Lectures courses by Daniel G Quillen

D. Cyclic Homology II: Cyclic cohomology and Karoubi Operators, Hilary Term 1991

125 pages of notes. The lecture course is concerned with cyclic homology and traces and considers the following topics. The differential graded algebra of noncommutative differential forms. The Karoubi operator and the analogue of Hodge theory. Connes B operator, and the Greens operator. The Hodge decomposition. Augmented algebras. Morita equivalence of algebras. Noncommutative harmonic forms. Hochschild homology and cyclic homology. The double complex and cyclic homology. Spectral sequences. Connes Tsygan bicomplex. Connes exact sequence. Reduced Hochschild homology. Universal properties of tensor algebra and free algebra. The Fedosov product. Cuntz's algebra. Filtrations with respect to ideals and products. Traces on RA. Bianchi's identities. Characterisations of traces. Karoubi's operator on cochains. Cohomology formulas for cochains. From $(IA)^n$-adic traces to odd cyclic cohomology. Intermission: the analogue of the de Rham complex in noncommutative geometry. The Lefschetz, Atiyah–Hodge and Grothendieck theorem on nonsingular maximal ideal spaces. The smooth algebra is defined via the lifting process for nilpotent extensions. Periodic cyclic homology, homology of smooth and commutative algebras. Quasi free algebras and lifting. Analogue of Zariski–Grothendieck. Universal differential algebra for RA; passage to linear functionals. The complex $X(RA)^*$. The noncommutative analogues of nonsingular varieties. Connes' connections, and Chern character classes. Splitting of connection sequence. Connections on $\Omega^1 R$. Fedosov's construction. Poisson structures on manifolds. Weyl algebras and commutative algebras. Index theorems on $\mathbf{R}^n$. Fedosov product and the Stone–von Neumann relations.

Editor's remark The lecture notes were taken during lectures at the Mathematical Institute on St Giles in Oxford. There have been subsequent corrections, by whitening out writing errors. The pages are numbered, but there is no general numbering system for theorems and definitions. For the most part, the results are in consecutive order, although in one course the lecturer interrupted the flow to present a self-contained lecture on a topic to be developed further in the subsequent lecture course. The note taker did not record dates of lectures, so it is likely that some lectures were missed in the sequence. The courses typically start with common material, then branch out into particular topics. Quillen seldom provided any references during lectures, and the lecture presentation seems simpler than some of the material in the papers.

- D. Quillen, Cyclic cohomology and algebra extensions, *K-Theory* **3**, 205–246.

- D. Quillen, Algebra cochains and cyclic cohomology, *Inst. Hautes Etudes Sci. Publ. Math.* **68** (1988), 139–174.
- J. Cuntz and D. Quillen, Cyclic homology and nonsingularity, *J. Amer. Math. Soc.* **8** (1995), 373–442.

Commonly used notation

k a field, usually of characteristic zero, often the complex numbers

A an associative unital algebra over k, possibly noncommutative

$\bar{A} = A/k$ the algebra reduced by the subspace of multiples of the identity

$\Omega^n A = A \otimes (\bar{A} \otimes \ldots \otimes \bar{A})$

$\omega = a_0 da_1 \ldots da_n$ an element of $\Omega^n A$

$\Omega A = \oplus_{n=0}^{\infty} \Omega^n A$ the universal algebra of abstract differential forms

e an idempotent in A

d the formal differential (on bar complex or tensor algebra)

b Hochschild differential

b', B differentials in the sense of Connes's noncommutative differential geometry

λ a cyclic permutation operator

K the Karoubi operator

$\circ$ the Fedosov product

G the Greens function of abstract Hodge theory

N averaging operator

P the projection in abstract Hodge theory

D an abstract Dirac operator

∇ a connection

I an ideal in A

V vector space

M manifold

E vector bundle over manifold

τ a trace

$T(A) = \oplus_{n=0}^{\infty} A^{\otimes n}$ the universal tensor algebra over A

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Cyclic cohomology & Karoubi Operator

A algebra over $\mathbb{C}$ associative with 1

A cochain $f(a_0, \ldots, a_n)$ is normalised when it vanishes whenever $a_i = 1$ for some $i \geq 1$.

These cochains with differential b compute $HH^n(A)$.

Not cyclically invariant. The Karoubi operator is the substitute for the cyclically permuting with sign for normalized ~~[illegible]~~ cochains.

Begin with chains.

Prop: There is a unique DG algebra

$$\Omega A = \bigoplus_n \Omega^n A$$

with $d : \Omega^n A \to \Omega^{n+1} A$ and $\Omega^0 A = A$ and with following properties:

1) Given any differential graded algebra $\underline{R}$,

and a homomorphism $u: A \to R^0$ there is a unique homomorphism of differential graded algebras $\Omega A \to R$ extending $\underline{u}$.

2) The map $A \otimes \bar{A}^{\otimes n} \longrightarrow \Omega^n A$ where $\bar{A} = A/\mathbb{C}$

$$(a_0, a_1, \ldots, a_n) \longmapsto a_0 da_1 \ldots da_n$$

is an isomorphism for all $\underline{n}$.

Each of these properties characterises ΩA up to canonical isomorphism.

Now ΩA is called the differential graded algebra of non commutative differential forms.

