebra structure on ΩA satisfying (1).

Proof of 2) Uniqueness

$$u_*(a_0, \dots, a_n) = u_*(a_0 da_1 \dots da_n)$$
$$= u(a_0)\, du(a_1) \dots du(a_n)$$

Existence follows from the formulae for the products.

<u>Applications</u>: The universal extension of A and Cuntz alg.

Fedosov product. If $\Gamma = \bigoplus \Gamma^n$ is a DG algebra, let

~~tst. ry~~ $$x \circ y = xy - (-1)^{|x|} dx dy$$

extended from the homogeneous x, y by linearity.

This is an associative and makes Γ into $\Gamma = \Gamma^{ev} \oplus \Gamma^{odd}$ into a super algebra.

For the case of $\Gamma = \Omega A$ let QA be ΩA with the Fedosov product and let RA be the even subalgebra i.e. $\Omega^{ev} A$ with product $\circ$.

Claim that given any linear map $\rho : A \to R$ such that $\rho(1) = 1$ where $\underline{R}$ is an algebra. Then there is a unique homomorphism of algebras

$$\rho_* : RA \to R$$

such that $\rho_*(a) = \rho(a)$

Corollary: $$RA \cong TA / (1_A - 1_{TA})$$

$$(TA = \mathbb{C} \oplus A \oplus A^{\otimes 2} \oplus \dots \text{ tensor algebra})$$

RA = free algebra with generators the non identity elements in a basis for $\underline{A}$ containing 1_A

Proof: Put $\omega(a_1, a_2) = \rho(a_1 a_2) - \rho(a_1)\rho(a_2)$ 'curvature of ρ', and put

$$\rho_*(a_0\, da_1\, da_2 \dots da_{2n-1} da_{2n})$$

\#

$$= \rho(a_0)\, \omega(a_1, a_2) \dots \omega(a_{2n-1}, a_{2n})$$

This is well defined because if any $a_j = 1$ $(j \geq 1)$ both sides vanish:

$$\Omega^{2n} A = A \otimes \bar{A}^{\otimes 2n}$$
and $\rho(1) = 1$

$$\rho_* : \Omega^{ev} A = R$$

Uniqueness of ρ_* such that ρ_* is a hom $\rho_*(a) = \rho(a)$

$$\begin{tikzcd} a_1 \circ a_2 = a_1 a_2 - da_1 da_2 \arrow[d, "\rho_*"] \\ \rho(a_1)\rho(a_2) = \rho(a_1 a_2) + \rho_*(da_1 da_1) \end{tikzcd}$$

$\therefore \quad \rho_*(da_1 da_2) = \omega(a_1, a_2)$

Note that $x \circ y = xy$ if either of x, y is closed.

$\therefore \quad \rho_*(a_0 da_1 \dots da_n) = \rho_*(a_0 \circ (da_1 da_2) \circ \dots \circ (da_{2n-1} da_{2n}))$

$$= \rho(a_0)\,\omega(a_1, a_2) \dots \omega(a_{2n-1}, a_{2n}).$$

Note that RA is generated by the elements $a \in \Omega^0 A = A$

To show that ρ_* is a homomorphism when defined by (#). Consider

$$\{x \in RA : \rho_*(x \circ y) = \rho_*(x)\rho_*(y) \ \forall y\}$$

This is a subalgebra of RA. It suffices to show that $a \in A$ is in this subalgebra, since these elements generate.

$$\rho_*(a \circ (a_0 da_1 \dots da_{2n})) \overset{?}{=} \rho_*(a)\,\rho_*(a_0 \dots da_{2n})$$

$$\|$$

$$\rho_*(a a_0 da_1 \dots da_{2n}) - \rho_*(da\, da_0 \dots da_{2n})$$

$$= \rho(a a_0)\,\omega(a_1, a_2) \dots \omega(a_{2n-1}, a_{2n}) - \omega(a, a_0)\,\omega(a_1, a_2) \dots \omega(a_{2n-1}, a_{2n})$$

$$= \rho(a)\rho(a_0)\,\omega(a_1, a_2) \dots \omega(a_{2n-1}, a_{2n})$$

$$= \rho_*(a)\,\rho_*(a_0 da_1 \dots da_{2n})$$

Point of describing RA like this?

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] & R \arrow[r] & A \arrow[r] \arrow[l, bend right] & 0 \end{tikzcd}$$

Choose a linear lifting $\rho : A \to R$ $\quad \rho(1) = 1$.

$$\begin{tikzcd} 0 \arrow[r] & IA \arrow[r] & RA \arrow[r] \arrow[d, "\rho_*"] & A \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & I \arrow[r] & R \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

Claim $\quad (IA)^n = \bigoplus_{k \geq n} \Omega^{2k} A$

$$IA = \bigoplus_{k > 0} \Omega^{2k} A$$

$$gr^{IA} RA = \bigoplus (IA)^n / (IA)^{n+1} \cong \Omega^{ev} A$$

with the usual product.

Exact forms: $\quad d\Omega^n A = \bar{A}^{\otimes n+1}$

$\qquad da_0 \dots da_n \longleftrightarrow (a_0, \dots, a_n)$

Exact sequence of vector spaces

$$0 \to \mathbb{C} \to A \to \bar{A} \to 0$$

$$(n \geq 1) \qquad 0 \to \bar{A}^{\otimes n} \to A \otimes \bar{A}^{\otimes n} \to \bar{A}^{\otimes n+1} \to 0$$

$$0 \to d\Omega^{n-1}A \to \Omega^n A \xrightarrow{d} d\Omega^n A \to 0$$

$$H^n(\Omega A, d) = \begin{cases} 0 & n \neq 0 \\ \mathbb{C} & n = 0 \end{cases}$$

The operator $\underline{b}$ and Hochschild homology

M – A-bimodule

$$H_n(A, M) = \mathrm{Tor}_n^{A \otimes A^{op}}(M, A)$$

$$\mathrm{Tor}_n^{A \otimes A^{op}}(M, A) = H_n(M \otimes_A P \otimes_A)$$

circular tensor product

P – any projective A-bimodule resolution of A

$\otimes_A$ the right A-module structure on P is coupled to the left A-module structure on M.

Standard resolutions:
unnormalized

$$\xrightarrow{b'} A \otimes A \otimes A \xrightarrow{b'} A \otimes A \xrightarrow{b'} A$$

$$b'(a_0, a_1) = a_0 a_1$$
$$b'(a_0, a_1, a_2) = (a_0 a_1, a_2) - (a_0, a_1 a_2)$$

Normalized resolution:

$$A \otimes \bar{A}^{\otimes 2} \otimes A \xrightarrow{b'} A \otimes \bar{A} \otimes A \xrightarrow{b'} A \otimes A \xrightarrow{\overset{m}{\shortparallel}\, b'} A \to 0$$

where $\underline{m}$ is multiplication

We can write the unnormalised resolution in the form

$$\xrightarrow{b'} \Omega^2 A \otimes A \xrightarrow{b'} \Omega^1 A \otimes A \xrightarrow{b'} \Omega^0 A \otimes A \to$$

where
$$b'(\omega da \otimes a') = (-1)^{|\omega|}(\omega a \otimes a' - \omega \otimes a a')$$

$$b(a_0 da_1 \dots da_n \otimes a_{n+1}) \longleftrightarrow b'(a_0, \dots, a_n$$

Using this resolution P we have that

$$H_n(A, A) = H_n(A \otimes_A (\Omega A \otimes A) \otimes_A)$$
$$= H_n((\Omega A \otimes A) \otimes_A) = H_n(\Omega A, b)$$

where $\underline{b}$ is the differential of degree (-1) given by

$$b\,\omega da = (-1)^{|\omega|}(\omega a - a\omega).$$

Write $HH_n(A) = H_n(A, A)$ and call it the Hochschild homology of A. This often can be calculated

$$H_n(\Omega A, b) = HH_n(A)$$

$$A \underset{d}{\overset{b}{\rightleftarrows}} \Omega^1 A \underset{d}{\overset{b}{\rightleftarrows}} \Omega^2 A \underset{d}{\overset{b}{\rightleftarrows}}$$

Consider the Laplacian $bd + db$

$$(bd+db)(\omega da) = b(d\omega da) + d((-1)^{|\omega|}[\omega,a])$$

$$= (-1)^{|\omega|+1}[d\omega, a] + (-1)^{|\omega|}[d\omega,a] + (-1)^{|\omega|2}[\omega, da]$$

$$= [\omega, da] = \omega da - (-1)^{|\omega|} da\,\omega$$

Karoubi operator $\underline{K}$ is the degree 0 operator on ΩA given by

$$K(\omega da) = (-1)^{|\omega|} da\,\omega$$

for positive degree forms $\qquad K \equiv 1$ on $\Omega^0 A = A$

$$bd + db = 1 - K$$

$Kb = bK$; $Kd = dK$
follow from this formula

K - is homotopic to the identity in the $\underline{b}$-complex

$$K(a_0 da_1 \dots da_n) = (-1)^{n-1} da_n a_0 da_1 \dots da_{n-1}$$

$$= (-1)^{n-1} a_n da_0 da_1 \dots da_{n-1} + (-1)^{n-1} d(a_n a_0) da_1 \dots da_{n-1}$$

$$K(a_0, \dots, a_n) = (-1)^n (a_n, a_0, \dots, a_{n-1}) + (-1)^{n-1}(1, a_n a_0, \dots, a_{n-1})$$

$$= \lambda(a_0, \dots, a_n) + (-1)^{n-1}(1, a_n a_0, \dots, a_{n-1})$$

$\underline{\lambda}$ is only defined on unnormalised chains.

$\underline{K}$ is a substitute for $\underline{\lambda}$ on normalized chains.

$$0 \to d\Omega^{n-1}A \to \Omega^n A \to d\Omega^n A \to 0$$

$$0 \to \bar{A}^{\otimes n} \to \Omega^n A \to \bar{A}^{\otimes n+1} \to 0$$

$K = \lambda$ on $d\Omega A$

$K = \lambda_{n+1}$ on $d\Omega^n A$ $\qquad$ (order of λ)

Hence $(K^n - 1)(K^{n+1} - 1) = 0$ on $\Omega^n A$ $\qquad$ by linear algebra

Hence $\underline{K}$ has a spectral decomposition and is invertible since it satisfies a polynomial equation with constant term 1.

$$K^j(a_0 da_1 \dots da_n) = K^{j-1}(-1)^{n-1} da_n a_0 da_1 \dots da_{n-1}$$

$$\vdots$$

$$= (-1)^{(n-1)j} da_{n-j+1} \dots da_n a_0 da_1 \dots da_{n-j}$$

for $j \le n$.

$$K^n(a_0 da_1 \dots da_n) = da_1 \dots da_n a_0$$
$$= [da_1 \dots da_n, a_0] + a_0 da_1 \dots da_n$$
$$= a_0 da_1 \dots da_n + (-1)^n b(da_1 \dots da_n da_0)$$

$$K^n = 1 + bK^{-1}d \quad \text{on } \Omega^n A$$

$$K^{n+1} = K(1 + bK^{-1}d) = K + bd = 1 - db$$

This leads to
$$(K^{n+1}-1)(K^n-1) = 0$$
as well.

<u>Connes' Operator B</u> is defined by
$$B(a_0 da_1 \dots da_n)$$
$$= \sum_{j=1}^{n} (-1) da_j \dots da_n da_0 \dots da_{j-1}$$

so
$$B = \sum_{j=0}^{n} K^j d \quad \text{on } \Omega^n A$$
$$B^2 = 0 \quad \text{since } d^2 = 0$$

$$K^{n(n+1)} - 1 = \sum_0^n K^{nj}(K^n - 1)$$
$$= \sum_{j=0}^{n} bK^{nj-1} d$$

Since $K^n - 1 = bK^{-1}d$.

K is of order $(n+1)$ on Ω^n this gives.
$$K^{n(n+1)} - 1 = \sum_0^n bK^{nj-1} d = bB$$

Also
$$K^{n(n+1)} - 1 = \sum_{j=0}^{n-1} K^{(n+1)j}(K^{n+1} - 1)$$
$$= -\left(\sum_{j=0}^{n-1} K^{(n+1)j} d\right) b = -Bb$$

Since $K^{n+1} - 1 = -db$ and K has order n on $db\Omega^n \subseteq d\Omega^{n+1}$

$$bB + Bb = 0$$
$$K^{n(n+1)} = 1 - Bd$$

K is never of finite order, except in trivial cases

Let $\Omega^n = \Omega^n A = A \otimes \bar{A}^{\otimes n}$
$$\Omega^0 \underset{d}{\overset{b}{\leftrightarrows}} \Omega^1 \underset{d}{\overset{b}{\leftrightarrows}} \Omega^2 \underset{d}{\overset{b}{\leftrightarrows}}$$

Analogy with theory of harmonic forms

Laplacian $\quad bd + db = 1 - K$

$K \quad$ Karoubi operator

Recall that on a compact Riemannian manifold one has the Laplacian $\Delta = dd^* + d^*d$ which has a spectral decomposition

$$\Omega(M) = \operatorname{Ker}\Delta \oplus \bigoplus_{\lambda>0} \operatorname{Ker}(\Delta - \lambda)$$

$\operatorname{Ker}\Delta$ – space of harmonic forms on manifold M

$\bigoplus_{\lambda>0} \operatorname{Ker}(\Delta - \lambda) = \operatorname{Im}\Delta$

and one can introduce the Green's operator G which is the inverse of Δ on the image of Δ

Let P = spectral projection on $\operatorname{Ker}\Delta$

G = Green's operator $\begin{cases} \Delta^{-1} & \text{on } \operatorname{Im}\Delta \\ 0 & \text{on } \operatorname{Ker}\Delta \end{cases}$

$$\Omega(M) = \operatorname{Im}P \oplus (\operatorname{Im}P)^{\perp} \qquad P^{\perp} = I - P$$

$$[\Delta, d] = 0 \quad [\Delta, d^*] = 0$$

Spaces are invariant under $\underline{d}, \underline{d}^*$. (In this case $d = d^*$ on $\operatorname{Ker}\Delta$).

$$\operatorname{Im}(P^{\perp}) = \cancel{\operatorname{Im}}\, d\Omega(M) \oplus d^*\Omega M$$

$d^* : d\Omega(M) \xrightarrow{\simeq} d^*\Omega M$
with inverse Gd
$d : d^*\Omega M \xrightarrow{\simeq} d\Omega M$
with inverse Gd^*

Difference in noncommutative case is that d, d^* needn't be zero on $\operatorname{Ker}\Delta$. This uses positivity of Δ via

$$\langle \Delta x | x \rangle = \|dx\|^2 + \|d^*x\|^2$$

K on Ω^n satisfies $(K^n - 1)(K^{n+1} - 1) = 0$
By theory of Jordan normal form, Ω^n splits into generalized eigenspaces corresponding to the distinct roots of the polynomial

$$(x^n - 1)(x^{n+1} - 1)$$

which are the n^{th} roots of 1 and the $(n+1)^{st}$ roots of 1.
The roots are distinct apart from the double root 1. i.e. $\gamma^n = 1$, or $\gamma^{n+1} = 1$ $(\gamma \neq 1)$ are simple.

$$\Omega^n = \operatorname{Ker}(K-1)^2 \oplus \bigoplus_{\substack{\gamma^n = 1 \\ \text{or } \gamma^{n+1} = 1 \\ \gamma \neq 1}} \operatorname{Ker}(K - \gamma)$$

Can do this for each $\underline{n}$.
Hence

$$\Omega = \operatorname{Ker}(K-1)^2 \oplus \bigoplus \operatorname{Ker}(K - \gamma)$$

The eigenspaces are stable under any operator commuting with $\underline{K}$ i.e. $\underline{d}, \underline{b}$. (They are all subcomplexes)

Let $\underline{P}$ be the spectral projection of $\underline{K}$ for the eigenvalue $\underline{1}$ (= generalized null space of $1-K$) projection

$$\Omega = \cancel{P\Omega}\ \mathrm{Ker}\,(K-1)^2 \oplus \mathrm{Im}(K-1)^2$$
$$= P\Omega \oplus P^\perp\Omega$$

Let $\underline{G}$ be the Green's operator $G = \begin{cases} (K-1)^{-1} \text{ on } P^\perp\Omega \\ 0 \text{ on } P\Omega \end{cases}$

P, G commute with any operator commuting with $\underline{K}$.

$$PG = GP = 0$$
$$P^\perp = G(I-K) = (I-K)G$$

$$I - P = P^\perp = G(I-K) = G(bd+db) = b(Gd) + (Gd)b$$
$$= (Gb)d + d(Gb)$$

Hence $P^\perp$ is homotopic to zero with respect to either differential $\underline{d}$, $\underline{b}$.

Hence $H(P^\perp\Omega, b) = H(P^\perp\Omega, d) = 0$

Also
$$P^\perp\Omega = \mathrm{Im}_{P^\perp\Omega}(b) + \mathrm{Im}_{P^\perp\Omega}(d)$$
$$= bP^\perp\Omega + dP^\perp\Omega$$

where
$$d : bP^\perp\Omega \xrightarrow{\sim} dP^\perp\Omega \qquad \text{has inverse } Gb$$
$$b : dP^\perp\Omega \xrightarrow{\sim} bP^\perp\Omega \qquad \text{has inverse } Gd.$$

$$K^{n(n+1)} = 1 - Bd \qquad \text{where } (Bd)^2 = 0$$

$\underline{K}$ is not usually of finite order

Monodromy operators

"Quasi unipotent operators"

<u>Finite order operators</u>

Proposition: Let T be an operator on a vector space V satisfying $T^m = 1$ with some $m>1$. Then

$$P_T = \frac{1}{m}\sum_{i=0}^{m-1} T^i$$

$$G_T = \frac{1}{m}\sum_{i=0}^{m-1}\left(\frac{m-1}{2} - i\right)T^i$$

are the null space projection and Green's operator for $I-T$.

Proof: $P_T^\perp = G_T(I-T)$; $P_T G_T = 0$

$$I - P_T = \frac{1}{m}\sum_{i=0}^{m-1}(1-T^i) = \frac{1}{m}\sum_{i=0}^{m-1}\sum_{j=0}^{i-1} T^j\,(I-T)$$
$$= \frac{1}{m}\left(\sum_{j=0}^{m-1}(m-1-j)T^j\right)(I-T)$$

Note $0 = I - T^m = (I-T)\sum_0^m T^j$

so that we can introduce G_T as above with the desired properties.

$$0 \to \underset{\bar{A}^{\otimes n}}{d\Omega^{n-1}} \to \cancel{H}\,\Omega^n \to \underset{\bar{A}^{\otimes n+1}}{d\Omega^n} \to 0$$

Recall the identification $d\Omega^n \cong \bar{A}^{\otimes n+1}$

$$da_0 \ldots da_n \longleftrightarrow (a_0, a_1, \ldots, a_n)$$
$$K \longleftrightarrow \lambda$$

$P \longleftrightarrow P_\lambda \qquad G \longleftrightarrow G_\lambda$

Consequently we have $P d\Omega^n \cong P_\lambda \bar{A}^{\otimes n+1} = (\bar{A}^{\otimes n+1})^\lambda$

$$P^\perp d\Omega^n = P_\lambda^\perp \bar{A}^{\otimes n+1} = (1-\lambda)\bar{A}^{\otimes n+1}$$

and on Ω^n we have

$$Pd = \frac{1}{n+1}\sum_{j=0}^{n} K^j d$$

$$Gd = \frac{1}{n+1}\sum_{j=0}^{n+1}(n/2 - j)K^j d$$

This formula for Gd gives

$$1 = P + b(Gd) + (Gd)b$$

Terminology - call $P\Omega$ the space of harmonic forms.

Proposition ω is harmonic if and only if $d\omega$ and $db\omega$ are fixed under $\underline{K}$

Proof: $(\Rightarrow)$ $\underline{\omega}$ harmonic means that $\underline{\omega}$ belongs to the $\neq 1$ * eigenspace for K. * generalized Hence $d\omega$, $db\omega$ belong to the $+1$ generalized eigenspace for K. But this means that $d\omega$, $db\omega$ are $\underline{K}$-invariant because $\underline{K}$ is of finite order on $d\Omega^n$.

$(\Leftarrow)$ Use the identity

$$\omega = P\omega + bGd\omega + Gdb\omega$$

$$Pd = \frac{1}{n+1}\sum_{j=0}^{n}K^j d = \frac{1}{n+1}B$$

Use the definition of $\underline{B}$ to get $B = NPd$ where

$$N\omega = |\omega|\omega$$

Calculate the homology $H(P^\perp\Omega) = 0$ with respect to $\underline{b}, \underline{d}$. Hence

$$H(P\Omega, b) = H(\Omega, b) = HH(A)$$

$$H(P\Omega, d) = H(\Omega, d) = \mathbb{C}[0] \quad \text{(degree 0)}$$

$B = 0$ on $P^\perp\Omega$ since $B = NPd$

$$H(P^\perp\Omega, B) = P^\perp\Omega$$

$H(P\Omega, B) = H(P\Omega, d)$ because $B = d$ up to non-zero scalar factors so

$$H(P\Omega, B) = \mathbb{C}[0]$$

i.e. $\underline{B}$ is almost exact on $P\Omega$

Recall $\quad \Omega = \operatorname{Ker}(1-K)^2 \oplus \operatorname{Im}(I-K)^2$

the harmonic decomposition $\quad \Omega = P\Omega + P^\perp\Omega$

$H(P^\perp\Omega, b, d) = 0$; $B = 0$ on $P^\perp\Omega$

$$H(P\Omega, b) = H(\Omega, b) = HH(A)$$

$$H(P\Omega, B) = H(P\Omega, d) = H(\Omega, d) = \mathbb{C}[0]$$

$\underline{B}$ is nearly exact on $P\Omega$.

Introduce the reduced space of differential forms $\bar\Omega = \bar\Omega A$ given by

$$0 \to \Omega\mathbb{C} \to \Omega A \to \bar\Omega A \to 0$$

$$\Omega\mathbb{C} = \mathbb{C}$$

So that $\bar\Omega A = \Omega A$ except in degree 0 where $\bar\Omega^0 = \bar A$.

This makes $H(\bar\Omega, d) = 0$

All the operators b, B, K, P, G descend from Ω to $\bar\Omega$. Hence we have a harmonic decomposition

$$\bar\Omega = P\bar\Omega \oplus P^\perp\bar\Omega$$

where $P\bar\Omega = P\Omega/\mathbb{C}$; $P^\perp\bar\Omega = P^\perp\Omega$

Now we have $\underline{B}$ exact on $P\bar\Omega$, $B = 0$ on $P^\perp\bar\Omega$.

Corollary: (Connes' Lemma) In the case of $\bar\Omega$ the inclusion $\operatorname{Im} B \to \operatorname{Ker} B$ is a

quasi-isomorphism with respect to $\underline{b}$.

(A quasi-isomorphism is a map inducing an isomorphism on homology).

Proof: Recall $\bar\Omega = P\bar\Omega \oplus P^\perp\bar\Omega$

$$\operatorname{Ker} B = \operatorname{Im} B \oplus P^\perp\bar\Omega$$

since $\underline{B}$ is exact on $P\bar\Omega$.

Hence

$$\operatorname{Ker} B / \operatorname{Im} B = P^\perp\bar\Omega$$

which is acyclic with respect to $\underline{b}$ since

$$b(Gd) + (Gd)b = 1$$

is a homotopy to zero.

Typical application (Connes)

Definition: $\bar{H}H_n(A) = H_n(\bar\Omega, b)$ reduced Hochschild homology.

$$\bar{H}C_n(A) = H_n(\bar\Omega A/\operatorname{Ker} B, b)$$ reduced cyclic homology

Application (Connes exact sequence)

$$\to \bar{H}C_1(A) \to \bar{H}H_2(A) \to \bar{H}C_2(A) \xrightarrow{S} \bar{H}C_0(A) \xrightarrow{B} \bar{H}H_1(A) \to \bar{H}C_1(A) \to 0$$

$$0 \to \operatorname{Ker} B \to \bar\Omega \to \bar\Omega/\operatorname{Ker}B \to 0$$

$$\begin{array}{c} \text{qis} \\ \operatorname{Im} B \hookleftarrow \end{array} \quad B \quad \text{shifts degree by } 1$$

The Connes exact sequence is the long exact sequence associated to this short exact sequence.

'qis' - quasi-isomorphism.

This uses the fact that

$$B : \bar\Omega/\operatorname{Ker} B \to \operatorname{Im} B \hookrightarrow \operatorname{Ker} B$$

is a quasi-isomorphism of degree one.

Want to relate reduced cyclic homology and homology

Augmented algebras and nonunital algebra

An augmented algebra $\underline{A}$ is a unital algebra equipped with a homomorphism $\underline{A} \xrightarrow{\varepsilon} \mathbb{C}$

$$A = \mathbb{C} \oplus \operatorname{Ker}\varepsilon$$

$\operatorname{Ker}\varepsilon$ is the augmentation ideal. It is a non unital algebra.

If $\underline{\mathcal{A}}$ is non unital, we can adjoin an identity to get $\tilde{\mathcal{A}} = \mathcal{A} \oplus \mathbb{C}$ so that $\mathcal{A}$ is augmented.

Get equivalence of categories.

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Suppose that $\underline{A}$ is augmented. $A = \tilde{\mathcal{A}}$ an augmented. Identify $\bar A = \mathcal{A} = A/\mathbb{C}$.

$$\Omega^n A = A \otimes \bar A^{\otimes n} \cong (\mathbb{C} \oplus \mathcal{A}) \otimes \mathcal{A}^{\otimes n}$$

More precisely we have the following isomorphism

$$\Omega^n A = \begin{array}{c} \mathcal{A}^{\otimes n+1} \\ \oplus \\ \mathcal{A}^{\otimes n} \end{array}$$

$$\mathcal{A}^{\otimes n+1} \oplus \mathcal{A}^{\otimes n} \quad \begin{pmatrix} (a_0, \ldots a_n) \\ (a_1' \ldots a_n') \end{pmatrix} \mapsto \begin{array}{c} a_0 da_1 \ldots da_n \\ + \\ da_1' \ldots da_n' \end{array}$$

where $a_i, a_i' \in \mathcal{A}$

Want to see what K, G, P etc look like on $\mathcal{A}$.

Operators on $\bigoplus_n \mathcal{A}^{\otimes n}$. We define:

$$b(a_0, \ldots, a_n) = \sum_{i=0}^n (-1)^i (\ldots, a_i a_{i+1}, \ldots) + (a_n a_0, \ldots a_{n-1})$$

$$\lambda(a_0, \ldots, a_n) = (-1)^n (a_n, a_0, \ldots, a_{n-1})$$

$$b'(a_0, \ldots, a_n) = \sum_{i=0}^n (-1)^i (\ldots, a_i a_{i+1}, \ldots)$$

$$N_\lambda(a_0, \ldots, a_n) = \sum_{j=0}^n \lambda^j (a_0, \ldots, a_n)$$

b - not quite the same as before

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Claim that the operators b, d on ΩA are given by the following matrices on $\bigoplus \mathbb{C}^{\otimes n}$.

$$d = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \text{obvious}$$

$$b = \begin{pmatrix} b & 1-\lambda \\ 0 & -b' \end{pmatrix} \qquad \text{check}$$

$$db = \begin{pmatrix} 0 & 0 \\ b & 1-\lambda \end{pmatrix} \qquad bd = \begin{pmatrix} 1-\lambda & 0 \\ -b' & 0 \end{pmatrix}$$

$$1-K = db + bd = \begin{pmatrix} 1-\lambda & 0 \\ b-b' & 1-\lambda \end{pmatrix}$$

$$K = \begin{pmatrix} \lambda & 0 \\ b'-b & \lambda \end{pmatrix}$$

where $b'-b$ is the cross-over term.

$$K^j d = \begin{pmatrix} 0 & 0 \\ \lambda^j & 0 \end{pmatrix} \qquad Pd = \begin{pmatrix} 0 & 0 \\ P_\lambda & 0 \end{pmatrix}$$

$$Gd = \begin{pmatrix} 0 & 0 \\ G_\lambda & 0 \end{pmatrix}$$

ΩA/

$$\begin{array}{ccccc}
\mathbb{C}^{\otimes 3} & & & & \\
\mathbb{C}^{\otimes 2} & \xleftarrow{1-\lambda} & \mathbb{C}^{\otimes 2} & & \\
\downarrow & & \downarrow -b' & & \\
\mathbb{C} & \longleftarrow & \mathbb{C} & \cdots & \cdots \\
 & & \downarrow & & \\
 & & \mathbb{C} & \Omega^1 A & \Omega^2 A
\end{array}$$

$$I = P + b(Gd) + (Gd)b$$

$$0 = b^2 = \begin{pmatrix} b & 1-\lambda \\ 0 & -b' \end{pmatrix}^2 \quad \text{so} \quad \begin{array}{l} b^2 = 0 \\ b'^2 = 0 \\ b(1-\lambda) = (1-\lambda) b' \end{array}$$

on $\mathbb{C}^{\otimes n}$

$$B = NPd = \begin{pmatrix} 0 & 0 \\ N_\lambda & 0 \end{pmatrix}$$

$$bB + Bb = 0 \Rightarrow b' N_\lambda = N_\lambda b$$

Connes identities.

<u>Augmented algebras</u>

$$A = \mathbb{C} \oplus \mathbb{C}$$

$$A/\mathbb{C} = \bar{A} = \mathbb{C}$$

ΩA is an augmented DG algebra $\mathbb{C} \leftrightarrows A$ gives

$$\mathbb{C} = \Omega\mathbb{C} \leftrightarrows \Omega A$$

Identify $\bar\Omega A = \Omega A/\mathbb{C}$ with $\mathrm{Ker}(\Omega A \to \Omega\mathbb{C})$

Then $\Omega A = \mathbb{C} \oplus \bar\Omega A$

It is easy to see that $\bar\Omega A$ defined thus is the universal nonunital DG algebra generated by the nonunital algebra $\mathbb{C}$.

(This is Connes' starting point — adjoining a unit to unital algebra).

$$(*) \quad \begin{matrix} \psi & \mathbb{C}^{\otimes n+1} \\ \oplus & \oplus \\ \phi & \mathbb{C}^{\otimes n} \end{matrix} \xrightarrow{\sim} \bar\Omega^n A$$

$(*)$ holds for all $n \in \mathbb{Z}$ so long as we set $\mathbb{C}^{\otimes 0} = 0$ (note that we are in nonunital category).

On $\bar\Omega A$ we have d, b, κ, P, B, G

On $\mathbb{C}^{\otimes n}$ we have operators $b, \lambda, b', P_\lambda, G_\lambda, N_\lambda$ (different $\underline{b}$).

$$b_{\bar\Omega A}\psi = \psi b_a$$

$$b_{\bar\Omega A}\phi = -b'\phi b'_A + \psi(1-\lambda)$$

$$b_{\bar\Omega A} = \begin{pmatrix} b_a & 1-\lambda \\ 0 & -b' \end{pmatrix}$$

$$b_{\bar\Omega A}(a_0 da_1 \ldots da_{n+1}) = \sum_1^n (-1)^{n+i} a_0 da_1 \ldots d(a_i a_{i+1}) \ldots da_{n+1}$$

$$+ (-1)^n a_0 a_1 da_2 \ldots da_{n+1}$$

$$+ (-1)^n a_{n+1} a_0 da_1 \ldots da_n$$

$$b(a_0, \ldots a_n) = \sum_{i=1}^n (-1)^{n-i} (a_0, \ldots a_i a_{i+1}, \ldots a_{n+1}) + (a_0 a_1, \ldots, a_{n+1}) + (-1)^{n+1}(a_{n+1} a_0, \ldots a$$

$\mathbb{C}$ is a subalgebra of $\underline{A}$ so the formula follows by restriction.

$$b^2_{\bar\Omega A} = 0 \implies \begin{pmatrix} b_a & 1-\lambda \\ 0 & -b' \end{pmatrix}^2 = 0$$

$$\text{so} \quad b_a^2 = 0, \quad b'^2 = 0, \quad (1-\lambda)b' = b(1-$$

$$d = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \kappa = 1 - (db + bd)$$

$$\kappa = \begin{pmatrix} \lambda & 0 \\ b' - b & \lambda \end{pmatrix}$$

$$K^j d = \begin{pmatrix} 0 & 0 \\ \lambda^j & 0 \end{pmatrix}$$

$$B = \cancel{\Omega\lambda} \, (n+1) P d = \sum_{j=0}^{n} K^j d = \begin{pmatrix} 0 & 0 \\ N_\lambda & 0 \end{pmatrix}$$

$$Gd = \frac{1}{n+1} \sum_0^n \left(\frac{n}{2} - j \right) K^j d = \begin{pmatrix} 0 & 0 \\ G_\lambda & 0 \end{pmatrix}$$

$$b_{\bar\Omega A} B + B_{\bar\Omega A} b_{\bar\Omega A} = 0 \qquad \text{so that}$$

$$b'_a N_\lambda = N_\lambda b_a$$

<u>Connes</u> <u>Tsygan</u> <u>double</u> <u>complex</u>

The above identities establish that the following is a periodic double complex.

$$\begin{tikzcd}
 & \mathcal{A}^{\otimes 3} \arrow[d, "b"] & \mathcal{A}^{\otimes 3} \arrow[l, "1-\lambda"'] \arrow[d, "-b'"] & \mathcal{A} \arrow[l, "N_\lambda"'] \arrow[d, "b"] & \arrow[l, "1-\lambda"'] \\
\mathcal{A}_\lambda^{\otimes 2} & \mathcal{A}^{\otimes 2} \arrow[l, dotted] \arrow[d, "b"] & \mathcal{A}^{\otimes 2} \arrow[l, "1-\lambda"'] \arrow[d, "-b'"] & \mathcal{A}^{\otimes 2} \arrow[l, "N_\lambda"'] \arrow[d, "b"] & \arrow[l, "1-\lambda"'] \\
\mathcal{A}_\lambda & \mathcal{A} \arrow[l, dotted] & \mathcal{A} \arrow[l, "1-\lambda"'] & \mathcal{A} \arrow[l, "N_\lambda"'] & \arrow[l, "1-\lambda"']
\end{tikzcd}$$

$$\underbrace{\qquad\qquad\qquad}_{\bar\Omega A} \xleftarrow{\ B\ } \underbrace{\qquad\qquad\qquad}_{\bar\Omega A}$$

$$1 - P = P^\perp = b(Gd) + (Gd)b$$

$$I = P + b(Gd) + (Gd)b$$

orthogonal idempotents

$$b(Gd) = \begin{pmatrix} b & 1-\lambda \\ 0 & -b' \end{pmatrix} \begin{pmatrix} 0 & 0 \\ G_\lambda & 0 \end{pmatrix} = \begin{pmatrix} P_\lambda^\perp & 0 \\ -b' G_\lambda & 0 \end{pmatrix}$$

$$(Gd)b = \begin{pmatrix} 0 & 0 \\ G_\lambda & 0 \end{pmatrix} \begin{pmatrix} b & 1-\lambda \\ 0 & -b' \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ G_\lambda b & P_\lambda^\perp \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ G_\lambda b & P_\lambda^\perp \end{pmatrix} \begin{pmatrix} P_\lambda^\perp & 0 \\ -b' G_\lambda & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ G_\lambda b P_\lambda^\perp - P_\lambda^\perp b' G_\lambda & 0 \end{pmatrix}$$

$$G_\lambda b P_\lambda^\perp = G_\lambda b (1-\lambda) G_\lambda$$

$$P_\lambda^\perp b' G_\lambda = G_\lambda (1-\lambda) b' G_\lambda$$

$b(Gd)$, $(Gd)b$ are annihilating idempotents

so

$$P = 1 - b(Gd) - (Gd)b$$

$$P = \begin{pmatrix} P_\lambda & 0 \\ b' G_\lambda + G_\lambda b & P_\lambda \end{pmatrix}$$

is an idempotent

This corresponds to a diagram chase in the bicomplex.

Recall the reduced cyclic complex of $\underline{A}$ is

$$\bar{C}^\lambda(A) = \bar{\Omega} A / \ker B \xrightarrow[\text{degree } 1]{\sim} B\bar{\Omega}A \subset \Omega A$$

Connes' lemma gives that the rows are exact.

$\bar{C}^\lambda(A)$ can be identified with the quotient complex

$$(\mathcal{A}^{\otimes(\cdot+1)}, b)/(1-\lambda)(\mathcal{A}^{\otimes(\cdot+1)}, b')$$
$$\cong ((\mathcal{A}^{\otimes(\cdot+1)})_\lambda, b)$$

or with the image of $N_\lambda(\mathcal{A}^{\otimes(\cdot+1)}, b)$

$$\cong ((\mathcal{A}^{\otimes\cdot+1})^\lambda, b')$$

ie $\bar{C}^\lambda(A)$ = cyclic complex $C^\lambda(\mathcal{A})$ of $\mathcal{A}$

Facts (Standard facts from cyclic theory based on the cyclic bicomplex)

1) There is an obvious augmentation from the total complex of the (first quadrant) cyclic bicomplex to $C^\lambda(\mathcal{A})$ which is a quasi isomorphism (because $\operatorname{Im} N_\lambda = \operatorname{Ker}(1-\lambda)$, $\operatorname{Im}(1-\lambda) = \operatorname{Ker} N_\lambda$)

Definition $HC_n(\mathcal{A}) = H_n(C^\lambda(\mathcal{A}))$

$= H_n$ (cyclic bicomplex of $\mathcal{A}$)

Hochschild homology $HH(\mathcal{A})$ is the homology of $(\bar{\Omega}A, b)$ or equivalently the homology of the total complex of the first two columns of the cyclic bicomplex.

<u>Connes Exact Sequence</u>

$$\to HC_{n+2} \xrightarrow{S} HC_n \xrightarrow{B} HH_{n+1} \xrightarrow{I} HC_{n+1} \xrightarrow{S} HC_{n-1}$$

This results from the following

$CC(\mathcal{A})$ total complex of the cyclic bicomplex

$$0 \to (\bar{\Omega}A, b) \to CC(\mathcal{A}) \to \Sigma^2 CC(\mathcal{A}) \to 0$$

Now take the long exact sequence in homology.

$$0 \to \Sigma \bar{C}^\lambda(A) \to \underset{\underset{\textstyle \bar{\Omega}A}{\cup\!\!|}}{P\bar{\Omega}A} \to \bar{E}^\lambda(A) \to 0$$

Mixed complex — graded vector space $M = \oplus_n M_n$ together with operators b of degree -1, B of degree 1 satisfying $b^2 = B^2 = bB + Bb = 0$

Example $M = \Omega A$ with b, B as before
$= \bar{\Omega} A$

$\Omega A = P\Omega A \oplus P^{\perp}\Omega A$ since the decomposition is stable.

Suppose M is a chain complex $M_n = 0$ $(n<0)$. We regard b as the primary differential, B as extra structure on the complex (M,b).
Introduce the differential graded algebra $\mathbb{C}[B]$

$$\mathbb{C}[B] = \mathbb{C}^0 \oplus \mathbb{C}^1 B$$

A mixed complex is the same as a DG module over $\mathbb{C}[B]$.