Think of $A \otimes \bar{A}^{\otimes n}$ as normalized n-chains. So we have a DG algebra structure on all normalized n-chains.

Operator $\underline{b}$: Notice that $\Omega^{n+1}A \cong \Omega^n A \otimes \bar{A}$

$$\omega da \longleftarrow\!\shortmid\, \omega \otimes a$$

Definition: $b: \Omega^{n+1}A \to \Omega^n A$ is given by

$$b(\omega da) = (-1)^n(\omega a - a\omega)$$

In chain notation this becomes

$$b(a_0, \ldots, a_n) = \sum_0^{n-1}(-1)^i(\ldots, a_i a_{i+1}, \ldots)$$
$$+ (-1)^n(a_n a_0, \ldots, a_{n-1})$$

e.g. $\omega = a_0 da_1$

$$b(a_0 da_1 da_2) = (-1)\{a_0 da_1 a_2 - a_2(a_0 da_1)\}$$

$$= (-1)(a_0 d(a_1 a_2) - a_0 a_1 da_2 - a_2 a_0 da_1)$$

$$= a_0 a_1 da_2 - a_0 d(a_1 a_2) + a_2 a_0 da_1$$

Proof of "$b^2 = 0$" (Put $b = 0$ on $\Omega^n A$, $n \leq 0$)

$\Omega^{n+2}A$ has generators $\omega da_1 da_2$ where $\omega \in \Omega^n A$.

Now $b^2(\omega da_1 da_2) = b((-1)^{n+1}(\omega da_1 a_2 - a_2 \omega da_1))$

$$= (-1)^{n+1} b(\omega d(a_1 a_2) - \omega a_1 da_2 - a_2 \omega da_1)$$

$$= (-1)^{n+1}(-1)^n(\omega a_1 a_2 - a_1 a_2 \omega - \omega a_1 a_2 + a_2 \omega a_1$$
$$+ a_1 a_2 \omega + a_2 \omega a_1)$$

$$= 0$$

Fact: $H_n(\Omega A, b) =$ Hochschild homology $HH_n(A)$

Definition: Karoubi operator $K : \Omega^n A \to \Omega^n A$ is defined by

$$\begin{cases} K(\omega da) = (-1)^{|\omega|} da\,\omega & (|\omega| \geq 1) \\ K = id & \text{on } \Omega^0 A \end{cases}$$

In terms of chains,

$$K(a_0 da_1 \ldots da_n) = K(\omega da_n)$$

$$= (-1)^{n-1}(da_n a_0 da_1 \ldots da_{n-1})$$

$$= (-1)^{n-1}(d(a_n a_0) da_1 \ldots da_{n-1} - a_n da_0 \ldots da_{n}$$

$$K(a_0, \ldots, a_n) = (-1)^{n-1}(1, a_n a_0, a_1, \ldots, a_{n-1})$$

$$+ (-1)^n (a_n, a_0, \ldots, a_{n-1}) \qquad (n \geq 1)$$

Formula $\quad bd + db = 1 - K$

In degree $n \geq 1$ we calculate

$$(bd + db)(\omega da) = b(d\omega da) + d((-1)^{|\omega|}(\omega a - a\omega))$$

$$= (-1)^{|\omega|+1}(d\omega a - a d\omega) + (-1)^{|\omega|}\{d\omega a - da\omega + (-1)^{|\omega|}\omega da - a d\omega\}$$

$$= \omega da - (-1)^{|\omega|} da\,\omega$$

$$= \omega da - K(\omega da)$$ hence the formula.

Write Ω^n for $\Omega^n A$

$$\begin{tikzcd} \Omega^0 \arrow[r, shift left, "d"'] & \Omega^1 \arrow[l, shift left, "b"'] \arrow[r, shift left, "d"'] & \Omega^2 \arrow[l, shift left, "b"'] \arrow[r, shift left, "d"'] & \Omega^3 \arrow[l, shift left, "b"'] \arrow[r, shift left] & \arrow[l, shift left] \end{tikzcd}$$

Analogue of Hodge theory "$1 - K =$ Laplacian"

Use the 'spectral theory' of the Laplacian $1-K$ to decompose and construct an algebraic analogue of the Hodge decomposition.

Properties of K:

1) $\underline{K}$ commutes with $\underline{b}$, $\underline{d}$

2) K is automorphism

On Ω^n we have

3) $K^n = 1 + bK^{-1}d$
4) $K^{n+1} = 1 - db$
5) $(K^n - 1)(K^{n+1} - 1) = 0$

Proof:

1) $bd + db = 1 - K$
$b(1-K) = b^2d + bdb = bdb$
$(1-K)b = bdb + db^2 = bdb$
$\therefore$ $1-K$ and hence $\underline{K}$ commutes with b
similarly with $\underline{d}$

2) Consider the subspace $d\Omega^{n-1} \subseteq \Omega^n$

$$d(a_0 da_1 \ldots da_{n-1}) = da_0 \ldots da_{n-1}$$

One has the isomorphism

$$\bar{A}^{\otimes n} \to d\Omega^{n-1}$$
$$(a_1, \ldots a_n) \longmapsto da_1 \ldots da_n$$

and $K = \lambda$ relative to this isomorphism.

$\therefore$ $K^n = 1$ on $d\Omega^{n-1}$

(λ - cyclic permutation with sign).