We now introduce various types of homology groups associated to a mixed complex which are invariant with respect to maps of mixed complexes which are quasi-isomorphisms with respect to b.

$H_n^b M$ — ordinary (M,b) homology.

Cyclic homology $H_n^c M = H_n(\mathbb{B}, M)$

where $\mathbb{B}M$ = total complex of the bicomplex

$$\begin{tikzcd}
 & \arrow[d, "b"] & \arrow[d, "b"] & \arrow[d, "b"] \\
 & M_2 \arrow[d, "b"] & M_1 \arrow[l, "B"'] \arrow[d, "b"] & M_0 \arrow[l, "B"'] \\
M_2 & M_1 \arrow[d, "b"] & M_0 \arrow[l, "B"'] \arrow[dl, "S"] & \\
M_1 & M_0 & &
\end{tikzcd}$$

Spectral $E^1 = H^b \Rightarrow H(\mathbb{B}M)$

$$0 \to (M,b) \to \mathbb{B}M \xrightarrow{S} \Sigma^2 \mathbb{B}M \to 0$$

This leads to a Connes exact sequence. (M,b) is a subcomplex due to the periodicity.

Extend the bicomplex in the negative direction to get a periodic bicomplex. There are two choices for the total complex.

$$(\mathbb{B}^{per} M)_n = \oplus_{p\in\mathbb{Z}} M_{n-2p}$$

$$(\mathbb{B}M)_n = M_n \oplus M_{n-2} \oplus \cdots = \bigoplus_{p\geq 0} M_{n-2p}$$

$$(\hat{\mathbb{B}}^{per} M)_n = \prod_{p\in\mathbb{Z}} M_{n-2p} \quad \text{(completed)}$$

$$H_n(\mathbb{B}^{per} M) = H_{n+2\mathbb{Z}}\Big(\oplus M^{ev} \underset{b+B}{\rightleftarrows} M^{odd}\Big)$$

$$H_n(\hat{\mathbb{B}}^{per} M) = H_{n+2\mathbb{Z}}(\hat{M}^{ev} \rightleftarrows \hat{M}^{odd})$$

$$M^{ev} = \bigoplus_{\mathbb{Z}} M_{2n} \quad \text{etc.}$$

The completed version is invariant with respect to $\mathcal{B}$ - quasi-isomorphism. The uncompleted one is not.

The periodic cyclic homology

$$H_i^p M = H_i(\hat{M}, b+B)$$

$$H^p_{n+2\mathbb{Z}} M = H_{n+2\mathbb{Z}}(\hat{\mathbb{B}}^{per} M)$$

The first quadrant is a quotient complex of a larger complex so that

$$\hat{\mathbb{B}}^{per} M = \varprojlim_n \Sigma^{-2n} \mathbb{B}(M)$$

Surjective inverse system of complexes. This gives a Milnor exact sequence

$$\cancel{H_i^p M \to \varprojlim_{n\in i} H_n}$$

$$0 \to \varprojlim_{n\in i+1}{}^1 H_n^c M \to H_i^p M \to \varprojlim_{n\in i} H_n^c M \to 0$$

<u>Negative cyclic homology.</u> $H_n^{c-} M = H_n(\prod_{p\le 0} M_{n-2p}, b+B)$

(overlap on the first column)

Call the family of homology groups $H^b M$, $H^c M$, $H^p M$ $H^{c-} M$ the cyclic theory associated to the mixed complex $\underline{M}$.

Alternative construction of the cyclic theory of M using $\mathbb{Z}(\mathbb{Z})$ graded complexes.

Consider the $\mathbb{Z}(\mathbb{Z})$ graded complex $(M, b+B)$

Introduce the Hodge filtration of M

$$F^n M = bM_{n+1} \oplus M_{n+1} \oplus$$

which is stable under $b+B$

$$M = F^{-1} M \supset F^0 M \supset \dots$$

$$F^n(M)/F^{n+1} M = \cancel{bM_{n+1} \oplus M_{n+1}}$$

$$\begin{array}{ccc|cc} bM_{n+1} & \leftarrow & M_{n+1} & 0 & bM_{n+2} \\ \oplus & & \oplus & \oplus & \oplus \\ M_{n+2} & & M_{n+3} & M_{n+2} & M_{n+3} \\ \vdots & & \vdots & & \end{array}$$

$$bM_{n+1} \underset{B}{\overset{b}{\rightleftarrows}} M_{n+1}/bM_{n+2}$$

$$H_i(F^nM/F^{n+1}) = \begin{cases} 0 & i = n+\mathbb{Z} \\ H^b_{n+1}M & i = n+1+2\mathbb{Z} \end{cases}$$

This gives the Connes exact sequence

<u>Definition</u> $H^{\tilde{c}}_n M = H_{n+2\mathbb{Z}}(M/F^nM)$

$$H^d_{n-1}M = H_{n+1+2\mathbb{Z}}(M/F^nM)$$

Exercise: Show that $H^{\tilde{c}}_n M$ is the cyclic homology

$$H^{\tilde{c}}_n M \cong H^c_n M$$

$$M/F^nM = M_0 \oplus \dots \oplus M_n/bM_{n+1}$$

How to derive the Connes exact sequence

$$0 \longrightarrow F^nM/F^{n+1}M \longrightarrow M/F^{n+1}M \longrightarrow M/F^nM \rightarrow$$

$$0 = H_{n+2\mathbb{Z}}(F^n/F^{n+1}) \rightarrow H_{n+2\mathbb{Z}}(M/F^{n+1}) \rightarrow H_{n+2\mathbb{Z}}(M/F^n)$$

$$H_{n+1+2\mathbb{Z}}(M/F^n) \leftarrow H_{n+1+2\mathbb{Z}}(M/F^{n+1}) \leftarrow H_{n+1+2\mathbb{Z}}(F^n/F^{n+1})$$

$$H^d_{n-1} \qquad H^c_{n+1} \qquad H^b_{n+1}$$

$$0 \rightarrow H^d_n \rightarrow H^c_n \rightarrow H^b_{n+1} \rightarrow H^c_{n+1} \rightarrow H^d_{n+1} \rightarrow 0$$

$$0 \rightarrow H^d_{n-1} \rightarrow H^c_{n-1} \rightarrow H^b_n \rightarrow H^c_n \rightarrow H^d_{n-2} \rightarrow 0$$

$$H^d_n M = \operatorname{Im} S : H^c_{n+2} \rightarrow H^c_n$$

From $M/F^nM = M_0 \oplus M_1 \oplus \dots \oplus M_{n-1} \oplus M_n/bM_{n+1}$

one has $\varprojlim_n M/F^nM = \prod_n M_n = \hat{M}$

Define $F^n\hat{M} = \varprojlim_k F^nM/F^{n+k}M$

Check that

$$H^{c-}_n M = H_{n+2\mathbb{Z}}(F^{n-1}\hat{M})$$

One gets a cyclic theory associated to any tower of $\mathbb{Z}(2)$ graded complexes

$$X_{n+1} \twoheadrightarrow X_n \twoheadrightarrow X_{n-1} \twoheadrightarrow X_{n-2}$$

such that $H_i(\operatorname{Ker}(X_{n+1} \rightarrow X_n)) = 0 \quad i \in n+2\mathbb{Z}$

$\Omega = (\Omega A, b, B)$ is a mixed complex

$P\Omega$ and ΩA are quasi-isomorphic w.r.t. b (and <u>d</u>)

Had $(\bar{\Omega}, b, B)$ is quasi-isomorphic to $(P\bar{\Omega}, b, B)$ with respect to <u>b</u>

Also B is exact on $P\bar{\Omega}$ (since on this subcomplex $\underline{B}$ is a multiple on $\underline{d}$).

Recall for a mixed complex $H_n^c(M) = H_n(\mathcal{B}M)$
$= H_{n+2r}(M/F^r M)$

$$H^d_{n}(M) = H_{n+2r}(M/F^{n+1}M)$$

$$= \mathrm{Im}\{S : H^c_{n+2} \to H^c_n\} = \mathrm{Ker}\{B : H^c_n \to H^b_{n+1}\}$$

Proposition: Assume $\underline{M}$ is a mixed complex so that B is exact on M. Then

$$H_n^c(M) = H_n(M/BM, b)$$
$$H_n^d(M) = H_n(M/bM, B)$$

Proof: Proof via the double complex $\mathcal{B}M$

$$\begin{array}{ccc} \downarrow b & & \\ M_1 & \xleftarrow{B} & M_0 \\ \downarrow b & & \\ M_0 & & \end{array} \qquad E^2 = H^v H^h \Rightarrow H(\mathcal{B}M)$$

This gives the first isomorphism from the spectral sequence.

Lemma: Let $X \to Y \to Z$ be an exact sequence of complexes with differentials denoted $\underline{b}$.

$$X/bX \longrightarrow Y/bY \longrightarrow Z/bZ$$

$$\frac{\mathrm{Ker}(Y/bY \to Z/bZ)}{\mathrm{Im}(X/bX_{n+1} \to Y/bY_{n+1})} = \mathrm{Ker}\{H_n(J) \to H_n(Z)\}$$

Put $J = \mathrm{Coker}(X \to Y) = \mathrm{Im}(Y \to Z)$.
Then

Assuming this, consider the exact sequence of complexes

$$\Sigma M \xrightarrow{B} M \xrightarrow{B} \Sigma^{-1} M$$

Here $\dfrac{\mathrm{Ker}\{\ \}}{\mathrm{Im}\{\ \}} = H_n(M/bM, B)$

$J = M/BM$ so $H_n(J) = H_n(M/BM, b) = H_n^c M$

$$\mathrm{Ker}(H_n(J) \to H_n(\Sigma^{-1}M))$$
$$= \mathrm{Ker}\{H_n^c M \xrightarrow{B} H_{n+1}^b M\}$$

Recall a mixed complex is a DG module over the DG algebra $\mathbb{C}[B] = \mathbb{C} \oplus \mathbb{C}B$ $\quad d(B) = 0$

A DG algebra has a underline bar construction which is a DG co-algebra. The bar construction of $\mathbb{C}[B]$ is the co-algebra $\mathbb{C}[u]$ where $\underline{u}$ has degree 2

$$\overset{0}{\mathbb{C}} \oplus \overset{2}{\mathbb{C}u} \oplus \overset{4}{\mathbb{C}u^2} \oplus \ldots$$

i.e.

$$\mathbb{C} \oplus 0 \oplus \mathbb{C}u \oplus 0 \oplus \mathbb{C}u^2$$

differential is zero
Coproduct $\Delta u = u \otimes 1 + 1 \otimes u$

There are functors

$$\{\text{DG modules over } \mathbb{C}[B]\} \rightleftarrows \{\text{DG comodules over } \mathbb{C}[u]\}$$

$$\mathcal{A} \qquad\qquad\qquad \mathcal{C}$$

These functors compose to give quasi-isomorphic (co)modules

$$M \qquad\qquad C \otimes_\tau M = \mathbb{B}(M)$$

$$M \qquad\qquad M + u\otimes M + u^2\otimes M$$

$$\downarrow$$

$$M_1 \qquad M_0$$

$$\downarrow$$

$$M_0$$

A graded comodule over $\mathbb{C}[u]$ is equivalent to a module over the dual algebra $(\mathbb{C}[u])^{d*}$ = polynomial algebra with generator S of degree -2.

S is adjoint to $u\otimes$

$$M \longmapsto C\otimes_\tau M = \mathbb{B}M$$

$\underline{\mathcal{A}}$ Category of mixed complexes

$\underline{\mathcal{C}}$ Category of chain complexes equipped with an endomorphism S of degree -2.

$$N \longmapsto A\otimes_\tau N \qquad \text{(opposite map).}$$

$$N \longmapsto \mathbb{C}[B]\otimes_\tau N = N \otimes (B\otimes N)$$

N comes with d, S. These give $d + BS$ to be of degree -1 on other side.

1) Basic properties are that these functors are inverse up to quasi-isomorphism.

2) The $\mathbb{C}[B]$ side is adapted to the H^b, the $\mathbb{C}[u]$ comodule side is adapted to the H^c side.

3) Because $\mathbb{C}[S]$ is a principal ideal domain any DG comodule over $\mathbb{C}[u]$ is quasi-isomorphic to the one with differential zero. (Analogous to the fact that any complex of abelian groups is quasi-isomorphic to the complex of its homology groups). (cf Künneth Theorem).

Indecomposables ~ $\mathbb{C}[S]$ modules

$$\oplus\, \mathbb{C}^{n} \oplus 0 \oplus \mathbb{C}^{n-2} \xrightarrow{S} \oplus\, 0 \oplus \xrightarrow{S} \mathbb{C}^{n-2h} \longrightarrow$$

$$\mathbb{C}[S]/S^{h-1}$$

This means that everything is determined by the cyclic homology as a $\mathbb{C}[S]$ comodule.

Hochschild picture

$$0 \to \mathbb{C}^{2h} \xrightarrow{S} \mathbb{C} \xrightarrow{S} \mathbb{C} \xrightarrow{S} \mathbb{C}^{n-2h} \to 0$$

$$0 \quad \xrightarrow{d} 0 \xrightarrow{d} \quad 0 \xrightarrow{d} \quad 0$$

The corresponding mixed complex $\mathbb{C}[B]\otimes N$

$$\mathbb{C}^{2n+1} \underset{b=0}{\overset{B\,\sim}{\leftrightarrows}} \mathbb{C}^{2n} \underset{b\sim}{\overset{B=0}{\leftrightarrows}} \mathbb{C}^{2n-1} \underset{b=0}{\overset{B\,\sim}{\leftrightarrows}} \mathbb{C}^{2n-2} \underset{b\sim}{\overset{B=0}{\leftrightarrows}} \mathbb{C} \underset{b=0}{\overset{B\,\sim}{\leftrightarrows}} \mathbb{C}^{n-2k}$$

$b = d + BS$

$$H^b_{2n+1} = \mathbb{C} \quad ; \quad H^b_{n-2k} = \mathbb{C}$$

<u>Two models</u> for a cyclic theory

1) Mixed complexes Chain complex which is a DG module over a $\mathbb{C}[B]$

$$M_0 \underset{B}{\overset{b}{\leftrightarrows}} M_1 \underset{B}{\overset{b}{\leftrightarrows}}$$

minimal mixed complex: i) free as a graded $\mathbb{C}[B]$-module (which is an Artin algebra - unique maximal ideal which is nilpotent). (equivalent to $\underline{B}$ being exact).

ii) M/BM has $b = 0$

Recall that: B exact $\Rightarrow H^c(M) = H(M/BM, b)$

Add (ii) and $H^c(M) = M/BM$

Considered the case

$$\mathbb{C}^{2n+1} \xleftarrow{B} \mathbb{C}^{2n} \underset{b}{\leftrightarrows} \mathbb{C} \qquad \xleftarrow{B} \mathbb{C} \xrightarrow[b]{} \mathbb{C}^{2n-2k} \leftarrow$$

2) Chain DG modules over $\mathbb{C}[S]$ $\deg S = -2$

$$\begin{tikzcd} & & N_2 \arrow[r, "S"] \arrow[dr] & N_0 \\ \arrow[urr, "S"] & N \arrow[ur] \arrow[rr, "S"] & & N_1 \arrow[u] \end{tikzcd}$$

Replace by the dual picture — cochain DG modules over $\mathbb{C}[S]$

$$\begin{tikzcd} X^0 \arrow[r, "S"] \arrow[dr] & X^2 \arrow[r, "S"] \arrow[dr, "d"] & X^4 \\ & X^1 \arrow[u, "d"] \arrow[r, "S"] & X^3 \arrow[u] \end{tikzcd}$$

$\deg S = +2$

Minimal cochain DG $\mathbb{C}[S]$ modules:

(i) Free as a graded $\mathbb{C}[S]$-module i.e. $\underline{S}$ injective (if finitely generated)

(ii) X/SX 'space of generators' has $d = 0$.

Claim is that a minimal DG cochain DG S-module is the same as a ~~tower~~ inductive system $\mathbb{Z}_2$ graded complex Y equipped with an increasing filtration

$$0 \subset F_0 Y \subset F_1 Y \subset \dots$$

Such that

$$Y = \bigcup_n F_n Y$$

$F_n Y / F_{n-1} Y$ is zero in degree $n+1+2\mathbb{Z}$

Description of the correspondence

$$\begin{tikzcd}
X^0 \arrow[r, hook, "s"] \arrow[dr] & X^2 \arrow[r, hook] \arrow[dr] & X^4 \\
& X^1 \arrow[u] \arrow[r] & X^3 \arrow[u] \arrow[r] & {}
\end{tikzcd}$$

Set $Y^{ev} = \lim_{n \in [i]} X^{2n}$

$Y^{odd} = \lim_n X^{2n+1}$

$F_0 Y = X^0 \qquad F_n Y = X^n + X^{n-1}$

This is a subcomplex since $dX^n \subset SX^{n-1}$ as $\underline{d}$ is zero on X/SX.

Question – If one is given algebra with b coming from Hochschild theory, can one find B?

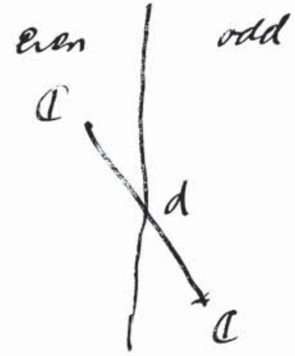
In the example $H^c_j = \begin{cases} \mathbb{C} & j = 2n-2k, \ldots, 2n \\ 0 & \text{else} \end{cases}$

the corresponding filtered complex Y will be

$\mathbb{C} \qquad \mathbb{C}$

The tower of $\mathbb{Z}/2$ graded complexes is

$k = 0$



$\} \ M/F^0M \ \} \ M/F^2M \ \} \ M/F^{n}M$

$M = \mathbb{C} \oplus \mathbb{C} \qquad F^n M =$

To go in the opposite direction

$$\longrightarrow M/F^2 \longrightarrow M/F^1 \longrightarrow M/F^0$$

$$\begin{matrix} M/F^0 \\ \uparrow\sim \\ Q^0 \\ \downarrow\sim \end{matrix}$$

$$0 \to F^0/F^1 \longrightarrow M/F^1 \longrightarrow M/F^0 \to 0$$

<u>Cyclic homology of A</u>

This consists of $HH_n A$, $HC_n A$, $HP_n A$, $HC^-_n A$ (Hochschild, cyclic, periodic cyclic, negative cyclic)

defined by the mixed complex $(\Omega A, b, B)$.

In terms of $\mathbb{Z}_2$ graded complexes

$$HC_n A = H_{n+2\mathbb{Z}}(\Omega A / F^n \Omega A)$$

$$= H_{n+2\mathbb{Z}}\left(\Omega^0 \oplus \ldots \oplus \Omega^n / b\Omega^{n+1}\right)_{,\, b+B}$$

$$HP\,A = H(\hat{\Omega}) \doteq H_i\left(\prod \Omega^{2n} \underset{b+B}{\rightleftarrows} \prod \Omega^{2n+1}\right)$$

$\Omega A / F^n \Omega A$ is the n^{th} approximation to the cyclic theory of A.

If $HH_k A = 0$ for $k \leq n$, then

$\Omega A \twoheadrightarrow \Omega A / F^n \Omega A$ is a quasi isomorphism wrt b

It induces an isomorphism of cyclic theories associated to these mixed complexes. The n^{th} order approximation is exact.

Example: Suppose that $\underline{A}$ is separable i.e. $A = \prod M_n \mathbb{C}$

$\underline{A}$ is projective as an $\underline{A}$-bimodule

In this case $\Omega A / F^0 \Omega A$ $\quad A/b\Omega^1 A \rightleftarrows 0$

gives the cyclic theory of $\underline{A}$.

$$A/b\Omega^1 A = A/[A,A]$$

Example 2: $\underline{A}$ is quasi-free i.e. $\Omega^1 A$ is projective as an $\underline{A}$-bimodule

$$0 \to \Omega^1 A \to A \otimes A \xrightarrow{m} A \to 0$$

so $\Omega^1 A$ projective $\iff$ $\underline{A}$ has projective dimension ≤ 1.

Eg $T(V) = \bigoplus V^{\otimes n}$, free group algebras

In this case $\Omega A / F^1 \Omega A$ gives the cyclic theory

$$\underbrace{A \underset{B=d}{\overset{b}{\rightleftarrows}} \Omega^1 A/[A,\Omega^1 A]}_{X(A)}$$

$X(A)$ plays role of the de Rham complex.

Study $X(R)$ where $A = R/I$ where $\underline{R}$ is quasi free. Study $X(R/I^n)$ and obtain the cyclic theory of A from it.

Step 1 - calculate $A = RA/IA$ i.e. the case of the universal extension of A.

Program is to calculate $X(RA)$

Structure on $X(R)$

i) product on R: $\quad R \otimes R \to R$
$\qquad x \otimes y \mapsto xy$

ii) pairing $\quad R \otimes R \to \Omega^1 R_\natural$
$\qquad x \otimes y \mapsto b(xdy)$

which satisfies the

$$\natural(xd(yz)) = \natural(xyd z) + \natural(zxdy)$$

- universal one Hochschild cocycle condition for

$$f(x,y) = \natural(xdy)$$

(iii) $\mathbb{Z}/2$ graded complex: $bd = db = 0$ i.e. $bB = Bb = 0$

Facts about RA:

Two descriptions

1) It is the universal algebra equipped with a linear map $A \to RA$ such that $1 \mapsto 1$

2) $RA = \Omega^{ev} A = \bigoplus_{k \geq 0} \Omega^{2k} A$

equipped with the Fedosov product
$$x \circ y = xy - dxdy$$

One has a canonical extension
$$IA = \mathrm{Ker}\,(RA \to A) = \bigoplus_{k>0} \Omega^{2k} A$$
(neglect the terms of degree ≥ 0)

The powers $(IA)^n = \sum_{k \geq n} \Omega^{2k} A$

4) $\Omega^1(RA) \simeq RA \otimes \bar{A} \otimes RA$
$x \delta a y \qquad x \otimes a \otimes y$

Let δ be the canonical derivation $\delta : RA \to \Omega^1 RA$ be the canonical derivation and let the differentials in $X(RA)$ be denoted

$$RA \underset{\delta}{\overset{\beta}{\rightleftarrows}} \Omega^1(RA)_\natural$$

$x \circ y - y \circ x \longleftrightarrow \natural(x \delta y)$

$x \longmapsto \natural(\delta x)$

In general, for a free algebra $T(V)$, there is
$$\Omega^1(T(V)) \simeq T(V) \otimes V \otimes T(V)$$
$$x\, dv\, y \longleftarrow x \otimes v \otimes y$$

$\Omega^1 R$ is the universal bimodule with respect to derivations from R to an R-bimodule.

5) $\Omega^1(RA)_\natural \xleftarrow{\sim} RA \otimes \bar{A}$ $\qquad (= \Omega^{odd} A$
$\natural(x \delta a) \longleftarrow x \otimes a$ $\qquad x \otimes a \mapsto x da)$

6) We have an isomorphism of vector spaces

$$\begin{array}{ccc} RA & & \Omega^1(RA)_\natural \\ \wr\| & & \wr\uparrow \\ \Omega^{ev} A & & \Omega^{ev} A \end{array}$$

Find the β operations in terms of isomorphisms.
$$x \circ y = xy - dxdy$$

Take β $\qquad$ Calculate $\qquad \beta : \Omega^{od} A \to \Omega^{ev} A$

$$\begin{array}{ccccc} x\circ a - a\circ x \; RA & \underset{\delta}{\overset{\beta}{\rightleftarrows}} & \Omega^{?}(RA)_{\natural} & & \natural(x\delta a) \\ \| & & \| & & \\ \Omega^{ev}A & & \Omega^{odd}A & & x\,da \end{array}$$

$[x,a] - dxda + dadx$

$$x\circ a - a\circ x = xa - dxda - ax + dadx$$
$$= [x,a] - dxda + dadx$$
$$= +b(xda) \cancel{+ d(xda)} - (1+K)d(xda)$$

Therefore $\beta = b + (1+K)d$

Claim: Let $x, y \in \Omega^{ev}A$, let $y \in \Omega^{2n}A$, then

$$\natural(x\delta y) = -\sum_{j=0}^{n-1} K^{2j} b(x\circ y) + \sum_{j=0}^{2n-1} K^j d(x\circ y) + K^{2n} x\,dy$$

Put $x = 1$ to give

$$\natural(\delta y) = -\sum_0^{n-1} K^{2j} b y + \underbrace{\left(\sum_{j=0}^{2n} K^j d\right) y}_{(=By)}$$

Proof by induction on $\underline{n}$

For $n=0$ we have simply
$$\natural(x\delta a) = x\,da.$$

Now consider $n>0$ and $\underline{y}$ is in standard form
$$y = z\,da_1 da_2 \quad \text{with } z \in \Omega^{2n-2}A$$

$$\natural(x\delta y) = \natural(x\,\delta(z\,da_1da_2)) = \natural(x\,\delta(z\circ da_1 da_2))$$
$$= \natural((x\circ z)\,\delta(da_1da_2)) + \natural((da_1da_2\circ x)\delta z)$$

By induction, the last term is
$$-\sum_{j=0}^{n-2} K^{2j} b(da_1da_2\circ x\circ z)$$
$$+ \sum_{j=0}^{2n-3} K^j d(da_1da_2\circ x\circ z) + K^{2n-2}$$
$$+ K^{2n-2}(da_1da_2\circ x)dz$$

$da_1da_2\circ x\circ z$ like $K^2((x\circ z)\circ da_1da_2)$
– induction applies.

$$\natural((x\circ z)\,\delta(da_1da_2))$$
$$= \natural((x\circ z)\,\delta(a_1a_2 - a_1\circ a_2))$$

$$= \natural\{(x\circ z \mid \delta(a_1a_2)\} - \natural\{(x\circ z\circ a_1 \mid \delta a_2\}$$
$$- \natural\{a_2 x\circ z\, \delta a_1\}$$

$$= (x\circ z)d(a_1a_2) - (x\circ z\circ a_1)da_2 - (a_2 x\circ z)da_1$$

$$= (x\circ z)da_1a_2 + \cancel{(x\circ z)a_1da_2} - \cancel{(x\circ z)a_1da_2} + d(x\circ z)da_1da_2$$
$$+ a_2(x\circ z)da_1 + da_2d(x\circ z)da_1$$

$$= -b\left((x\circ z)da_1da_2\right) + (1+\kappa)d(x\circ z)da_1da_2$$

$$= -b(x\circ y) + (1+\kappa)d(x\circ y)$$

$$\natural(\delta y) = -(N_{\kappa^2})by + By$$

"average over κ^2"

I-adic behaviour

Generalities: I ideal in R

I-adic tower : Inverse system of algebras

$$\to R/I^3 \to R/I^2 \to R/I$$

$$\to X(R/I^3) \to X(R/I^2) \to X(R/I)$$

Fact $\quad \Omega^1(R/I) = \Omega^1(R)/I\Omega^1R + (\Omega^1R)I + dI$

so $\quad \Omega^1(R/I^n) = \Omega^1R / I^n\Omega^1R + (\Omega^1R)I^n + dI^n$

<u>Theorem</u>

$$X(RA): \quad \begin{array}{ccc} RA & \underset{\delta}{\overset{\beta}{\rightleftarrows}} & \Omega^1(RA)_\natural \\ \wr\wr & & \wr\wr \\ \Omega^{ev}A & & \Omega^{odd}A \end{array}$$

0) Product in RA is the Fedosov product

1) The pairing $\quad RA \times RA \to \Omega^1(RA)_\natural$

$$(x,y) \mapsto \natural(x\,\delta y) = -\Big(\sum_{j=0}^{n-1}\kappa^{2j} b(x\circ y)\Big) + \sum_{j=0}^{2n-1}\kappa^j d(x\circ y)$$
$$+\kappa^{2n}(xdy)$$

for $y \in \Omega^{2n}A$

Write $\quad \sum_0^{2n-1} + \kappa^{2n} = \sum_{j=0}^{2n-1}\kappa^{j}(dxy) + \sum_{j=0}^{2n}\kappa^j(xdy)$

2) $\quad \beta = b + (1+\kappa)d$

$$\delta = -N_{\kappa^2}b + B$$

Here $\quad N_{\kappa^2}b = \sum_{j=0}^{n-1}\kappa^{2j}b \quad$ on $\Omega^{2n}A$

Recall that on $\Omega^n A$

$$K^n = 1 + bK^{-1}d$$
$$K^{n+1} = 1 - db$$

Consider the general case of $I \subset R$ an ideal
I-adic filtration of R

$$R = I^0 \supset I^1 \supset I^2 \supset \dots$$

Tower of algebras

$$R/I^2 \twoheadrightarrow R/I \twoheadrightarrow R/R$$

$$\hat{R} = \varprojlim R/I^{n+1} \qquad \text{the } I\text{-adic completion}$$

One gets a tower of $\mathbb{Z}/2$-graded complexes

$$\twoheadrightarrow X(R/I^2) \twoheadrightarrow X(R/I)$$

which are all quotients of $X(R)$.

Fact: i) $\Omega^1(R/I) \cong \Omega^1 R / I(\Omega^1 R) + (\Omega^1 R) I + dI$

ii) $(\Omega^1(R/I))_\natural \xleftarrow{\sim} \Omega^1 R / [R, \Omega^1 R] + I\Omega^1 R + dI$

$\therefore$

$$dI^{n+1} \subseteq \sum_{j=1}^{n} I^j (dI) I^{n-j}$$
$$\equiv I^n dI \qquad \operatorname{mod} [\,,\,]$$

$$b(I^n dI) \subseteq [I^n, I]$$

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$$\begin{array}{llcl}
X(R/I^{n+1}): & R/I^{n+1} & \rightleftarrows & \Omega^1 R / [R, \Omega^1 R] + I^{n+1} dR + dI^{n+1} \\
 & \| & & \downarrow \\
\mathcal{X}^{2n+1}(R, I) & R/I^{n+1} & \rightleftarrows & \Omega^1 R / [R, \Omega^1 R] + I^{n+1} dR + I^n dI \\
\quad \bigoplus_{k \le n} \Omega^{2k} A & \downarrow & & \downarrow \ \Omega^1 \oplus \dots \oplus \Omega^{2n+1}/\dots \\
\mathcal{X}^{2n}(R, I) & R/I^{n+1} + [I^n, R] & \rightleftarrows & \Omega^1 R / [R, \Omega^1 R] + I^n dR \\
\quad \bigoplus_{k \le n} \Omega^{2k} / b\Omega^{2n+1} & \downarrow & & \downarrow \ \bigoplus_{k < n} \Omega^{2k+1} A \\
X(R/I^n) & R/I^n & \rightleftarrows & \Omega^1 R / [R, \Omega^1 R] + I^n dR + dI^n
\end{array}$$

Note $\quad I\Omega^1 R \equiv IR\,dR = I\,dR$

Proposition: Under the isomorphism $X(RA) \simeq \Omega A$ one has an isomorphism of quotients

$$\mathcal{X}^k(RA, IA) \xrightarrow{\sim} \Omega A / F^k \Omega A$$

where $F^k \Omega A$ is the Hodge filtration

$$F^k \Omega A = b\Omega^{k+1} A \oplus \bigoplus_{j > k} \Omega^j A$$

Proof: Calculate each side.

$$(IA)^n = \bigoplus_{k \ge n} \Omega^{2k} A$$

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$\natural\,(IA)^n\delta(RA) = \bigoplus_{k\geq n}\Omega^{2k+1}A$. RA is generated by $\underline{a}\in A$

$\natural\,(z\,\delta(a_1\circ\ldots\circ a_q)) = \natural\sum_{j=1}^{q}(a_{j+1}\circ\ldots\circ a_q\circ z\circ a_1\circ\ldots\circ a_{j-1})\,\delta a_j$

$\therefore\ \natural((IA)^n\,\delta(RA)) = \natural((IA)^n\,\delta A) = \bigoplus_{k\geq n}\Omega^{2k+1}A$

since $\quad \natural(z\,\delta a) = \natural(z\,da)$

$[IA^n, RA] = \beta\,[IA^n\,\delta(RA)]$

$= (b-(1+K)d)\,(\bigoplus_{k\geq n}\Omega^{2k+1}A)$

$\cong b\,\Omega^{2n+1}A \quad$ modulo $IA^{n+1} = \bigoplus_{k\geq n}\Omega^{2k}A$

$\therefore\ RA/(IA)^{n+1} + [RA, (IA)^n]$

$= \bigoplus_n \Omega^{2k}A \big/ \bigoplus_{k\geq n}\Omega^{2k}A + b\,\Omega^{2n+1}A$

$\natural\,(I^n dIA) \cong \natural\,(I^n\,\delta\,(dAdA)) \quad$ mod $IA^{n+1}\,\delta\,RA$

$dAdA$

a_0 ...

$\delta(a_0\circ(da_1\,da_2)\circ\ldots\circ(da_{2n-1}\,da_{2n}))$

Let $z\in I^n$ $\quad da, da_2$

$\natural(z\,\delta(da_1 da_2))$

By the formula

$\natural(z\,\delta(da_1 da_2)) = \Sigma_{\text{etc}} = -b(z\,da_1 da_2) + (1+K)d(z\,da_1 da_2)$

Now we wish to calculate the homology of the X-complex for the differentials $\underline{\delta}$ and $\underline{\beta}$.

Apply the harmonic decomposition

$\Omega A = P(\Omega A) \oplus P^{\perp}(\Omega A)$

$= P(\Omega A) \oplus P_{-1}(\Omega A) \oplus P^{\perp}_{\{\pm1\}}(\Omega A)$

Recall that all the eigenvalues of $\underline{K}$ apart from 1 are simple.

$\beta^2 = (b-(1+K)d)^2 = -(1+K)(1-K)$

on $P^{\perp}_{\{\pm1\}}$ $\quad \delta=0,\quad \beta$ is an isomorphism

Let $X = X(RA) = RA \underset{\beta}{\overset{\delta}{\rightleftarrows}} \Omega^1(RA)_\natural$

$\beta = b-(1+K)d$

$\delta = -N_{K^2}\,b + K$

$X^q = X^q(RA, IA) \quad$ tower of quotient complexes of $X(RA)$

Relative to the above identification $X(RA) = \Omega A$

we have $\mathcal{X}^q = \Omega A / F^q \Omega A$.

Taking the inverse limit over q we have the identification

$$\hat{X} = \hat{X}(RA \,\&\, IA) \cong \hat{\Omega} A$$

To calculate the homology we use the harmonic decomposition of ΩA with respect to K to relate X to $(\Omega A, b+B)$.

$$\Omega A = P\Omega A \oplus P_{(-1)} \Omega A \oplus P_{(\pm 1)}^{\perp} \Omega A$$

$$P_{-1} \Omega A = \operatorname{Ker}(K+1)$$
$$P \Omega A = \operatorname{Ker}((K-1)^2)$$

$$P_{(\pm 1)}^{\perp} \Omega A = \oplus_{\zeta;\, \zeta^n = 1, \zeta \neq \pm 1} \operatorname{Ker}(K-\zeta)$$

This decomposition is stable under b, d, K etc.
Consider β, δ on $P_{(\pm 1)}^{\perp} \Omega A$.
Take $\beta = b + (1+K) d$ and consider β^2

$$(b - (K+1)d)^2 = (1+K)(bd+db) = K^2 - 1$$

But this is invertible on this space (so $P_{\pm}^{\perp} \Omega A$)
$\beta : \Omega^{odd} \to \Omega^{even}$
is an isomorphism on ~~$P_{\pm}^{\perp} \Omega A$~~
and $\delta = 0$

The decomposition is compatable with the Hodge filtration.

All the complexes $P_{\pm}^{\perp} X$, $P_{\pm}^{\perp} \mathcal{X}^q$, $P_{\pm}^{\perp} \hat{\mathcal{X}}$ have β an isomorphism from odd to even and δ = zero from even to odd.

Next consider $P_{-1} \Omega A$. Here $K = -1$.
Then $\beta = b - (1+K)d = b : P_{-1} \Omega^{odd} \to P_{-1} \Omega^{ev}$
and $\delta = -N_{K^2} b + B$

Recall $N_{K^2} b = \sum_{j=0}^{n-1} K^{2j} b$ on $\Omega^{2n} A$
so
$$N_{K^2} = nb \quad \text{on} \quad P_{-1} \Omega^{2n} A$$

$$B = \sum_{j=0}^{2n-1} K^j d = 0 \quad \text{on} \quad P_{-1} \Omega^{2n}$$

so
$$\delta = -nb \quad \text{on} \quad P_{-1} \Omega^{2n} A \to P_{-1} \Omega^{2n-1} A$$

Rescaling to remove the term '$-n$' we have
$P_{-1} X = P_{-1} \Omega A$ with b
But recall $bd + db = 1 - K = 2$ on $P_{-1} \Omega A$
so b is exact on $P_{-1} \Omega A$; homotopy operator $\frac{1}{2} d$.
The same argument works for the Hodge filtration.
The same conclusion for $P_{-1} \mathcal{X}^q = P_{-1}(\Omega A / F^q \Omega A)$
applies so that
$$P_{-1} \mathcal{X}, \; P_{-1} \mathcal{X}^q, \; P_{-1} \hat{X}$$
have zero homology.

On the harmonic space we consider $PX = P \Omega A$.
On $P \Omega A$ $\quad K^j d = d \quad K^j b = b$ so
$$\beta = b - (1+K) d = b - 2d$$
$$= b - \frac{2}{N} B$$

$$\left(B = \Sigma_0^n K^j d = (n+1)d \text{ on } \Omega^n\right)$$

$$\delta = -N_{K^2} b + B = -\sum_0^{n-1} K^{2j} b + B = -nb + B \quad \text{on } P\Omega^n$$

$$\beta = b - \tfrac{2}{2n+1} B \quad \text{on } P_1 \Omega^{2n+1} A \ ; \ \delta = -nb + B \text{ on } P\Omega^{2n} A$$

<u>Lemma</u>: Let c be the scaling transformation

$$c\omega = c_{|\omega|}\, \omega$$

where $c_{2n} = c_{2n+1} = (-1)^n n!$.
Then on $P\Omega A$ (in fact on $P\Omega A + P_{-1}\Omega A$) we have

$$c(\beta + \delta) c^{-1} = b + B$$

i.e.

$$c \beta c^{-1} = b + B \quad \text{from odd to even}$$
$$c \delta c^{-1} = b + B \quad \text{from even to odd}$$

On Ω^{2n+1},
$$c\beta c^{-1} = c \beta c_{2n+1}^{-1} = c\left(b + \tfrac{2}{2n+1} B\right) c_{2n+1}^{-1}$$
$$= \frac{c_{2n}}{c_{2n+1}} b - \frac{2}{2n+1} \frac{c_{2n+2}}{c_{2n+1}} B$$
$$= b + B$$

where $c_{2n}/c_{2n+1} = 1$; $\dfrac{c_{2n+2}}{c_{2n+1}} = -\dfrac{2n+2}{2}$

We find that there are isomorphisms of complexes
$$\begin{cases} c: PX \to P\Omega A \\ c: PX^q \xrightarrow{\sim} P(\Omega A / F^q \Omega A) \\ c: P\hat{X} \to P(\hat{\Omega} A) \end{cases}$$

The differentials on the right are $b+B$. We conclude
$cP: X \to \Omega A \qquad X^q \to \Omega A / F^q \Omega A$
$\qquad \hat{X} \to \hat{\Omega} A$
is a quasi-isomorphism of complexes.