But we have exact sequences

$$\begin{tikzcd}
0 \arrow[r] & d\Omega^{n-1} \arrow[r] & \Omega^n \arrow[r] & d\Omega^n \arrow[r] & 0 \\
0 \arrow[r] & \bar{A}^{\otimes n} \arrow[r] \arrow[u, "1"'] & A \otimes \bar{A}^{\otimes n} \arrow[r] \arrow[u, "2"'] & \bar{A}^{\otimes n+1} \arrow[r] \arrow[u, "2"'] & 0
\end{tikzcd}$$

$$x \longmapsto 1 \otimes x$$

$$\therefore \quad \Omega^n / d\Omega^{n-1} \xrightarrow{\sim} \bar{A}^{\otimes n+1}$$
$$K \longleftarrow\!\shortmid \lambda$$

K is an automorphism on the quotient space of order $n+1$.

This also proves (5)!

$\therefore$ $\underline{K}$ is an automorphism on the subspace and on the quotient space and hence is an isomorphism.

$$(K^n - 1)(K^{n+1} - 1)\Omega^n \subseteq (K^n - 1)d\Omega^{n-1} = 0$$

3) $K^n(a_0 da_1 \ldots da_n) = (-1)^{n-1}(da_n a_0 da_1 \ldots da_{n-1})$
$= (-1)^{n+2}(da_{n-1} da_n a_0 da_1 \ldots da_{n-2})$
$=$
$= (-1)^{n(n-1)}(da_1 \ldots da_n a_0)$

$= a_0 da_1 \dots da_n + (-1)^n b(da_1 \dots da_n da$

$= a_0 da_1 \dots da_n + bK^{-1}(da_0 \dots da_n)$

$= a_0 da_1 \dots da_n + bK^{-1}d(a_0 da_1 \dots da_n)$

4/ $K^n = 1 + bK^{-1}d$

$\Rightarrow K^{n+1} = K + bd = 1 - db$

Note $K(da_0 \dots da_n) = (-1)^{n} da_n da_0 \dots da$

$\therefore$ on $d\Omega^n$ one has that $K^{n+1} = 1$

Definition: Connes' B-operator

$$B = \sum_{j=0}^{n} K^j d$$

Proposition (as above)

On Ω^n we have $K^n = 1 + bK^n d$

and hence $K^{n+1} = 1 - db$

$\therefore (K^n - 1)(K^{n+1} - 1) = 0$

Hence the Karoubi operator is an automorphism

$K(\text{polynomial in } K) = I$

(*) $K^{n(n+1)} = 1 + bB = 1 - Bb$

$\left\{ \begin{array}{l} Bd = dB = 0, \quad B^2 = 0 \\ BK = KB = B \end{array} \right\}$ from the defn of $\underline{B}$.

Proof of (*)

$K^{n(n+1)} - 1 = \sum_{j=0}^{n} (K^n)^j (K^n - 1)$

$= \sum_{j=0}^{n} (K^n)^j bK^n d$

$= b\sum_{j=0}^{n} K^{nj+n} d$

But $K^n = K^{-1}$ on $d\Omega^n$ so we get

$= bB$

Also $K^{n(n+1)} - 1 = \sum_{j=0}^{n-1} (K^{n+1})^j (K^{n+1} - 1)$

$= -\sum_{j=0}^{n-1} (K^{n+1})^j db$

But $K^{n+1}=K$ on $d\Omega^{n-1}$, hence we get

$K^{n(n+1)}-1 = -Bb$

"<u>Hodge decomposition</u>"

$$\begin{tikzcd}
\Omega^0 \arrow[r, shift right, "d"'] & \Omega^1 \arrow[l, shift right, "b"'] \arrow[r, shift right, "d"'] & \Omega^2 \arrow[l, shift right, "b"'] \arrow[r, shift right, "d"'] & \Omega^3 \arrow[l, shift right, "b"'] \arrow[r, shift right, "d"'] & {} \arrow[l, shift right, "b"']
\end{tikzcd}$$

"Laplacian" $bd + db = (d+b)^2 = 1-K$

Recall $(K^n-1)(K^{n+1}-1) = 0$ so by linear algebra we can decompose into generalized eigenvalues.

The roots of $(x^n-1)(n^{n+1}-1)$ are the $\underline{n}$ n^{th} roots of unity and the $\underline{n+1}$ $(n+1)^{st}$ roots of unity. These two sets have only $\underline{1}$ in common. Hence the polynomial has simple roots given by ζ satisfying $\zeta \neq 1$, $\zeta^n=1$ or $\zeta^{n+1}=1$ and the double root 1.

We can conclude that Ω^n is the direct sum of the eigenspaces $Ker(\zeta-K)$,

here $\zeta \neq 1$ and the generalized eigenspace $Ker(I-K)^2$ corresponding to root 1.

Doing this for all $\underline{n}$ gives a decomposition

$\Omega = Ker((1-K)^2) \oplus \bigoplus_{\zeta} Ker(\zeta-K)$

where ζ runs over all roots of unity not equal to $\underline{1}$.

Each of these summands is stable under d & $\underline{b}$ since $\underline{K}$ commutes with $\underline{b}$ and d.

Consider $Ker(\zeta-K)$ with $\zeta \neq 1$ a root of unity. We have

$db + bd = 1-K = 1-\zeta$

on this complex.

$\therefore$ $Ker(\zeta-K)$ is contractible as a complex with either differential.

$d(\frac{1}{1-\zeta}b) + (\frac{1}{1-\zeta}b)d = 1$

contracts $\underline{d}$.

Picture of $Ker(\zeta-K)$: Let ζ be a primitive m^{th} root of unity. Then

$K^n_\zeta = \mathrm{Ker}\,(\zeta - K)^n \neq 0$ if $n \equiv 0$ or -1

$$K^{2m}_\zeta \underset{d}{\overset{b}{\rightleftarrows}} K^{2m-1}_\zeta \quad 0 \quad \dots \quad 0 \quad K^m_\zeta \rightleftarrows K^{m-1}_\zeta$$

$\underline{b}$, $\underline{d}$ isomorphisms, but not inverse to one another.

Define P to be the spectral projection associate to $\underline{K}$ and the eigenvalue 1. Put $P^\perp = I$

$$\Omega = \mathrm{Ker}(1-K)^2 \oplus \bigoplus_{\zeta \neq 1} \mathrm{Ker}(\zeta - K)$$

Define G, the analogue of the Green's operator, to be the inverse of $1-K$ on $P^\perp\Omega$ and extend it by ze on $P\Omega$.

Theorem One has a decomposition
$$\Omega = P\Omega \oplus P^\perp\Omega$$
stable under b, d such that $P^\perp\Omega$ is contractible with respect to either differential b, $\underline{d}$. Further, one has a decomposition

$$P^\perp\Omega = bP^\perp\Omega \oplus dP^\perp\Omega$$
where
$$\begin{cases} b: & dP^\perp\Omega \xrightarrow{\sim} bP^\perp\Omega \\ d: & bP^\perp\Omega \xrightarrow{\sim} dP^\perp\Omega \end{cases}$$
isomorphisms.

of: We have $P^\perp = G(1-K) = G(bd+db)$ so that
$$P^\perp = b(Gd) + d(Gb)$$
so that the $\underline{b}$-complex is contractible
$$P^\perp = (bG)d + d(Gb)$$
and the $\underline{d}$-complex is contractible. So that on $P^\perp\Omega$ we have the
$$1 = bGd + dGb$$
with bGd and dGb orthogonal. Hence they are orthogonal idempotents. Clearly
$$\mathrm{Im}\,b = \mathrm{Ker}\,b, \quad \mathrm{Im}\,d = \mathrm{Ker}\,d$$
and
$$P^\perp\Omega = bP^\perp\Omega \oplus dP^\perp\Omega$$

$\therefore$ $b : dP^{\perp}\Omega \to bP^{\perp}\Omega$ is onto and hence an isomorphism.

Hence bGd projects onto $bP^{\perp}$ and similarly dGb projects onto $dP^{\perp}$

To be more concrete we use the fact $K^n = 1$ on $d\Omega^n$ so that

$$Pd\Omega = \begin{cases} \text{fixed point subspace} \\ (d\Omega)^K \quad \text{for } \underline{K} \end{cases}$$

$$P^{\perp}d\Omega = (1-K)\,d\Omega$$

Hodge decomposition takes the following form

$$\Omega = P\Omega \oplus (1-K)d\Omega \oplus b(1-K)\Omega$$

so $\Omega^n = P\Omega^n \oplus (1-\lambda)\bar{A}^{\otimes n} \oplus b(1-\lambda)\bar{A}$

using the isomorphism $d\Omega^n \cong \bar{A}^{\otimes n+1}$

$$da_0 \dots da_n \longmapsfrom (a_0, \dots, a_n)$$
$$K \longleftrightarrow \lambda$$

Lemma: Let T be an operator of finite order on vector space V with $T^m = I$. Then we have a decomposition

$$V = V^T \oplus (I-T)V.$$

Let $\underline{P}$ be the projection on V^T with kernel $(I-T)V$ and let $G = 0$ on V^T and $G = (I-T)^{-1}$ on $(I-T)V$. Then

$$P = \frac{1}{m}\sum_{j=0}^{m-1} T^j$$

$$G = \frac{1}{m}\sum_{j=0}^{m-1}\left(\frac{m-1}{2} - j\right)T^j$$

Formula for P on Ω^n is

$$I - P = (Gd)b + b(Gd)$$

$$Gd = \frac{1}{n+1}\sum_{j=0}^{n}\left(\frac{n}{2} - j\right)K^j d$$

Hodge theory for Ω

$$\Omega^0 \underset{d}{\overset{b}{\rightleftarrows}} \Omega^1 \underset{d}{\overset{b}{\rightleftarrows}} \Omega^2 \rightleftarrows$$

$L = db + bd = 1 - K$ Laplacian

On Ω^n, K satisfies $(K^n-1)(K^{n+1}-1) =$

$$0 \to d\Omega^{n-1} \to \Omega^n \to d\Omega^n \to 0$$

$$\bar{A}^{\otimes n+1} \qquad\qquad \bar{A}^{\otimes n+1}$$
$$\lambda \qquad\qquad \lambda$$

Now 1 is a double root so that we have

$$\Omega = \mathrm{Ker}\,(\underline{I-K})^2 \oplus \mathrm{Im}\,(1-K)^2$$

Define $\underline{P}$ to be the projection on Ker $(I-K$

"harmonic forms"

Green's function G $\begin{pmatrix} 0 & 0 \\ 0 & (1-K)^{-1} \end{pmatrix}$

$P^\perp$: $I-P$

Now $I-P = G(I-K) = G(bd+db)$

commutes with $\underline{b}$ and $\underline{d}$ so that $\underline{G}, \underline{P}$ as well.

$$b(Gd) + (Gd)b = I-P$$
$$(Gb)d + d(Gb) = I-P$$

So that $\underline{P}$ is homotopic to the identity respect to either differential $\underline{b}$ or $\underline{d}$.