<u>Theorem</u>
$$H_i(\hat{X}(RA, IA)) = H_i(\hat{\Omega} A, b+B)$$
$$= HP_i(A)$$

$$H_i(X^q(RA, IA)) = \begin{cases} HC_q(A) & i \in q + 2\mathbb{Z} \\ \operatorname{Im} S : HC_{q+1} A \to HC_{q-1} A & i = q-1+2\mathbb{Z} \end{cases}$$

$$H_i(X(RA)) = H_i(\Omega A, b+B) = \begin{cases} \mathbb{C} & i \in 2\mathbb{Z} \\ 0 & i \in \mathbb{Z}+1 \end{cases}$$

Proof: The final equality follows from
$$H_i(\Omega A, b+B) = H_i(P\Omega A, b+B)$$
because
$$P^\perp = b(Gd) + (Gd)b$$
$$= (b+B)(Gd) + (Gd)(b+B)$$

Therefore $\quad H_i(P^\perp \Omega A, b+B) = 0$

Also $\underline{B}$ is almost exact on $P\Omega A$ and $B = Nd$
On $\underline{B}$ is exact on $P\bar{\Omega} A$. Then

Suppose M is a mixed complex with b exact.

$$\begin{tikzcd}
 & M_2 & M_1 \arrow[l, "B"'] & M_0 \\
 & M_1 \arrow[u, "b"'] \arrow[l] & M_0 \arrow[l, "B"'] & \\
 & M_0 \arrow[u, "b"'] & &
\end{tikzcd}$$

Total homology of this $\mathcal{B}^{per}(M)$ is the homology of M with differential $b+B$. Since the rows are exact there is a spectral sequence which converges $H^{hor}(M,B)=0$.

If $A = R/I$ where R is quasi free then the tower $X^q(R,I)$ gives the cyclic theory of the algebra A. In particular $\hat{X}(R,I) = \varprojlim X^q(R,I)$ $= \varprojlim X(R/I^{n+1})$ gives $HP_\bullet(A)$.

<u>Cartan homotopy formula for the X complex</u>

Let $u : R \to S$ be a homomorphism of algebras and
$\dot{u} : R \to S$ be a derivation relative to u

We have $u_* : \Omega R \to \Omega S$ DG alg. hom

$$u_*(a_0 da_1 \dots da_n) = u(a_0)\, d\, u(a_1) \dots d\, u(a_n)$$

Lie derivative $L = L(u,\dot{u}) : \Omega R \to \Omega S$

$$L(x_0 dx_1 \dots dx_n) = \dot{u}x_0\, d(ux_1) \dots d(ux_n) + \sum_1^n u(x_0) d(ux_1) \dots d(\dot{u}x_i) \dots d(ux_n)$$

u_*, L are compatible with d, b, K etc.

Consider $L : X(R) \to X(S)$

$$\begin{tikzcd}
\arrow[r] & \Omega^1 R_\natural \arrow[r, "\bar{b}"] \arrow[d, "L_1"] & R \arrow[r, "\natural d"] \arrow[d, "L_0"] \arrow[dl, dashed, "h_0"'] & \Omega^1 R_\natural \arrow[r, "\bar{b}"] \arrow[d, "L_1"] \arrow[dl, dashed, "h_1"'] & R \arrow[dl, dashed, "h_0"'] \\
\arrow[r] & \Omega^1 S_\natural \arrow[r, "\bar{b}"'] & S \arrow[r, "\natural d"'] & \Omega^1 S_\natural \arrow[r, "\bar{b}"'] & S
\end{tikzcd}$$

Here $\natural : \Omega^n \to \Omega^n / b\Omega^{n+1} = \Omega^n_\natural$
and $\bar{b} : (\Omega^n)_\natural \to \Omega^{n-1}$ is the map induced
by $b : \Omega^n \to \Omega^{n-1}$ $(b^2 = 0)$

The problem is to construct a homotopy for $L : X(R) \to X(S)$ assuming that $\underline{R}$ is quasi free.

Definition: (Connes) Let $\underline{E}$ be a right R-module. A connection on $\underline{E}$ is a map $\nabla : E \to E \otimes_R \Omega^1 R$ which is linear and satisfying the Leibniz formula

$$\nabla(\xi x) = (\nabla\xi)x + \xi dx \qquad (\xi\in E,\ x\in R)$$

Fact: ∇ is the same as a ~~left~~ right module splitting

$$0 \to E\otimes_R \Omega^1 R \xrightarrow{j} E\otimes R \xrightarrow{mult} E \to 0$$

$$j(\xi\, dx) = \xi x\otimes 1 - \xi\otimes x$$

Splitting $\quad (\nabla\xi)x \longleftrightarrow \xi\otimes x$

Hence ∇ is the same as a section of the multiplication $E\otimes R \to E$ which is a right module map

∇ exists if and only if E is projective.

Definition: If E is an R-bimodule, then a right connection on E is a $\nabla: E\to E\otimes_R\Omega^1 R$ such that

$$\nabla(x\xi) = x\,\nabla\xi$$
$$\nabla(\xi x) = (\nabla\xi)x + \xi dx$$

For the Cartan homotopy formula we need a right connection ∇ on $\Omega^1 R$. This is an operator

$$\nabla: \Omega^1R \to \Omega^1R\otimes_R\Omega^1R = \Omega^2R$$

Such a connection is equivalent to a splitting of the exact sequence

$$0 \longrightarrow \Omega^2R \xrightarrow{j} \Omega^1R\otimes R \xrightarrow{m} \Omega^1R \to 0$$

$$j(\omega dx) = \omega x\otimes 1 - \omega\otimes x$$

as a sequence of bimodules. ∇ is equivalent to a bimodule lifting of Ω^1R into $\Omega^1R\otimes R$ $(= R\otimes\bar{R}\otimes R)$ which is free so it exists if and only if Ω^1R is a projective bimodule (i.e. $\underline{R}$ is quasi free)

<u>Lemma:</u> If ∇ is a right connection on Ω^1R we have a split exact sequence of vector spaces.

$$0 \to (\Omega^2R)_\natural \underset{\natural\nabla}{\overset{-b}{\rightleftarrows}} (\Omega^1R) \underset{k}{\overset{\natural}{\rightleftarrows}} (\Omega^1R)_\natural \to 0$$

where $\underline{k}$ is the lifting given by

$$k\natural = 1 - (-b)\natural\nabla$$

Return to $u, \bar{u}, L$ and define an odd degree operator $h: X(R) \to X(S)$ by

$$h_0: R \xrightarrow{d} \Omega^1R \xrightarrow{\nabla} \Omega^2R \xrightarrow{i} \Omega^2S$$
$$\downarrow\natural$$
$$(\Omega^2R)_\natural \xrightarrow{i} \Omega^2S_\natural$$

$$h_0: \Omega^1R_\natural \xrightarrow{k} \Omega^1R \xrightarrow{i} S$$

The interior product $i = i(u,\dot u) : \Omega R \to \Omega S$ of degree (-1) is defined by

$$i(x_0 dx_1 \dots dx_n) = u x_0 \dot u x_1 \, d(ux_2) \dots d(ux_n)$$

This satisfies $bi + ib = 0$

So that $\underline{i}$ induces maps $\Omega^n R_\natural \to \Omega^{n-1} S_\natural$

Claim that
$$\begin{cases} \bar b h_0 + h_1 \natural d = L_0 \\ \natural d h_1 + h_0 \bar b = L_1 \end{cases}$$

Proof: $\bar b h + h_1 \natural d = \bar b(\natural i \nabla d) + (ik)\natural d$

$= bi \nabla d + ik\natural d$

$= i(-b)\nabla d + ik\natural d$

$= id$ by def^n of k

$id(x) = i(dx) = \dot u x = L_0 x$

$\natural d h_1 + h_0 \bar b = (i \natural \nabla d)\bar b + \natural d(ik)$

$= + i \natural \nabla(-\bar b B) + di\natural k \quad (\Omega^2 R)_\natural \to (\Omega^2 \ldots)_\natural$

$= iB + di$

Check that $iB + di = L_1$ from $(\Omega^2 R)_\natural \to (\Omega^1 S)_\natural$

$di(xdy) = d(ux\,\dot u y) = d(ux)\dot u y + (ux)\, d(\dot u y)$

$d(uy)$

$iB(xdy) = i\{dxdy - dy\,dx\} = \dot u x\, d(uy) - \dot u y\, d(ux)$

$\therefore \quad (iB + di)(xdy) = \dot u x\, d(uy) + ux\, d(\dot u y)$
$= L_1(xdy)$

$\nabla : \Omega^1 R \to \Omega^2 R$ right connection

$$\begin{cases} \nabla(x\omega) = x \nabla\omega \\ \nabla(\omega x) = (\nabla\omega)x + \omega dx \end{cases} \qquad \begin{cases} \omega \in \Omega^1 R \\ x \in R \end{cases}$$

$h = h^\nabla(u,\dot u) \quad : X(R) \to X(S)$ odd operator

$h_0 : \quad R \xrightarrow{d} \Omega^1 R \xrightarrow{\natural\nabla} \Omega^2 R_\natural \xrightarrow{i} \Omega^1 S_\natural$

$h_1 : \quad (\Omega^1 R)_\natural \xrightarrow{k} \Omega^1 R \xrightarrow{i} S$

Splitting defined by ∇ $\quad 0 \to \Omega^2 R_\natural \underset{-b}{\overset{\natural\nabla}{\rightleftarrows}} \Omega^1 R \overset{k}{\rightleftarrows} \Omega^1 R_\natural \to 0$

$$\begin{cases} \natural \nabla(-b) = 1 & \natural k = 1 \\ (-b)\natural\nabla + k\natural = 1 \end{cases}$$

Homotopy formula for the X complex

$$L(u,\dot u) = [\partial, h^\nabla(u,\dot u)]$$

where ∂ means the differential in the X-complex.

Adic behaviour of the Cartan homotopy formula

Suppose $I \subset R$, $J \subset S$ such that

$u(I) \subset J$ and also $\bar{u}(I) \subset J$.
Recall the torus of quotient complexes of $\underline{X(R)}$

$$\mathcal{X}^{2n+1}(R, I): \quad R/I^{n+1} + [I^n, R] \longleftarrow \Omega^1 R/[\Omega^1 R, R] + I^{n+1} dR + I^n dI$$

$$\mathcal{X}^{2n}(R, I): \quad R/I^{n+1} + [I^n, R] \rightleftarrows \Omega^1/[R, \Omega^1 R] + I^n dR$$

We have induced maps of complexes from $\underline{u}, \underline{L}$.

$$u_*, h_*: \mathcal{X}^q(R, I) \longrightarrow \mathcal{X}^q(S, J)$$

$$L(x dy) = \bar{u} x dy + u x d(\bar{u} y)$$

$$L(I^n dI) \in \sum u(I)^j \bar{u}(I) u(I)^{n+1-j} * du(I) + u(I)^n d\bar{u}(I)$$

$$\subset J^n dJ$$

Theorem 1) h^∇ induces maps from $\mathcal{X}^q(R, I)$ to $\mathcal{X}^{q-1}(S, J)$ for each q.
2) If $\bar{u}(R) \subset J$ (This means the map $R/I \to S/J$ induced by $\underline{\bar{u}}$ is zero). Then h^∇ induces maps $\mathcal{X}^q(R, I)$ to $\mathcal{X}^q(S, J)$ $\forall q$.

Call (1) the general case and (2) the restricted case.

h_0 on $\mathcal{X}^{2n+1}_{even} = R/I^{n+1}$ $\quad h_0(I^{n+1})$?

$$I^{n+1} \xrightarrow{d} \sum_{j=0}^{n} I^{n-j}(dI) I^j \xrightarrow{\nabla} \sum I^{n-j} \nabla((dI) I^j)$$

$$= \sum I^{n-j} (\nabla(dI) I^j + dI \, \nabla(I^j))$$ Now $\nabla(dI) \subseteq dR dR$

$$= \sum_{0 \le j \le n} I^{n-j} dR dR I^j + \sum_{0 \le k \le j \le n} I^{n-j} dI I^k dI I^{j-1-k}$$

$$\xrightarrow{i} \sum_{0 \le j \le n} J^{n-j} \bar{u}(R) dS J^j + \sum J^{n-j} \bar{u}(I) J^k dJ J^{j-1-k}$$

$$\xrightarrow{q} J^n \bar{u}(R) dS + J^n dJ$$ since $\bar{u}(I) \subseteq J$

Have $h_0(I^{n+1}) \subseteq J^n \bar{u}(R) dS + J^n dJ$.

In case (2), $\bar{u}(R) \subseteq J$ so $h_0(I^{n+1}) \subseteq J^{n+1} dS + J^n dJ$
In general case $h_0(I^{n+1}) \subseteq J^n dS$

Other cases are done using the various identities.

Integrated forms of the Cartan homotopy formula

Let $u_t : R \longrightarrow S[t] = S \otimes \mathbb{C}[t]$
be a homomorphism i.e. a one parameter polynomial family of homomorphisms from $\underline{R}$ to $\underline{S}$

$$u_t(x) = \sum_n u_n(x) t^n$$

$$(u_t)_* : X(R) \to X(S)$$

We have a family of maps of complexes. Differentiating

$$\partial_t (u_{t*}) = L(u_t, \dot u_t)$$

where $\dot u_t = \partial_t u_t$. If right connection ∇ is given on R

$$L(u_t, \dot u_t) = [d\ell, h^\nabla(u_t, \dot u_t)]$$

h^∇ polynomial family of odd maps $X(R) \to X(S)$.

Integrating we have

$$(u_1)_* - (u_0)_* = \int \partial_t (u_t)_* \, dt$$

$$(u_1)_* - (u_0)_* = [d\ell, \int_0^1 h^\nabla(u_t, \dot u_t) dt]$$

The conclusion is that two homomorphisms u_0, u_1 from R to S which can be joined by a polynomial family u_t induce homotopic maps $X(R) \to X(S)$.

Application Take $R = T(V)$ free algebra

$u_0 = id : R \to R$

$u_1 : T(V) \to \mathbb{C} \hookrightarrow T(V)$

$v \mapsto 0$

$u_t : T(V) \to T(V)$ $\quad u_t(v) = tv \quad (v \in V)$

So $X(R)$ contracts to $X(\mathbb{C})$.

(This uses the fact that ∇ exists for a free algebra)

Consider $J \subset S$ an ideal

$$S[t]^\wedge = \varprojlim (S/J^{n+1})[t] \subset \hat S[[t]]$$

This is the subalgebra consisting of power series $\sum s_n t^n$ where $s_n \in \hat S$ and $s_n \to 0$ and w.r.t the ℓ $\forall k$ $s_n \in \mathrm{Ker}(\hat S \to S/J^{k+1})$ for large n.

(Avoid the error of $(\hat J)^n \neq$ $\mathrm{Ker}(\hat S \to S/J^n) = \overline{\hat J^n}$

$A = R/I$

$A \searrow S/J$

$u(I) \subseteq J$

$$\begin{array}{ccc}
R & \xrightarrow{\exists u} & S \\
\downarrow & & \\
R/I^2 & & S/J^2 \\
\downarrow & & \\
R/I & & S/J \\
\| & & \| \\
A & \xrightarrow{\varphi_0} & A'
\end{array}$$

Given $\varphi_0 : A \to A'$ there exists a lifting $u : R \to S$ assuming that R is a free algebra. Also $u(I) \subseteq J$ and we have an induced map of towers

$$\mathcal{X}(R, I) \to \mathcal{X}(S, J)$$

also induced map $\hat{\mathcal{X}}(R, I) \to \hat{\mathcal{X}}(S, J)$.

Given two liftings $u_0, u_1 : R \to S$ of ϕ, we can join them by a homotopy — a one parameter polynomial family $u_t : R \to S[t]$ such that

$$u_t(x) = (1-t)u_0(x) + t\,u_1(x)$$

for all generators x of R.

For each $t \in \mathbb{C}$ $\quad (u_t, \dot{u}_t)$

u_t lies over ϕ i.e. modulo J, u_t is constant in t. Hence $\dot{u}_t$ carries R into J.

i.e. we have a restricted homotopy

$(u, \dot{u}) \qquad u(I) \subseteq J$

$\qquad\qquad\quad \dot{u}(R) \subseteq J$ restricted case

Then $L(u,\dot{u}) = [d, h(u,\dot{u})]$ where $h = h^\nabla$ is defined $\mathcal{X}^q \to \mathcal{X}^{q+1}$ (restricted case).

Conclusion is that if $\underline{R}$ is free, then ~~choose~~ for any lifting $\underline{u}$ of ϕ one has a map of towers $u_* \mathcal{X}(R/I) \to \mathcal{X}^q(S/J)$ which is independent of the choice of $\underline{u}$ up to homotopy

$$(u_1)_* - (u_0)_* = \left[d, \int_0^1 h^\nabla(u_t, \dot{u}_t)\,dt\right]$$

Recall derived functors $\quad (L_p F)(M)$

R

$$\begin{array}{ccc} P & \overset{u}{\dashrightarrow} & Q \\ \downarrow & & \downarrow \\ M & \xrightarrow{\ \phi\ } & M' \end{array}$$

resolution

projective resolution complex

Basic result is that $\underline{u}$ exists and is unique up to homotopy

Hence def. $(L_pF)(M) = H_p(F(P))$

where P is any projective resolution of M.

Then L_pF is well defined and functorial in the module.

The steps in the proof are formal

$$\begin{array}{ccc} P & \leftarrow\!\!-\,-\,-\!\!\rightarrow & Q \\ & \searrow \quad \swarrow & \\ & M & \end{array} \qquad \text{here} \qquad \begin{array}{ccc} R & \leftrightarrows & S \\ & \searrow \quad \swarrow & \\ & A & \end{array}$$

Conclusion: If $A = R/I$ with $\underline{R}$ free then $X(R)$, $\mathcal{X}^q(R,I)$, $\hat{X}(R,I)$ up to homotopy type are independent of the choice of $\underline{R}$ and depend only on $\underline{A}$.

Now we do our calculations in the case ~~$R/I = A$~~

$A = RA/IA$

Theorem (free version) If $A = R/I$ with R free then we have

0) $\quad H_i(X(R)) = \begin{cases} \mathbb{C} & i \text{ even} \\ 0 & i \text{ odd} \end{cases}$

1) $\quad H_i(\hat{X}(R,I)) = HP_i(A)$

2) $\quad H_i(\mathcal{X}^q(R,I)) = HC_q(A) \qquad i = q + 2\mathbb{Z}$

$\mathrm{Ker}\{B : HC_{q-1}(A) \to HH_q(A)\} \quad i - q \in 2\mathbb{Z}+1$

$$H_i(\mathrm{Kar}\, \mathcal{X}^{q+\alpha} \to \mathcal{X}^q) = \begin{cases} 0 & i = q+2\alpha \\ HH_{q+\alpha}A & i = q+2\alpha+1 \end{cases}$$

One checks that

$$\mathcal{X}^q(\hat{S}, J) \xrightarrow{\sim} \mathcal{X}^q(S/J^{n+1}, J/J^{n+1}) \quad (q \le 2n+1)$$

Refinement:

$$\begin{tikzcd} R \arrow[d] & \hat{S} \arrow[d] \\ R/I^2 \arrow[d] & S/J^2 \arrow[d] \\ R/I \arrow[d, equal] & S/J \\ A \arrow[r, "\phi"'] & A' \end{tikzcd}$$

All we need in order to obtain a map of towers $\mathcal{X}^q(R,I) \to \mathcal{X}^q(S,J)$ is a compatible family or precisely one of the following equivalent things

1) A homomorphism $u: R \to \hat{S} = \varprojlim S/J^{n+1}$ lying over ϕ

2) A compatible family of homomorphisms from $R \to S/J^{n+1}$ $\forall n$ lying over ϕ

3) A compatible family $R/I^{n+1} \to S/J^{n+1}$ lying over ϕ.