$P^\perp\Omega$ is contractable with respect to $\underline{b}$ & $\underline{d}$

have the following decomposition

$$P^\perp\Omega = dP^\perp\Omega \oplus bP^\perp\Omega$$

b: $dP^\perp\Omega \xrightarrow{\sim} bP^\perp\Omega$ with inverse Gd and

d: $bP^\perp\Omega \xrightarrow{\sim} dP^\perp\Omega$ with inverse bG

But these maps are not inverses of one another.

That is $dP^\perp\Omega = P^\perp(d\Omega)$?

On $(d\Omega)^n = d\Omega^{n-1}$ $K^n = 1$

we can decompose into eigenspaces

$$d\Omega = (d\Omega)^K \oplus \mathrm{Im}\,(1-K)$$
$$= \mathrm{Ker}\,(\underline{I-K}) \oplus \mathrm{Im}\,(1-K)$$

$$(d\Omega)^n \cong (\bar{A}^{\otimes n})^\lambda \oplus (1-\lambda)\bar{A}^{\otimes n}$$

On this space $(d\Omega)^n$, $\underline{P}$ and $\underline{G}$ are given by

$$P_\lambda = \frac{1}{n}\sum_{j=0}^{n-1} \lambda^j$$

$$G_\lambda = \frac{1}{n}\sum_{j=0}^{n-1}\left(\frac{n-1}{2}-j\right)\lambda^j$$

We have the following formula

$$P = 1-(Gd)b - b(Gd)$$

where

$$Gd = \frac{1}{n+1}\sum_{j=0}^{n}\left(\frac{n}{2}-j\right)K^j d$$

Important case: $\underline{A}$ augmented i.e. equipped with a homomorphism $\varepsilon : A \to \mathbb{C}$. Put $\mathcal{A} = \operatorname{Ker}\varepsilon$. Then

$$A = \mathbb{C}\oplus\mathcal{A} = \tilde{\mathcal{A}}$$

$\mathcal{A}$ is a non-unital algebra

Then $\underline{A}$ is the ~~algebra~~ obtained by adjoining an identity to the non-unital algebra $\mathcal{A}$.

The categories of augmented algebras and non-unital algebras are thus equivalent.

Morita aspect of cyclic theory:

A $\quad$ $M_n(A)$

are Morita equivalent algebras

$$a \mapsto \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$$

non-unital homomorphism.

Not-Morita invariant is ΩA $\quad$ H^{DR}

$$H_n^{DR}(A) = H_n(\Omega A/[\,,\,], d)$$

Something which is Morita invariant is

$$\operatorname{Ker}\{\Omega(\tilde{A}) \longrightarrow \Omega(\mathbb{C})\}$$

- defined for (unital &) non-unital algebras

Take $A = \tilde{\mathcal{A}} = \mathbb{C}\oplus\mathcal{A}$

Then

$$\Omega^n A \cong A\otimes\bar{A}^{\otimes n}$$

$$= \mathcal{A}^{\otimes n+1}\oplus\mathcal{A}^{\otimes n}$$

Denote an element of $\Omega^n A$ by a pair (x,y)
$x \in A^{\otimes n+1}$, $y \in A^{\otimes n}$.

$$b(x,y) = (bx + (1-\lambda)y, -b'y)$$

$$d(x,y) = (0,x)$$

$$A^{\otimes n+1} \oplus A^{\otimes n} \xrightarrow{\sim} \Omega^n A$$

$$(a_1, \dots a_n) \longmapsto da_1 \dots da_n$$

$$(a_0, \dots a_n) \longmapsto a_0 da_1 \dots da_n$$

$$b(1, a_1, \dots a_n) = (a_1, \dots a_n) - (1, b' \quad)$$

$$+ (-1)^n (a_n, a_1, \dots a_{n-1})$$

Achieve a mapping cone construction

$$\begin{tikzcd}
{} \arrow[d] & {} \arrow[d] & {} \\
A^{\otimes 3} \arrow[d, "b"'] & A^{\otimes 3} \arrow[l, "1-\lambda"'] \arrow[d, "b'"] & {} \\
A^{\otimes 2} \arrow[d, "b"'] & A^{\otimes 2} \arrow[l, "1-\lambda"'] \arrow[d, "b'"] & {} \arrow[l, "N"'] \\
A & A \arrow[l, "1-\lambda"'] \arrow[d] & {} \arrow[l, "N"'] \\
{} & \mathbb{C} & {}
\end{tikzcd}$$

N - averaging operator

Hence $bd + db = 1 - K$ suggests

$$bd(x,y) = b(0,x) = ((1-\lambda)x, -b'x)$$

$$db(x,y) = (0, bx + (1-\lambda)y)$$

so

$$(1-K)(x,y) = ((1-\lambda)x, (1-\lambda)y + (b-b')x)$$

$$\therefore \quad K(x,y) = (\lambda x, \lambda y - (b-b')x)$$

$b-b'$ - cross-over term in b.