It is sufficient that $\underline{R}$ has the following lifting property with respect to nilpotent extensions

Definition An algebra $\underline{R}$ is quasi free when it has

the lifting property with respect to nilpotent extensions

$$\begin{tikzcd} & S \arrow[d] \\ R \arrow[ur, dashed, "\exists"] \arrow[r] & S/J \end{tikzcd} \qquad J^N = 0$$

Lemma Let $\underline{R}$ be quasi free, $p: S' \to S$ a nilpotent extension. Given $u_0, u_1 : R \to S'$ and v_t a $v_t : R \to S[t]$

$$\begin{tikzcd} & S'[t] \arrow[r, "(0,1)"] \arrow[d, "p"] & S' \times S' \\ R \arrow[urr, bend left, "(u_0,u_1)"] \arrow[r, "v_t"'] & S[t] \arrow[r, "(0,1)"'] & S \times S \end{tikzcd}$$

Then there is a $u_t : R \to S'[t]$ such that $p u_t = v_t$ and u_t joins u_0 and u_1. (covering homotopy property)

Proof:

$$\begin{tikzcd} R \arrow[r, "(u_0, v_t, u_1)"'] \arrow[dr, "u_t"'] & S' \times_S S[t] \times_S S' \\ & S'[t] \arrow[u, "(ev_0, p, ev_1)"'] \end{tikzcd}$$

The extension is nilpotent $\quad S' \to S$ nilpotent

$S' \otimes \mathbb{C}[t] \to S \otimes \mathbb{C}[t]$ is nilpotent.

Lemma 2 Given R quasi free, $J \subset S$

$$v_t : R \to (S/J)[t]$$

$$u_0, u_1 : R \to \hat{S} = \varprojlim S/J^n$$

then there exists $u_t : R \to \widehat{S[t]} = \varprojlim (S/J^{n+1}[t])$

Proof

$$\begin{tikzcd}
R \arrow[r, dotted] \arrow[dr, dotted] \arrow[ddr] & \widehat{S[t]} \arrow[r, "(0,1)"] & \hat{S} \times \hat{S} \arrow[d] \\
 & S/J^2[t] \arrow[r, "(0,1)"] & S/J^2 \times S/J^2 \arrow[d] \\
 & S/J[t] \arrow[r, "(0,1)"'] & S/J \times S/J
\end{tikzcd}$$

Theorem Let $A = R/I$, $A' = \cancel{R'/I'}\ S/J$ and R is quasi free

1) Given a homomorphism $v: A \to A'$, we obtain an induced map of towers

$$v_* : \mathcal{X}^q(R,I) \to \mathcal{X}^q(S,J)$$

well defined up to homotopy by choosing a lifting of v to a hom: u $R \to \hat{S}$ and taking v_* to be the map induced by u.

2) Given $v_t : A \to A'[t]$ then $(v_0)_*$ and $(v_1)_*$: $\mathcal{X}^q(R,I) \to \mathcal{X}^q(S,J)$ are homotopic

There is a homotopy $h: \mathcal{X}^q(R,I) \to \mathcal{X}^{q-1}(S,J)$

such that $(v_1)_* - (v_0)_* = [\partial, h]$

$$\begin{tikzcd}
R \arrow[d] \arrow[dr] & S \\
\hat{R} \arrow[d, dotted] & \hat{S} \arrow[d, dotted] \\
R/I^2 \arrow[d] & S/J^2 \arrow[d] \\
R/I \arrow[d, equal] & S/J \arrow[d, equal] \\
A \arrow[r, "v"'] \arrow[d] & A' \arrow[d] \\
0 & 0
\end{tikzcd}$$

Corollary 1: Two quasi free extensions

$A = R/I = S/J$, R, S quasi free have $\mathcal{X}(R, I)$ and $\mathcal{X}(S, J)$ homotopy equivalent.

Corollary 2: $A = R/I$, R quasi free then the tower $\mathcal{X}(R, I)$ computes the cyclic theory of A

$$HP_i(A) = H_i(\hat{\mathcal{X}}(R, I))$$

Proof: Apply Corollary 1 to $R/I = RA/IA = A$.

Corollary 3: Suppose $v_t : A \to A'[t]$, then

$$(v_1)_* = (v_0)_* : HP_i(A) \to HP_i(A')$$

$$HD_n(A) \to HD_n(A')$$

where $HD_n(A) = \mathrm{Im}\{S : HC_{n+2}(A) \to HC_n(A)\}$

$= \mathrm{Ker}\{B : HC_n(A) \to HH_{n+1}(A)\}$

Proof: Take $A = \cancel{RA/IA}\ R/I$, $A' = \cancel{RA'/IA'}\ S/J$ and apply (2) of the theorem

$$\begin{tikzcd}
HD_{2n}(A) \arrow[d, phantom, "\cap"] \arrow[r, phantom, "="] & H_0(X^{2n+1}(R,I)) \arrow[r] \arrow[d] & H_0(X^{2n}(S,J)) \arrow[d, phantom, "\|"] \\
HC_{2n}A \arrow[r, phantom, "="] & H_0(X^{2n}(R,I)) \arrow[ur] & HC_{2n}(A')
\end{tikzcd}$$

Look at H_1 for the odd case.

Corollary 4 (Goodwillie) If $A \to A'$ is a surjective nilpotent extension then $HP_i(A) \xrightarrow{\sim} HP_i(A')$

Proof: Choose $A = R/I$, R quasi free and let ~~$A' = A/K$~~. Let J be the ideal of R such that $A' = R/J$. Then $I \subset J$ and J/I is nilpotent so $J^N \subset I$ for some $\underline{N}$

$$\hat{X}(R,I) = \varprojlim X(R/I^n) = \varprojlim X(R/J^n) = \hat{X}(R,J)$$

and $\hat{X}$ calculates the periodic cyclic homology.

<u>Review of Hochschild cohomology</u>

$H^n(A,M)$ $\qquad$ M an A-bimodule

$\Updownarrow$

left (or right) module over $A^e = A \otimes A^{op}$

Define $H^n(A,M) = \mathrm{Ext}^n_{A^e}(A,M)$

$= H^n\{\mathrm{Hom}_{A^e}(P,M)\}$

where $\underline{P}$ is (any) projective bimodule resolution of A

$$H_n(A,M) = \mathrm{Tor}_n^{A^e}(A,M)$$

$$(= H_n(P \otimes_{A^e} M))$$

Let $\underline{P}$ be the standard normalized resolution

$$\xrightarrow{b'} A \otimes \bar{A} \otimes A \xrightarrow{b'} A \otimes A \to A \to 0$$

In this case we have

$$\mathrm{Hom}_{A^e}(P_\cdot, M) = C^\cdot(A, M)$$

where $C^n(A, M) = \mathrm{Hom}_{A^e}(A \otimes \bar{A}^{\otimes n} \otimes A, M)$

$$= \mathrm{Hom}_{\mathbb{C}}(\bar{A}^{\otimes n}, M)$$

$= \{ f(a_1, \ldots, a_n) :$ multilinear in $a_1, \ldots, a_n$; normalized: vanish if $a_i = 1$; take values in $M \}$

differential $\delta : C^n(A, M) \to C^{n+1}(A, M)$

$$(\delta f)(a_1, \ldots, a_{n+1}) = a_1 f(a_2, \ldots, a_{n+1}) + \sum_{i=1}^{n} (-1)^i f(\ldots, a_i a_{i+1}, \ldots) + (-1)^{n+1} f(a_1, \ldots, a_n) a_{n+1}$$

$$H^0(A, M) = \mathrm{Ker}\{ C^0(A, M) \xrightarrow{\delta} C^1(A, M) \}$$

$$(\delta m)(a) = am - ma$$
$$= \{ m \in M : am = ma \ \forall a \in A \}$$

$=$ centre of M.

$$H^0(A, M) = \mathrm{Hom}_{A^e}(A, M)$$

As a bimodule, $\underline{A}$ is universal for central elements.

$Z^1(A, M) = \{ f : \bar{A} \to M \} : \delta f(a_1, a_2) = 0 \} =$ Derivations

$H^1(A, M) = \{$derivations $D : A \to M\} / \{$inner derivations$\}$

$H^2(A, M) = \{$ Isomorphism classes of square zero algebra extensions of A by $M \}$

$$0 \to M \to R \to A \to 0$$

Map: Given $0 \to M \xrightarrow{i} R \xrightarrow{p} A \to 0$

Choose a linear lifting $\rho : A \to R$ such that $p \circ \rho = \mathrm{id}$ that $\rho(1) = 1$.

Put

$$\omega(a_1, a_2) = \rho(a_1 a_2) - \rho(a_1)\rho(a_2)$$

Then $\omega \in C^2(A, M)$. Also $\delta\omega = 0$

The class of $\underset{\sim}{\omega}$ in $H^2(A, M)$ is independent of the choice of $\underset{\sim}{\rho}$.

Basic exact sequence of A-bimodules

$$0 \to \Omega^1 A \xrightarrow{j} A \otimes A \xrightarrow{m} A \to 0$$

$$j(a_0\, da_1) = a_0(a_1 \otimes 1 - 1 \otimes a_1)$$

$$m(a_0 \otimes a_1) = a_0 a_1$$

$$H^n(A, M) = \mathrm{Ext}^n_{A^e}(A, M) \xleftarrow{\sim} \mathrm{Ext}^{n-1}_{A^e}(\Omega^1 A, M)$$

for $n \geq 2$

Definition & proposition $\underline{A}$ is separable if it satisfies the equivalent conditions:

1) $\underline{A}$ has cohomological dimension ≤ 0 w.r.t. Hochschild cohomology i.e.
$$H^n(A,M) = 0 \quad \forall n \geq 1, \ \forall M$$
$\Updownarrow$

$\Leftarrow$ 2) $\underline{A}$ is a projective A-bimodule

$H^1(A,\cdot) = 0 \Leftrightarrow$ 3) Any derivation $D: A \to M$ is an inner derivation

Definition and proposition: $\underline{A}$ is quasi free if it satisfies the following equivalent conditions:

1) $\underline{A}$ has cohomological dimension ≤ 1 w.r.t. Hochschild cohomology i.e.
$$H^n(A,M) = 0 \quad \forall n \geq 2, \ \forall M$$
$\Updownarrow$

$H^2(A,M) = 0 \Leftrightarrow$ 2) $\Omega^1 A$ is a projective bimodule

3) Any square zero extension of A is trivial i.e. there exists a lifting homomorphism in which case one has an isomorphism of R with
$$A \oplus M \xrightarrow{\sim} R$$
the semi direct product algebra.

Fact: $\underline{P}$ is projective $\Leftrightarrow$ $\mathrm{Hom}_{A^e}(P, \cdot)$ is exact

$\Leftrightarrow$ $\mathrm{Ext}^1_\Lambda(P, \cdot) = 0$

$\Leftrightarrow$ $\mathrm{Ext}^n_\Lambda(P, \cdot) = 0 \ (n \geq 1)$

Concrete versions of $(2) \Leftrightarrow (3)$:

Any derivation is inner if and only if the universal derivation $d: A \to \Omega^1 A$ is inner

$$\begin{tikzcd} A \arrow[r, "d"] \arrow[dr, "D"'] & \Omega^1 A \arrow[d, dashed, "u"] \\ & M \end{tikzcd}$$

$da = [a, Y] \quad Y \in \Omega^1 A$

$\underline{u}$ bimodule map

$Da = [a, uY]$

Any square zero extension of $\underline{A}$ is trivial i.e. has a lifting homomorphism if and only if there is a lifting homomorphism $A \to RA/IA^2$ where $RA/IA^2 = A \oplus \Omega^2 A$ under the Fedosov product.

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] & R \arrow[r] & A \arrow[l, bend right, "\rho"'] \arrow[r] & 0 \\ 0 \arrow[r] & IA \arrow[r] & RA \arrow[u, "\rho_*"] \arrow[r] & A \arrow[u, equal] \arrow[r] & 0 \end{tikzcd}$$

ρ linear lifting

$\rho(1) = 1$

ρ_* is the unique algebra homomorphism such that $a \mapsto \rho(a)$.

Recall
$$\rho_*(a_0 da_1 \ldots da_{2n}) = \rho(a_0)\, w(a_1,a_2) \ldots w(a_{2n-1}, a_{2n})$$
where $w(a_1,a_2) = \rho(a_1 a_2) - \rho(a_1)\rho(a_2)$

$a_1 \circ a_2 = a_1 a_2 - da_1 da_2$ so $da_1 da_2 = a_1 a_2 - a_1 \circ a_2$ so that $da_1 da_2$ is the curvature of the canonical linear map $A \to RA$.

In the case $I^2 = 0$ we have ρ_* kills IA^2. This means that $A = (RA/IA^2)/(IA/IA^2)$ is the universal square-zero extension of $\underline{A}$.

$$\begin{tikzcd}
0 \arrow[r] & IA/IA^2 \arrow[r] \arrow[d, equal] & RA/IA^2 \arrow[r] \arrow[d, equal] & A \arrow[r] \arrow[dd, equal] & 0 \\
& \Omega^2 A \arrow[d] & A \oplus \Omega^2 A \arrow[d, "\rho_*"] & & \\
0 \arrow[r] & M \arrow[r] & R \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

Remark $Z^1(A,M) = \mathrm{Der}(A, M) = \mathrm{Hom}_{A^e}(\Omega^1 A, M)$

says that $(\Omega^1 A, d)$ is universal with respect to 1-cocycles.

Similarly one can show that

$$Z^n(A,M) = \mathrm{Hom}_{A^e}(\Omega^n A, M)$$

i.e. the n-cochain

$$(a_1, \ldots, a_n) \longmapsto da_1 \ldots da_n \in \Omega^n A$$

is an n-cocycle and $\Omega^n A$ with this n cocycle

is universal with respect to n-cocycles.

How to see the equivalence of 'A projective' and '$d: A \to \Omega^1 A$' inner.

$$0 \to \Omega^1 A \xrightarrow[j]{} A \otimes A \xrightarrow[m]{} A \to 0$$

This sequence of bimodules splits if and only if $\underline{A}$ is projective. A splitting can be specified by either (i) a bimodule section $s: A \to A \otimes A$ for m or ii) a bimodule retraction $p: A \otimes A \to \Omega^1 A$ for j.

$$p(a_1 \otimes a_2) = a_1\, p(1 \otimes 1)\, a_2 = a_1 Y a_2$$

where $Y = p(1 \otimes 1)$.

$$pj\,(a_0 da_1) = a_0\, pj(da_1) = a_0\, p(a_1 \otimes 1 - 1 \otimes a_1)$$

$$pj\,(a_0 da_1) = a_0 (a_1 Y - Y a_1)$$

$$\therefore \quad pj = id \iff da = [a, Y] \quad \forall a$$

Cup product of cochains

L, M, N are bimodules and we have a bimodule map

$$L \otimes_A M \to N$$

$x \otimes y \mapsto x \cdot y$

Then we have a map $C^p(A,L) \otimes C^q(A,M) \xrightarrow{\smile} C^{p+q}(A,N)$

$f(a_1,\dots,a_p) \otimes g(a_1,\dots,a_q) \mapsto f(a_1,\dots,a_p)\cdot g(a_{p+1},\dots,a_{p+q})$

$(f \smile g)(a_1,\dots,a_{p+q}) = f(a_1,\dots,a_p)\, g(a_{p+1},\dots,a_{p+q})$

$\delta(f \smile g) = (\delta f) \smile g + (-1)^{|f|} f \smile \delta g$

$C^{\cdot}(A,L) \otimes C(A,M) \to C(A,N)$

Ex $(a_1,a_2) \mapsto da_1 da_2$
is the cup product $d \smile d$
$d \in C^1(A, \Omega A)$

$\overbrace{d \smile \dots \smile d}^{n}$ is the universal $\underline{n}$ cocycle.

Separable algebras
The following are equivalent

i) A bimodule splitting of

$0 \to \Omega^1 A \to A \otimes A \to A \to 0$

$a_0 da_1 \mapsto a_0(a_1 \otimes 1 - 1 \otimes a_1)$

$a_1 \otimes a_2 \mapsto a_1 a_2$

2) An element $Y \in \Omega^1 A$ such that $da = [a, Y]$ $(a \in A)$

3) An element $Z \in A \otimes A$ such that $Z \in (A \otimes A)^{A}$ (centre of $A \otimes A$) such that $m(Z) = 1$, called a separability element for $\underline{A}$.

Proof:

Z is equivalent to a bimodule map $s: A \to A \otimes A$ such that $ms = 1$ so $3) \Leftrightarrow 1)$

Y is equivalent to a bimodule map $p: A \otimes A \to \Omega^1 A$ with $pj = 1$, so $2) \Leftrightarrow 1)$

Note that Y, Z correspond to the same splitting if and only if
$1 = jp + sm$
$1 \otimes 1 = j(Y) + Z$

Examples: 1) $A = M_n \mathbb{C}$
$Z = \sum_{i=1}^{n} e_{i1} \otimes e_{1i}$

or $Z = \frac{1}{n} \sum_{i,j} e_{ij} \otimes e_{ji}$

2) $A = \mathbb{C}[G]$ group algebra $\underline{G}$ finite

$Z = \frac{1}{|G|} \sum_{g \in G} g^{-1} \otimes g \in A \otimes A$

$Y = \frac{1}{|G|} \sum_{g \in G} g^{-1} dg$

Check $j(Y) = \frac{1}{|G|} \sum_{g \in G} g^{-1}(g \otimes 1 - 1 \otimes g)$
$= 1 \otimes 1 - Z$

Recall that A is quasi-free if and only if

$\iff$ $\Omega^1 A$ is a projective bimodule

$\iff$ Any square-zero extension $A = R/M$, $M^2 = 0$ has a lifting homomorphism $A \to R$.

$\iff$ There is a lifting homomorphism $A \to RA/IA^2$

Connections on right modules E

$$0 \to \Omega^1 A \to A \otimes A \to A \to 0$$

$\otimes E_A$

$$(*) \quad 0 \to E \otimes_A \Omega^1 A \to E \otimes A \to E \to 0$$

$$\xi \otimes a \mapsto \xi a$$

$$\xi\, da \mapsto \xi a \otimes 1 - \xi \otimes a$$

(write $\xi . \omega$ for $\xi \otimes \omega \in E \otimes_A \Omega A$).

There is an obvious section $\xi \mapsto \xi \otimes 1$ on $\underline{m}$

There is a one-to-one correspondence between $\mathbb{C}$-linear sections $s : E \to E \otimes A$ of $\underline{m}$ and $\mathbb{C}$-linear maps $\nabla : E \to E \otimes_A \Omega^1 A$ given by

$$j(\nabla \xi) = \xi \otimes 1 - s(\xi)$$

(regard $E \otimes \Omega^1 A$ as fibred over E)

$$j\big(\nabla(\xi a) - (\nabla \xi) a\big) = \xi a \otimes 1 - \xi \otimes a - s(\xi a) + s(\xi) a$$

$$\therefore \quad j\big(\nabla(\xi a) - (\nabla \xi) a - \xi\, da\big) = -\{ s(\xi a) - s(\xi) a \}$$

$\underline{s}$ is a right module map if and only if

$$\nabla(\xi a) = (\nabla \xi) a + \xi\, da$$

Following Connes, such a ∇ is called a connection on E.

The following data are equivalent

3) connection $\nabla : E \to E \otimes_A \Omega^1 A$

1) right module splitting of $(*)$

2) right module section $s : E \to E \otimes A$

Such things exist if and only if E is projective.

Slight refinement: Let E be a bimodule. Then TFAE

1) bimodule splitting of $(*)$

2) bimodule section $s : E \to E \otimes A$ of $E \otimes A \to E$

3) A linear map $\nabla_r : E \to E \otimes_A \Omega^1 A$

$$j(\nabla(a\xi) - a\nabla\xi) = -(s(a\xi) - as(\xi))$$

where $\quad \nabla_r(\xi a) = (\nabla_r\xi)(a) + \xi da$

$$\nabla_r(a\xi) = a\nabla_r\xi$$

such a thing is called a right connection on E.

Assuming that E is projective as a left module, then the above data exist if and only if E is projective as a bimodule.

Proposition: The following data are equivalent

1) A one cochain $\phi: \bar{A} \to \Omega^2 A$ such that $-\delta\phi = dud$ i.e.

$$a_1\phi(a_2) - \phi(a_1a_2) + \phi(a_1)a_2 + da_1\,da_2 = 0$$

2) A lifting homomorphism of $A \to RA/IA^2$

3) A bimodule splitting of

$$0 \to \Omega^2 A \to \Omega^1 A \otimes A \xrightarrow[m_r]{} \Omega^1 A \to 0$$

$$j(\omega da) = \omega a \otimes 1 - \omega \otimes a$$

4) A right connection $\nabla_r: \Omega^1 A \to \Omega^2 A$

$(\Omega^2 A = \Omega^1 A \otimes_A \Omega^1 A)$

$$\nabla_r(a\omega) = a\nabla_r\omega \; ; \quad \nabla_r(\omega a) = (\nabla_r\omega)a + \omega da$$

(3) $\Leftrightarrow$ (4) seen before.

(1) $\Leftrightarrow$ (2) Recall $RA/IA^2 = A \oplus \Omega^2 A$ with the Fedosov product. A linear lifting $A \to RA/IA^2$ sending $1 \mapsto 1$ is of the form $a \mapsto a + \phi(a)$ where ϕ is a linear map $A \to \Omega^2 A$.

When is this lifting a homomorphism?

$$(a_1 - \phi(a_1)) \circ (a_2 - \phi(a_2))$$

$$= a_1a_2 - da_1da_2 - \phi(a_1)a_2 + (d\phi(a_1))da_2 - a_1\phi(a_2)$$

(modulo forms of degree ≥ 4)

This $= a_1a_2 - \phi(a_1a_2)$

if and only if $\quad \delta\phi + dud = 0$.

$$\nabla_r : \Omega^1 A \to \Omega^2 A$$

$$A \otimes \bar{A}$$

If ∇_r is linear and compatible with left multiplication

$$\nabla_r(a_0 da_1) = a_0\phi(a_1)$$

where $\phi : \bar{A} \to \Omega^2 A$ is arbitrary linear map.

Check that $-\delta\phi = d\nabla d \iff \nabla_r$ satisfies the Leibniz rule with respect to right multiplication

Note $\Omega^1 A \otimes A = A \otimes \bar{A} \otimes A$ is free as an $\underline{A}$ bimodule so that these data exist if and only if $\Omega^1 A$ is a free projective bimodule.

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Volume $\underline{\overline{XV}}$

Examples of quasi free algebras:

$$0 \to \Omega^1 A \to A \otimes A \to A \to \mathbb{C}$$

so that A projective $\Rightarrow$ $\Omega^1 A$ projective

A separable $\Rightarrow$ A quasi free

If $Y \in \Omega^2 A$ satisfies $da = [a, Y]$
equivalently $\delta Y = d$

Then $\phi = d \cup Y$ ie $\phi(a) = (d \cup Y)(a)$
i.e. $\phi(a) = da . Y$
satisfies

$$\begin{aligned} \delta \phi &= \delta(d \cup Y) \\ &= \cancel{\delta(d) \cup Y}^{\,0} + (-1)\, d \cup \delta(Y) \\ &= -\, d \cup d \end{aligned}$$

Can also take $\phi = - Y \cup d$
or
$Y = \frac{1}{2}(d \cup Y - Y \cup d)$

Concrete ways of saying $\underline{A}$ is quasifree

1) $\exists\ \phi : \overline{A} \to \Omega^2 A \quad - \delta\phi = d \cup d$
2) $\exists$ lifting homomorphism $\underline{\ell}$ $A \to RA / IA^2$
3) $\exists$ bimodule splitting $\underline{s}$ of

$$0 \to \Omega^2 A \hookrightarrow \Omega^1 A \otimes A \hookrightarrow \Omega^1 A \to 0$$

4) $\exists$ right connection $\nabla_r : \Omega^1 A \to \Omega^2 A$

$$\nabla_r(a\omega) = a\,\nabla_r \omega$$
$$\nabla(\omega a) = (\nabla\omega)a + \omega da$$

In fact these data are equivalent and

$$\ell(a) = a - \phi(a)$$
$$p(a_0 \otimes a_1 \otimes a_2) = a_0\,\phi(a_1)a_2$$
$$\nabla(a_0 da_1) = a_0\,\phi(a_1)$$
$$\text{i.e.} \quad \phi = \nabla d$$

Examples: 1) A separable: Let $Y \in \Omega^2 A$ be such that $\delta Y = d$ i.e. $da = [a, Y]$

Then $\phi = d \cup Y$ or $-Y \cup d$ or $(1-t)\, d\cup Y + t(-Y\cup d)$

satisfies $-\delta\phi = d \cup d$

2) $A = T(V)$ Here there is a canonical lifting homomorphism $\ell : A \to RA$

which is the unique homomorphism such that $\ell(v) = v \quad \forall v \in V$

Look at $\ell : A \to RA/IA^2$. This has the form
$$\ell(a) = a - \phi(a)$$
where ϕ is unique such that
$$-\delta\phi = d \cup d$$

and $\phi(v) = 0 \quad v \in V$

Let ∇ be the connection
$$\nabla(a_0 da_1) = a_0\,\phi(a_1)$$
$$\phi(v_1 \dots v_n) = \nabla d(v_1 \dots v_n)$$
$$= \nabla\left(\textstyle\sum_{j=1}^n v_1 \dots v_{j-1}\, dv_j\, v_{j+1} \dots v_n\right)$$
$$= \textstyle\sum_{j=1}^n v_1 \dots v_{j-1}\left((\nabla dv_j) v_{j+1} \dots v_n + dv_j\, d(v_{j+1} \dots v_n)\right)$$

Now $\nabla dv_j = \phi(v_j) = 0$ so

$$= \textstyle\sum_{j=1}^{n-1} v_1 \dots v_{j-1}\, dv_j\, d(v_{j+1} \dots v_n)$$
$$= \textstyle\sum_{j<k} v_1 \dots v_{j-1}\, dv_j\, v_{j+1} \dots dv_k\, v_{k+1} \dots v_n$$

<u>Fact</u> Let $A = R/I$ be a square zero extension. Then
1) If A is quasi free then there is a lifting homomorphism $\ell : A \to R$
2) If A is separable then $\exists$ a lifting homomorphism and any two lifting homomorphisms are conjugate by an element of $1 + I$
$$\ell : A \to R \;;\; \ell' : A \to R'$$

The universal case of (1) is $A = RA/IA^2$

The universal case of (2) is

$$A = (QA/J)/J^2/I$$

where
$$QA = A * A \quad \text{and} \quad J = \mathrm{Ker}(QA \to A)$$
the kernel of the folding map.

$$\begin{array}{c} A * A \\ \theta \uparrow \ \ \nearrow \theta^{\gamma} \\ A \end{array}$$

$A = R/I \qquad I^2 = 0$

$A \underset{\ell'}{\overset{\ell}{\rightrightarrows}} R$ gives $QA \xrightarrow{u} R$

$\theta a \mapsto \ell a \qquad u(J) \subseteq I$

$\theta^{\gamma} a \mapsto \ell' a$

$QA/J^2 \to R \qquad \theta a \mapsto \ell a$

$\qquad\qquad\qquad \theta^{\gamma} a \mapsto \ell' a$

Fact $RA = \Omega^{ev} A$ under the Fedosov product

$QA = \Omega A$ under $\circ$

$\theta a = a + da \qquad \theta^{\gamma} a = a - da$

$\gamma(\omega) = (-1)^{|\omega|}\omega$

Check $(a_1 + da_1)(a_2 + da_2) = a_1 \circ a_2 + a_1 da_2 + da_1 a_2 + da_1 da_2$

$= a_1 a_2 + d(a_1 a_2)$

$J^n = \bigoplus_{k \geq n} \Omega^k A$

$\therefore \ QA/J^2 = A \oplus \Omega^1 A$ where on this space the ordinary product is the Fedosov product

Look for $y \in \Omega^1 A$ such that

$(1+y)(\theta a)(1+y)^{-1} = \theta^{\gamma} a$

$\iff (1+y)(a + da)(1-y) = a - da$

$\iff a + ya + da - ay = a - da$

$da = \frac{1}{2}[a, y]$

can solve this in separable algebra.

Refinements of higher order: Let $I \subset R$ be an ideal

$\hat{R} = \varprojlim R/I^{n+1}$

$\widehat{I^k} = \varprojlim I^k/I^{n+1}$

$(\neq (\hat{I})^k)$

<u>Proposition</u> 1) Assume A quasi free. Then any hom $u_0 : A \to R_0 = R/I$ lifts to $u : A \to \hat{R}$

2) Assume A separable and suppose given hom. $u : A \to \hat{R}$, $u' : A \to \hat{R}$ such that $g_0 \in R/I$

$$g_0 u_0 g_0^{-1} = u_0'$$

where $u_0, u_0' : A \to R_0 = R/I$ are the reductions of u, u' modulo $\hat{I}$ ($R_0 = R/I = \hat{R}_0/\hat{I}$).

Then there exists a lifting g of g_0 to an invertible element $g \in R$ such that
$$g u g^{-1} = u'$$

$$\begin{tikzcd} A \arrow[r, shift left] \arrow[r, shift right] \arrow[dr, shift left] \arrow[dr, shift right] & \hat{R} \arrow[d] \\ & R_0 \end{tikzcd}$$

Proof: (Formal consequence of the facts for square zero extensions).

Idea is to use induction on the tower R/I^n which reduces to the case $I^2 = 0$. Pull-back reduces to the case of an extension.

$$\begin{tikzcd} A \arrow[r, shift left] \arrow[r, shift right] \arrow[dr, shift left] \arrow[dr, shift right] \arrow[ddr, shift left] \arrow[ddr, shift right] & \hat{R} \arrow[d] \\ & R_1 \arrow[d] \\ & R_0 \end{tikzcd}$$

g_1, g_1, g_0

R_1 $(= R/I^2)$

Consider

$$\begin{tikzcd} A \times_{R/I} R \arrow[r] \arrow[d] & R \arrow[d] \\ A \arrow[r] \arrow[ur, dashed] & R/I \end{tikzcd}$$

Universal cases

1) If A quasi free, there exists a lifting homomorphism
$$A \to \widehat{RA} = \widehat{\Omega^{ev} A} \quad \text{(Fedosov)}$$

2) If A is separable, there exists an invertible element $g \in \widehat{QA}$ $(= (\widehat{QA}, \sigma))$ such that $g \in 1 + \hat{J}$

$$g\, \theta(a)\, g^{-1} = \theta^{\gamma}(a) \quad (a \in A).$$

To construct (explicitly) a lifting

1) when $\underline{A}$ is quasi free a lifting homomorphism $l : A \to \widehat{RA}$

Existence

$$\begin{tikzcd} & \widehat{RA} \arrow[d] \\ & RA/IA^2 \arrow[d] \\ A \arrow[ur] \arrow[r, "\cong"] & RA/IA \end{tikzcd}$$

2) When $\underline{A}$ is separable to construct an invertible $g \in \widehat{QA}$ such that $g \theta g^{-1} = \theta^{\gamma}$

$\widehat{RA} = \widehat{\Omega^{ev}A}$ with $\circ$; $\widehat{QA} = \widehat{\Omega A}$ with $\circ$

Starting with (1), $\underline{l}$ is a compatible family of lifting homomorphisms $l_n : A \to RA / IA^{n+1}$ $(n \geq 0)$

$$\mathrm{Hom}_{alg}(RA, R) = \{ \rho : A \to R : \rho \text{ linear}, \rho(1) = 1 \}$$

called the based linear maps.

$$\rho_* \longleftrightarrow \rho$$

where $\rho_* : RA \to R$ is the unique homomorphism such that $\rho_*(a) = \rho(a)$ and

$$\rho_*(a_0 da_1 \dots da_{2n}) = \rho(a_0)\,\omega(a_1, a_2) \dots \omega(a_{2n-1}, a_{2n})$$

Such elements span $(IA)^k$ for $n \geq k$.

$$\mathrm{Hom}_{alg}(RA/IA^{n+1}, R) = \{ \rho : A \to R : \rho \text{ based linear map } \omega^{n+1} = 0 \}$$

where $\omega^{n+1}(a_1, \dots a_{2n+2}) = \omega(a_1, a_2)\,\omega(a_3, a_4) \dots \omega(a_{2n+1}, a_{2n+2})$

A lifting homomorphism $l_n : A \to RA / IA^{n+1}$ is the same by Yoneda's lemma as functorial way of associating to any based linear map $\rho : A \to R$ with $\omega^{n+1} = 0$ a homomorphism $\rho^\# : A \to R$ such that $\rho^\# = \rho$ if ρ

is already a homomorphism.

Idea is to mimic Yang Mills theory and look for a functorial vector field X on the space of based linear maps $\rho : A \to R$ such that $-X$ decreases the curvature. Then we show that for the ρ nilpotent curvature the trajectory $e^{+tX}\rho$ exists and its limit as $t \to -\infty$ exists also — $\rho^\#$ is this limit.

Instead of X we look for a derivation D on RA.

$$RA \xrightarrow{\rho_*} R$$

$$\dot{\rho} = \rho_* D \qquad \rho \mapsto \dot{\rho} \in \mathrm{Hom}(\bar{A}, R)$$

A natural vector field on $\mathrm{Hom}_{alg}(RA, R) = \{\rho\}$ is equivalent to a derivation $D : RA \to RA$

D is specified by its restriction to A, which can be any linear map $\bar{A} \to RA$. The possibilities are

$$Da = f_0(a) + f_1(a) + f_2(a) + \dots$$

where $f_n : \bar{A} \to \Omega^{2n} A$

We want $D(RA) \subseteq IA$ so that if $\underline{\rho}$ is a homomorphism from $A \to R$.

$\therefore\ f_0 = 0$

$$D(da_1da_2) = D(a_1a_2 - a_1\circ a_2)$$

$$= D(a_1a_2) - a_1\circ Da_2 - (Da_1)\circ a_2$$

$$= D(a_1a_2) - a_1 Da_2 - (Da_1)a_2 + da_1\, dDa_2 + dDa_1\, da_2$$

$$\equiv f_1(a_1a_2) - a_1f_1(a_2) - f_1(a_1)a_2 \mod (IA)^2$$

Choose f_1 so that this leading term is the curvature da_1da_2.

$$f_1(a_1a_2) - a_2 f_1(a_2) - f(a_1)a_2 = da_1da_2$$

Because $\underline{A}$ is quasi-free, there exists $\phi: A \to \Omega^2 A$ such that $-\delta\phi = dud$ i.e.

$$-a_1\phi(a_2) + \phi(a_1a_2) - \phi(a_1)a_2 = da_1da_2$$

Take $f_1 = \phi$, then $f_2 = f_3 = \dots = 0$.
This defines $\underline{D}$.

Thus having chosen ϕ we then have D the unique derivation of RA such that
$$Da = \phi(a) \quad (\forall a).$$

We have
$$D(da_1da_2) = \begin{cases} da_1da_2 + da_1\, d\phi a_2 \\ + d\phi a_1\, da_2 \end{cases}$$

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One would like to integrate to give e^{tD}. Claim that e^{tD} is defined on RA/IA^{n+1} for all $\underline{n}$.
Use spectral decomposition of D to give e^{tD}.
On $gr^{IA}(RA) = \Omega^{ev}A$ usual product
we have $\underline{D}$ is a derivation, $D = 1$ on $gr_1 = \Omega^2 A$, $D = 0$ on $gr_0 A$
$\therefore$ $D = n$ on $\Omega^{2n}A$

$$D(RA) \subset IA$$
$$(D-1)D(RA) \subset (D-1)(IA) \subseteq IA^2$$
$$\vdots$$
$$(D-n)(D-n-1)\dots(D-1)D(RA) \subset (IA)^{n+1}$$

$$(D-n)\dots(D-1)D = 0 \quad \text{on } RA/IA^{n+1}$$

$$\therefore \quad RA/IA^{n+1} = \bigoplus_{j=0}^{n} \mathrm{Ker}(D-j)$$

contained in $(IA)^j$
complementary to $(IA)^{j+1}$
mod $(IA)^{n+1}$

So we get a canonical isomorphism
$$RA/IA^{n+1} \xrightarrow{\sim} \bigoplus_{j=0}^{n} \Omega^{2j}A$$

$D \longleftrightarrow$ multiplication by j on Ω^{2j}

$$D = \begin{bmatrix} 0 & & & \\ & 1 & & \\ & & 2 & \\ & & & \ddots \end{bmatrix}$$

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$RA/IA^{n+1} = \bigoplus \mathrm{Ker}(D-j)$

This decomposition is compatible with the grading since $\underline{D}$ is a derivation.

Conclusion is that as $n \to \infty$ one has an isomorphism of filtered algebras

$$\widehat{RA} = (\widehat{\Omega^{ev}A}, \circ) \xrightarrow{\sim} (\widehat{\Omega^{ev}A}, \cdot)$$

such that D corresponds to multiplication by $\underline{n}$ on $\Omega^{2n}A$

$$e^{tD} \longleftrightarrow e^{tn} \text{ on } \Omega^{2n} \text{ on } \Omega^{2nA}$$

as $t \to \infty$

$\therefore \lim_{t \to -\infty} e^{-tD}$ is the lifting homomorphism

A quasi free $\quad \phi : \bar{A} \to \Omega^2 A$

$$\delta\phi = d \cup d$$

$$\phi(a_1 a_2) = \phi(a_1) a_2 + a_1 \phi(a_2) + da_1 da_2$$

Define a derivation $\underline{D}$ on RA by $Da = \phi a$ $\forall a \in A$. (Recall RA is free).

$$D(da_1 da_2) = D(a_1 a_2 - a_1 \circ a_2)$$

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$$= da_1 da_2 + d\phi(a_1) da_2 + da_1 d(\phi a_2)$$

$$D(a_0 da_1 da_2 \ldots da_{2n}) = D(a_0 \circ (da_1 da_2) \circ \ldots \circ (da_{2n-1} da_{2n}))$$

$$\begin{aligned} &= \phi(a_0) da_1 \ldots da_{2n} \\ &+ a_0 (d\phi(a_1) da_2 + da_1 d\phi(a_2)) da_3 \ldots da_{2n} \\ &+ a_0 da_1 \ldots da_{2n} \\ &+ \ldots \\ &+ a_0 da_1 \ldots da_{2n-2} (d\phi(a_{2n-1}) a_{2n} + da_{2n-1} d\phi(a_{2n})) \\ &+ a_0 da_1 \ldots da_{2n} \end{aligned}$$

Let $\underline{H}$ be the degree operator

$$H(a_0 da_1 \ldots da_{2n}) = n a_0 da_1 \ldots da_{2n}$$

and let

$$\begin{aligned} L(a_0 da_1 \ldots da_{2n}) &= \phi a_0 da_1 \ldots da_{2n} \\ &+ a_0 d\phi(a_1) da_2 \ldots da_{2n} \\ &+ a_0 da_1 d\phi(a_2) da_3 \\ &+ \ldots \\ &+ a_0 da_1 \ldots da_{2n-1} d\phi(a_{2n}) \end{aligned}$$

Then $\quad D = H + L$

$[H, L] = L \quad$ because $\underline{L}$ raises $\frac{1}{2}$ degree by one

recall from quantum mechanics — H is number operator

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So $e^{-L} H e^{L} = H + e^{-L}[H, e^{L}]$

$$= H + e^{-L}\int_0^1 e^{(1-t)L}[H,L]e^{tL}\,dt$$

$$= H + \int_0^1 e^{-tL}[H,L]e^{tL}\,dt$$

$$= H + \int_0^1 e^{-tL}\, L\, e^{tL}\,dt = H + L = D$$

(nrs ane b group: calculate in the Lie algebra).