Now to evaluate P we consider

$$I-P = (Gd)b + b(Gd)$$

$$(Gd)(x,y) = G(0,x) \quad (0, G_\lambda x)$$

$$b(Gd)(x,y) = b(0, G_\lambda x) = ((1-\lambda)G_\lambda x, -b'G_\lambda x)$$

$$= ((1-P_\lambda)x, -b'G_\lambda x)$$

$$Gd\,b(x,y) = d(bx + (1-\lambda)y, -b'y)$$

$$= (0, \; G_\lambda(bx + (1-\lambda)y))$$

$$= (0, \; G_\lambda bx + (1-P_\lambda)y)$$

$$P(x,y) = (P_\lambda x, \; P_\lambda y - G_\lambda bx + b'G_\lambda x)$$

$P(x,y) = (x,y)$ if and only if

$P_\lambda x = x$ i.e. x is λ-invariant

... i.e. $\lambda x = x$ (so $b'G_\lambda x = 0$)

and $y = P_\lambda y - G_\lambda bx$ i.e. $y + G_\lambda bx$ is λ invariant.

$$y + G_\lambda bx = P_\lambda y = P_\lambda(y + G_\lambda bx)$$

To study the complementary part, i.e. the harmonic forms we consider

$(n \geq 1)$

$$P\Omega) \quad 0 \to \underset{\bar{A}^{\otimes n}}{\underset{\lambda}{d\Omega^{n-1}}} \to \underset{K}{\Omega^n} \to \underset{\bar{A}^{\otimes n+1}}{\underset{\lambda}{d\Omega^n}} \to 0$$

$$0 \to Pd\Omega^{n-1} \to P\Omega^n \xrightarrow{d} Pd\Omega^n \to 0$$

direct summand of exact sequence

$$(\bar{A}^{\otimes n})^\lambda \qquad\qquad (\bar{A}^{\otimes n+1})^\lambda$$

Recall that on Ω^n, $Pd = \sum_{j=0}^{n} \frac{1}{n+1} K^j d$

$$= \frac{1}{n+1} B$$

$B^2=0$ and $Bb + bB = 0$

Hence we get the same exact sequence with B. Define the reduced cyclic complex $\bar{C}C(A)$ to be

$$P\Omega/\mathrm{Ker}(B : P\Omega \to P\Omega)$$

$= \mathrm{Im}\, B$ up to a dimension shift

$$\bar{C}C^n(A) = (\bar{A}^{\otimes n+1})^{\lambda}$$

we have the following exact sequence of complexes

$$\Sigma \bar{C}C \xrightarrow{B} P\Omega \to \bar{C}C \to 0$$
$$\oplus\, \mathbb{C}[0]$$

from which we derive the exact sequence of Connes. Define $\bar{HC}^n(A) = H^n(\bar{C}C(A), b)$.

Recall

$$H(\Omega, b) = HH(A) \quad \text{Hochschild}$$
$$\|$$
$$H(P\Omega, b)$$

because $P^{\perp}\Omega$ contractible

$$\bar{HC}_n(A) \xrightarrow{B} HH_{n+1}(A) \to \bar{HC}_{n+1}(A) \to \bar{HC}_{n-1}(A) \longrightarrow$$

<u>Mixed</u> <u>complexes</u>

Definition: A mixed complex is a graded vector space (homology setting)

$$M = \oplus M_n$$

equipped with operators b, B of degrees -1 and 1 respectively, satisfying

$$b^2 = B^2 = bB + Bb = 0$$

Ex/ $(\Omega A, b, B)$

We usually take $M_n = 0$ for $n < 0$. We have the point of view that in a mixed complex M, M is primarily a complex (M, b) with the map $\underline{B}$ as extra structure.

The ordinary homology of M is the usual b-homology $H(M) = H(M, b)$.

In our example $H(\Omega A, b) = HH(A)$ the Hochschild homology.

We introduce the double complex

$$\begin{tikzcd} q \\ {} \arrow[u] \arrow[r] & p \end{tikzcd}$$

$$\begin{tikzcd} & M_2 \arrow[d, "b"'] & M_1 \arrow[l] \arrow[d, "b"] \\ M_2 \arrow[d] & M_1 \arrow[l] \arrow[d, "b"] & M_0 \arrow[l, "B"'] \\ M_1 & M_0 \arrow[l, "B"'] & 0 \arrow[l, "B"'] \end{tikzcd}$$

$K_{pq} = M_{q-p}$

b - vertical differential
B - horizontal differential

Obvious automorphism by periodicity S of degree $(-1,-1)$ which shifts down and left

Set $\mathcal{B}_{\geq}(M)$ = total complex supported in $p \geq 0$
$q \geq p \geq 0$

$\mathcal{B}_{\geq}(M)_n = M_n \oplus M_{n-2} \oplus M_{n-4} \oplus \dots$
(finite sum)

with the differential $b + B$

Definition: The cyclic homology in M is defined to be $HC(M) = H(\mathcal{B}_{\geq 0}(M), b+B)$

We have an exact sequence of complexes

$$0 \to (M, b) \to \mathcal{B}_{\geq 0}(M) \xrightarrow[S]{} \mathcal{B}_{\geq 0}(M) \to 0$$

$\left\{ \begin{array}{l} \text{first column} \\ p = 0 \end{array} \right.$ induced by the shift in the periodic complex.

$$m_n + m_{n-2} + \dots \longmapsto m_{n-2} + m_{n-4} + \dots$$

Corresponding to this short exact sequence of complexes is the long exact sequence

$$\begin{tikzcd} H_n(M) \arrow[r, "I"] & HC_n(M) \arrow[r, "S"] & HC_{n-2}(M) \arrow[dll, "B"] \\ H_{n-1}(M) \arrow[r] & HC_{n-1}(M) & \end{tikzcd}$$

This is called the Connes' exact sequence of the mixed complex M.

Ex $HC(A) \overset{\text{def}}{=} HC_n(\Omega A, bB)$

is the cyclic homology of A

$$\to HH_n(A) \xrightarrow{I} HC_n(A) \xrightarrow{S} HC_{n-2}(A) \xrightarrow{B} HH_{n-1}(A)$$

Spectral sequences - standard constructions on double complex

$$E^2_{pq} = H^h_p\, H^v_q \longrightarrow H^{total}$$

$$H^v_p\, H^h_q \longrightarrow H^{total}$$

Consider the case where B is exact

$$H^h_p(K) = \begin{cases} 0 & p > 0 \\ M/BM & p = 0 \end{cases}$$

h - horizontal

v - vertical

In this case we ~~have~~ obvious a map (augmentation)

$$\mathcal{B}_{\geq 0}(M) \longrightarrow M/BM$$

is a quasi-isomorphism i.e. it induces isomorphisms on homology groups.

$$HC_n(M) = H_n(M/BM, b)$$

$$\begin{tikzcd}
0 & M_2/BM_1 \arrow[l] \arrow[d] & M_2 \arrow[l] \arrow[d, "b"] & M_1 \arrow[l] \arrow[d] & M_0 \arrow[l] \\
0 & M_1/BM_0 \arrow[l] & M_1 \arrow[l] \arrow[d] & M_0 \arrow[l] & \\
 & & M_0 & &
\end{tikzcd}$$

Consider again $M = \Omega A$ with b, B.
Recall that $\Omega A = P\Omega A \oplus P^{\perp}\Omega A$
compatible with b, B.

$$\begin{aligned} \Omega A &= P\Omega A \oplus P^{\perp}\Omega A \\ b &= b \oplus b \\ B &= (n+1)d \oplus 0 \end{aligned}$$

Since $B = \sum_1^n K^i d$ on Ω^n
$= (n+1)Pd$ on Ω^n

$P^{\perp}\Omega A$ is contractible with respect to $\underline{b}$ and $\underline{d}$
$P^{\perp} = (Gd)b + b(Gd)$

$H(P^\perp \Omega A, b) = 0$

$H(P\Omega A, b) = H(\Omega A, b) = HH(A)$

$H(P\Omega A, B) = H(P\Omega A, d) = H(\Omega A, d)$

$$H(\Omega A, d) = \begin{cases} \mathbb{C} & n=0 \\ 0 & \text{else} \end{cases}$$

$H(P\Omega A, B) = H(P\Omega A, d)$ because $B = (n+1)d$ on $P\Omega^n A$

$0 \to d\Omega^{n-1}A \to \Omega^n A \to d\Omega^n A \to 0$
gives the induction for the d-homology, or splitting the short exact sequences.

$$HC(P\Omega A) \underset{\uparrow}{=} HC(\Omega A) \underset{\text{def}}{=} HC(A)$$

$(b+B)(hd) + (hd)(b+B)$ $\quad Ph = hP = 0$
$= bhd + hdb = P^\perp$

hd is a contracting homotopy for the cyclic complex
$\mathcal{B}_{\geq 0}(P^\perp \Omega A)$

Conclude ΩA, $P\Omega A$ have the same ordinary homology, cyclic homology and $\underline{B}$ is contingent

What is true for non unital algebras $\mathcal{A}$?
Connes – Tsygan bicomplex

$$\begin{tikzcd}
\mathcal{A}^{\otimes 3}_{\lambda} \arrow[d] & \mathcal{A}^{\otimes 3} \arrow[l] \arrow[r, dash] \arrow[d, "b"] & \mathcal{A}^{\otimes 3} \arrow[d, "-b'"] & \mathcal{A}^{\otimes 3} \arrow[l, "N"'] \arrow[d, "b"] \\
\mathcal{A}^{\otimes 2}_{\lambda} \arrow[d] & \mathcal{A}^{\otimes 2} \arrow[l] \arrow[r] \arrow[d, "b"] & \mathcal{A}^{\otimes 2} \arrow[d, "-b'"] & \mathcal{A}^{\otimes 2} \arrow[l, "N"'] \arrow[d, "b"] \\
\mathcal{A} & \mathcal{A}^{\otimes 1} & \mathcal{A} \arrow[l, "1-\lambda"'] & \mathcal{A} \arrow[l, "N"']
\end{tikzcd}$$

~~$\mathcal{A}$~~

$$\lambda(a_0, \ldots a_n) = (-1)^n (a_n, a_0, \ldots a_{n-1})$$

$$N = \sum_{\substack{\text{cyclic}\\ \text{group}}} \lambda^i$$

Def^n $HC_n(\mathcal{A})$ = homology of this complex

Tsygan proved that this is a double complex

Task was to show that the squares anticommute. Complex is periodic via shift to the right.

$$\begin{cases} \operatorname{Ker} N = \operatorname{Im}(1-\lambda) \\ \operatorname{Im} N = \operatorname{Ker}(1-\lambda) \end{cases}$$

in the case of characteristic zero.