Last time we say that there is a canonical algebra isomorphism compatible with the filtration

$$\widehat{RA} \cong \widehat{\Omega^{ev} A} = \prod \Omega^{2n} A$$

$$D \longleftrightarrow H$$

- correspondence of eigenspaces: e'spaces of $\underline{H}$ corr. to those of $\underline{D}$.

Remark is that this isomorphism is given by e^{L}

Fact: D on RA extends to a derivation on QA

$QA = A * A = \Omega A$ with $\circ$

$a \mapsto \theta a = a + da$

$a \mapsto \theta^{\gamma} a = a - da$

A derivation of QA is given by a pair its restrictions to $\theta(A)$ and $\theta^{\gamma}(A)$ which can be arbitrary derivations from these subalgebras to QA.

Claim that $D(\theta a) = \frac{1}{2} da + \phi a + d\phi a$

$$D(\theta^{\gamma} a) = -\frac{1}{2} da + \phi a - d\phi a.$$

Check that $\cancel{D(\theta a_1 \circ \theta a_2)}$ $D\theta(a_1 a_2) = D\theta a_1 \circ \theta a_2 + \theta a_1 \circ D\theta a_2$

and also for θ^{γ}.

Claim that these formulae define a derivation D on ΩA.

$$Da = \tfrac{1}{2} D(\theta a + \theta^{\gamma} a) = \phi a$$

$\therefore$ D extends the D we have on RA.

$$D(da) = \tfrac{1}{2} da + d\phi a$$

On J^n / J^{n+1} $\quad D = n/2.$

As in the case of RA we have a canonical algebra isomorphism

$$\widehat{QA} \simeq \widehat{\Omega A} = \prod_n \Omega^n A$$

$$D \longleftrightarrow H$$

Assume that A is separable. Then we know θ, θ^{τ} $A \to \widehat{QA}$ become conjugate in $\widehat{QA}$. To construct such a conjugacy starting from a $Y \in$ ... with $da = [a, Y]$.

Define D' a derivation on QA by

$D'(\theta a) = [\theta a, Y]$

$D'(\theta^{\tau} a) = -[\theta^{\tau} a, Y]$

Substitute $\theta a = a + da$ etc

$$D'(a + da) = (a+da) \circ Y - Y \circ (a+da)$$
$$= aY + daY - dadY$$
$$- Ya - Yda - dYda$$
$$= [a, Y] + (dYda + dadY) + [Y, da + daY]$$

$$D'(a - da) = -(aY - Ya) + (\quad\quad) - (\quad\quad)$$

$$\phi = \tfrac{1}{2}(d \circ Y - Y \circ d)$$

Conclusion is that D' on QA is $2D$ where D was defined before using ϕ.

Solve the differential equation

$$D'g = -Yg$$

with $g \in \widehat{QA}$ $\quad g = 1 +$ (higher order)

Write $\quad Y = \sum_{n \geq 1}^{\infty} y_n \quad$ with $\quad D'y_n = ny_n$

and let $\quad g = \sum_{n=0}^{\infty} g_n, \quad D'g_n = ng_n, \quad g_0 = 1$

Then $\quad D'g = -Yg \quad$ amounts to the

$$-ng_n = y_1 g_{n-1} + y_2 g_{n-2} + \dots + y_n g_0$$

which may be solved recursively.

$$D'(g\theta g^{-1}) = (-g^{-1} D'g\, g^{-1})\theta g$$
$$+ g^{-1} D'\theta g + g^{-1}\theta D'g$$

$$D'\theta = -[Y, \theta] \qquad D'g = -Yg$$

So $\quad D'(g\theta g^{-1}) = 0$

$\therefore \quad g^{-1}\theta a g \quad$ lies in $D' = 0$ eigenspace and is congruent to a mod J.

$\therefore \quad g^{-1}\theta g = \ell \quad$ the lifting homomorphism $A \to \widehat{RA}$.

Similarly $\quad D'(g\theta^{\tau} g^{-1}) = 0 \quad$ and

$$g^{2}\theta^{\tau} g^{-1} = g\ell g^{-1} = \theta$$

Connections

E right module over A

connection $\nabla : E \to E \otimes_A \Omega^1 A$

$$\nabla(\xi a) = (\nabla \xi) a + \xi\, da$$

This is equivalent to a right module map

$$s : E \to E \otimes A$$

such that $ms = 1$ where m is the multiplication. Now ∇ exists if and only if E is projective.

The difference of two connections is a right module map $E \to E \otimes_A \Omega^1 A$.

If E is a left module, a connection

$$\nabla : E \to \Omega^1 A \otimes_A E$$

$$\nabla(a\xi) = a \nabla \xi + da\, \xi$$

$$s : E \overset{m}{\leftrightarrows} A \otimes E$$

If E is a bimodule, we define a ~~left~~ right connection $\nabla_r : E \to E \otimes_A \Omega^1 A$

$$\nabla_r(\xi a) = (\nabla_r \xi) a + \xi\, da$$

$$\nabla_r(a\xi) = a \nabla_r \xi$$

equivalent to a bimodule liftings with respect to

$$\begin{tikzcd} E \otimes A \arrow[r, "m_r"] & E \arrow[l, bend left, dashed, "s_r"] \end{tikzcd}$$

E projective implies that s, ∇_r exist.

If E is left projective then $E \otimes A$ is left-projective and (∇_r exists $\Rightarrow$ E is a projective bimodule).

Left connection $\nabla_l : E \to \Omega^1 A \otimes E$

$$\nabla_l(a_1 \xi a_2) = a_1 (\nabla_l \xi) a_2 + da_1\, \xi a_2$$

This is equivalent to a bimodule lifting s_l w.r.t.

$$\begin{tikzcd} A \otimes E \arrow[r, "m_l"'] & E \arrow[l, bend right, dashed, "s_l"'] \end{tikzcd}$$

Definition: If E is a bimodule, then a (bimodule) connection in E is a pair (∇_l, ∇_r) consisting of a left and a right connection.

$$\begin{tikzcd} A \otimes E \otimes A \arrow[r, "m_l \otimes 1"] \arrow[d, "1 \otimes m_r"] \arrow[d, bend right, dashed, "1 \otimes s_r"'] & E \otimes A \arrow[d, "m_r"] \\ A \otimes E \arrow[r, "m_l"] & E \arrow[u, bend right, dashed, "s_r"'] \arrow[l, bend left, dashed, "s_l"] \end{tikzcd}$$

$s_r \longleftrightarrow \nabla_r$

$s_l \updownarrow \nabla_l$

If we have (∇_ℓ, ∇_r) then E is a direct summand of $A \otimes E \otimes A$. Hence we have a connection if and only if $\underline{E}$ is projective.

A bimodule lifting of E into $A \otimes E \otimes A$ is equivalent to a connection in E considered as a $A^e = A \otimes A^{op}$ module. This is a finer structure than a connection (∇_ℓ, ∇_r)

Example : $\underline{A}$ separable

$$(*) \qquad \begin{tikzcd} 0 \arrow[r] & \Omega^1 A \arrow[r, "j"'] & A \otimes A \arrow[r, "m"] & A \arrow[l, bend right, "z"'] \arrow[r] & 0 \end{tikzcd}$$

Let z be a separability element. The corresponding retraction of $\underline{j}$ has form

$$a_1 Y a_2 \longleftrightarrow a_1 \otimes a_2 .$$

A free bimodule $A \otimes V \otimes A$ has a canonical connection

$$\nabla_r = 1 \otimes 1 \otimes d : A \otimes V \otimes A \to A \otimes V \otimes \Omega^1 A$$

$$\nabla_\ell = d \otimes 1 \otimes 1 : A \otimes V \otimes A \to \Omega^1 A \otimes V \otimes A$$

If we choose $Y \in \Omega^1 A$ such that $da = [a, Y]$ then we get induced connections on A and $\Omega^1 A$

$$\nabla_\ell a = \nabla_\ell (a 1) = a \nabla_\ell 1 + da$$

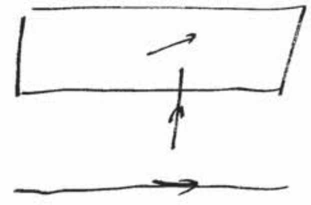
$$\nabla_\ell a = \nabla_\ell (1 a) = (\nabla_\ell 1) a$$

So one takes $\nabla_\ell (1) = -Y$.

$$\nabla_r a = \nabla_r (1 a) = (\nabla_r 1) a + da$$

$$\nabla_r a = \nabla_r (a 1) = a \nabla_r 1$$

So that $\nabla_r 1 = Y$.



E vector bundle

$C^\infty(M)$ M manifold

parallel transport on $\underline{M}$ is lifted to $\underline{E}$ by the connection
Swan's theorem – relates to projective

If M is a bimodule over $\underline{A}$ we consider

$$T_A M = A \oplus M \oplus (M \otimes_A M) \oplus \dots$$

$$\mathrm{Hom}_{alg}(T_A M, R) = \left\{ (u, v) : \begin{array}{l} u : A \to R \text{ is a hom.} \\ v : M \to R \text{ is a bimodule map related to } u \end{array} \right\}$$

polynomial functions on E

$$C^\infty(M) \oplus \Gamma(M,\check{E}) \oplus S_A^2(\Gamma(M,\check{E}))$$

$\Gamma(M,\check{E})$ sections of dual bundle
— obtain the symmetric algebra.

Think of $T_A M$ as the algebra corresponding to the vector bundle.
Idea is to extend derivations on A (manifold) to derivations on $T_A M$ (bundle)

<u>Proposition:</u> If (∇_ℓ, ∇_r) is a connection on E there is a canonical way to extend a derivation
$$D: A \to M$$
where M is a $T_A E$ bimodule to a derivation
$$T_A E \to M$$

Now D induces an A-bimodule map
$$D_*: \Omega^1 A \to M$$
$$D_*(da) = Da$$

$$\begin{tikzcd}
E \arrow[r, "{(\nabla_\ell,\nabla_r)}"] & \Omega^1 A \otimes_A E \arrow[d, "D_* \otimes 1"'] & \oplus & E \otimes_A \Omega^1 A \arrow[d, "1 \otimes D_*"] \\
& M \otimes_A E \arrow[dr] & & E \otimes_A M \arrow[dl] \\
& & M &
\end{tikzcd}$$

right multiplication of $T_A E$ on M

left multiplication

Call the composite $\tilde{D}: E \to M$.

Properties $\quad \tilde{D}(a_1 \xi a_2) = a_1 \tilde{D}\xi a_2 + Da_1 \xi a_2 + a_1 \xi Da_2$

This is what you need to see that

$$\tilde{D}(\xi_1 \ldots \xi_n) = \sum_{j=1}^n \xi_1 \ldots \xi_{j-1} (\tilde{D}\xi_j) \xi_{j+1} \ldots \xi_n$$

is well defined on $T_A E$.

$$\xi_1 \ldots \xi_n = \xi_1 \otimes_A \xi_2 \otimes_A \cdots \otimes_A \xi_n$$

Connections on the bimodule $\Omega^1 A$ are non-commutative analogues of connections on the tangent bundle TM to a manifold <u>M</u>.

E
$\downarrow \pi$ $\quad$ vector bundle $\leftarrow$ the algebra $S_A(\Gamma(M,\check{E}))$
M

$$\begin{array}{c} A = C^\infty(M) \\ \downarrow \\ T_A(E) \end{array}$$

E is a projective bimodule

$T_A(\Omega^1 A) = \Omega A$

(i.e. the non-commutative analogue of the tangent bundle of the manifold).

$$\overbrace{\Omega^1 A \otimes_A \Omega^1 A \otimes \dots \otimes \Omega^1 A}^{n} \xrightarrow{\sim} \Omega^n A$$

$$A\otimes \bar{A} \otimes_A (A \otimes \bar{A}) \otimes_A \cdots \otimes (A\otimes \bar{A})$$

Curious: In the commutative case one has two algebras attached to an A-module E, namely

$S_A(E)$ $\quad \Lambda_A E$

When $E = \Omega^1_A$ we have

$S_A(\Omega^1_A)$ = algebra of polynomial functions on the tangent bundle

$\Lambda\, \Omega^1 A = \Omega_A$ de Rham complex of differential forms.

In the non commutative case one only has $T_A(\Omega^1 A)$ which is ΩA.

Exercise Show that an algebra A is quasi free iff there is a derivation δ on ΩA (not a superderivation) such that $\delta a = da$

As $\Omega^1 A \otimes_A \Omega^1 A = \Omega^2 A$ a left connection ∇_ℓ, a right connection ∇_r and d are all operators from $\Omega^1 A$ to $\Omega^2 A$ satisfying

$$\nabla_\ell(a_1 \xi a_2) = a_1(\nabla_\ell \xi)a_2 + (da_1)\xi a_2$$

$$d(a_1 \xi a_2) = a_1 d\xi a_2 + (da_1)\xi a_2 + a_1 \xi da_2$$

$$\nabla_r(a_1 \xi a_2) = a_1(\nabla_r \xi)a_2 + a_1 \xi da_2$$

So that ∇_r a ~~left~~ right connection $\Rightarrow \nabla_r + d$ is a left connection

∇_ℓ a left connection $\Rightarrow \nabla_\ell - d$ a right connection.

Hence there is a 1-1 connection between ∇_ℓ and ∇_r given by $\nabla_\ell = d + \nabla_r$

Recall that a connection on $\Omega^1 A$ is a pair (∇_ℓ, ∇_r)
Define the torsion of the connection to be

$$\tau : \nabla_\ell - (d + \nabla_r) : \Omega^1 A \to \Omega^2 A$$

is a bimodule map.

We've seen that ∇_r is equivalent to a $\phi : \bar{A} \to \Omega^2 A$ such that $-\delta\phi = d \cup d$ via the formula $\nabla_r(a_0 da_1) = a_0\, \phi(a_1)$

Thus a left connection ∇_ℓ is equivalent to a $\phi : \bar{A} \to \Omega^1 A$ such that

$$\nabla_\ell(a_0 da_1) = d(a_0 da_1) + a_0 \psi(a_1)$$
$$= da_0 da_1 + a_0 \psi(a_1)$$

$$-\delta\psi = d \cup d$$

The torsion τ of (∇_ℓ, ∇_r) is

$$\tau(a_0 da_1) = a_0 (\psi - \phi)(a_1)$$

$\psi - \phi$ is a derivation

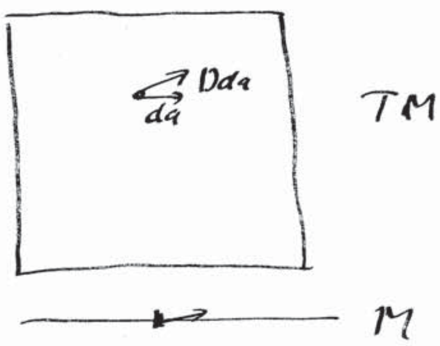
For a separable algebra

$$0 \to \Omega^1 A \underset{j}{\rightleftarrows} A \otimes A \to A \to 0$$

$$a_0 \psi a_1 \longleftarrow a_0 \otimes a_1$$

The torsion is non-zero.

Geodesic flow:



Recall that a connection on E allows one to extend derivations $D : A \to M$ a $T_A(E)$ bimodule to a derivation $T_A(E) \to M$.

$$\begin{tikzcd}
E \arrow[r, "(\nabla_\ell, \nabla_r)"] & \Omega^1 A \otimes_A E \arrow[d, "D_* \otimes 1"] & \oplus & E \otimes_A \Omega^1 A \arrow[d, "1 \otimes D^*"] \\
& M \otimes_A E \arrow[dr] & & E \otimes_A M \arrow[dl] \\
& & M &
\end{tikzcd}$$

Take $E = \Omega^1 A$ and $D = d : A \to \Omega A = T_A(\Omega^1 A)$

We then have, given (∇_ℓ, ∇_r) on $\Omega^1 A$ we then obtain a derivation $\underline{D}$ on ΩA, with $Da = da$

$$\Omega^1 A \xrightarrow{(\nabla_\ell + \nabla_r)}$$

$$D(a_0 da_1) = (\nabla_\ell + \nabla_r)(a_0 da_1)$$

$$D\, da = (\nabla_\ell + \nabla_r)(a)$$

$$\nabla_r(a_0 da_1) = a_0 \phi(a_1)$$

$$\nabla_\ell(a_0 da_1) = da_0 da_1 + a_0 \psi(a_1)$$

$$\begin{cases} \nabla_r(da) = \phi(a) \\ \nabla_\ell(da) = \psi(a) \end{cases}$$

$$\left.\begin{aligned} Dda &= \varphi(a) + \psi(a) \\ D(da) &= 2\left(\tfrac{\varphi+\psi}{2}a\right) \end{aligned}\right) \quad \theta = (\nabla_\ell - T/2) - (d + \nabla_r + T/2)$$

Torsion free connection – obtained by averaging.

<u>The exponential map</u>

$$TM \longrightarrow M \times M$$

$$(x, v) \longmapsto (x, \exp_x v)$$

$$\widehat{\Omega A} \xleftrightarrow[\sim]{?} \widehat{QA} = A * A$$

Define $QA \xrightarrow{\alpha} \widehat{\Omega A}$ by

$$\begin{aligned} \theta a &= a + da \longmapsto a \\ \theta^r a &= a - da \longmapsto e^X a \end{aligned}$$

where $\underline{X}$ is the geodesic flow

$$\begin{aligned} Xa &= da \\ Xda &= (\nabla_\ell + \nabla_r)(a) \end{aligned}$$

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To the first order we have

$$a \xmapsto{\alpha} \frac{1+e^X}{2} a = a + \tfrac{1}{2} da$$

$$da \longmapsto \frac{1-e^X}{2} a = -\tfrac{1}{2} da$$

So that $\underline{\alpha}$ induces on the associated graded algebras the map

$$\Omega A \to \Omega A : \quad \omega \mapsto (-1/2)^{|\omega|}\omega$$

when one completes, the conclusion is that $\underline{\alpha}$ induces an isomorphism

$$\widehat{QA} \xrightarrow{\sim} \widehat{\Omega A}$$

$$\widehat{\Omega^{u} A} \xrightarrow{\sim} \widehat{RA}$$

$$\widehat{\Omega A} \xrightarrow{\sim} \widehat{QA}$$

$$\widehat{QA} \xrightarrow{\sim} \widehat{\Omega A}$$

$$\theta a \mapsto e^{\frac{1}{2}X} a$$

$$\theta^r a \mapsto e^{-\frac{1}{2}X} a$$

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