It is also a resolution of the cyclic complex $(\mathcal{A}_\lambda^{\otimes n+1}, b)$ which is the vertical sequence at the extreme left. We can use the bicomplex to obtain the Connes exact sequence (basic idea is that the periodicity is manifested)

<u>Connes exact sequence</u>

$$\begin{tikzcd} HH_n(\mathcal{A}) \arrow[r, "I"] & HC_n(\mathcal{A}) \arrow[r, "S"] & HC_{n-1}(\mathcal{A}) \arrow[dll, bend left] \\ HH_{n-1}(\mathcal{A}) \arrow[r] & \cdots \end{tikzcd}$$

where $HH_n(\mathcal{A})$ the Hochschild homology of the first two columns which form a subcomplex of the periodic complex.

What is true for unital algebras? $\mathcal{A}$

Everything above is true. Also here the b' complex is acyclic because we can introduce a contracting homotopy s

$$s(a_1, \to a_n) = (1, a_1, \to a_n)$$

satisfying

$$b's + sb' = id$$

Define $\quad B = (1-\lambda)sN : \mathcal{A}^{\otimes n} \to \mathcal{A}^{\otimes n+1}$

$$\begin{tikzcd} {} & {} \arrow[l, "1-\lambda"'] \\ & {} \arrow[u, "s"] \arrow[r, dash, "N"] & {} \end{tikzcd}$$

It satisfies

$$\begin{cases} B^2 = 0 \\ Bb + bB = 0 \end{cases}$$

$$\begin{tikzcd}
A^{\otimes 3} \arrow[d] & A^{\otimes 3} \arrow[d, "b"] & \\
A^{\otimes 2} \arrow[d, "b"] & A^{\otimes 2} \arrow[d, "b"] \arrow[ul, "B"'] & \\
A & A \arrow[ul, "B"'] & {} \arrow[ul, "B"']
\end{tikzcd}$$

This complex may be mapped into the previous one. (at the level of total complexes)
This is the Connes (b,B) bicomplex

$$\begin{tikzcd}
A^{\otimes 3} \arrow[d, "b"] & A^{\otimes 2} \arrow[l, "B"'] \arrow[d, "b"] \\
A^{\otimes 2} \arrow[d, "b"] & A \arrow[l, "B"'] \arrow[d, "b"] \\
A^{\otimes 1} & {}
\end{tikzcd}$$

One has $B(a_0, \dots, a_n) = \sum_{i=0}^{n} (-1)^{ni} (1, a_i, \dots, a_n, a_0, \dots, a_{i-1})$
$+ \sum_{i=0}^{n} (-1)^{(i+1)n} (a_i, \dots, a_n, a_0, \dots, a_{i-1})$

given by the formula $(1-\lambda)sN = B$

Now we have the normalized (b,B) bicomplex

$$\begin{tikzcd}
A \otimes \bar{A}^{\otimes 2} \arrow[d, "-b"] & A \otimes \bar{A} \arrow[l, "B"'] \arrow[d, "b"] & A \arrow[l] \\
A \otimes \bar{A} \arrow[d, "b"] & A \arrow[l, "B"'] & \\
A & &
\end{tikzcd}$$

This is just $\mathcal{B}_{\geq 0}(\Omega A, b, B)$ the mixed complex

We have three bicomplexes in this case for the cyclic homology.
They are the Connes Tsygan, the

Connes (b,B) complex and the normalised (b,B) complex.

$$\begin{tikzcd}
\text{Connes Tsygan} \arrow[r, leftrightarrow] & \text{Connes } (b,B) \arrow[r, two heads] & \text{Connes normalised } (b,B)
\end{tikzcd}$$

These induce isomorphisms on the vertical (b) homology and also on cyclic homology.

[Relations: Given $\mathcal{A}$ non unital we can let $A = \tilde{\mathcal{A}} = \mathbb{C} \oplus \mathcal{A}$.]

What is the analogue of $(\mathcal{A}_{\lambda}^{\otimes n}, b)$ in the unital setting?

The $\underline{B}$ sequence is not exact but we recall

$$\Omega A = P\Omega A \oplus P^{\perp}\Omega A$$

giving

$$\mathcal{B}_{\geq 0}(\Omega A) \sim \mathbb{B}_{\geq 0}(P\Omega A)$$

$\underline{B}$ is almost exact on $P\Omega A$
$(H_0(P\Omega A, B) = \mathbb{C})$

If we take $P\bar{\Omega} A = P\Omega A/\mathbb{C}[0]$ then $\underline{B}$ becomes exact and we have a complete analogue so we have a reduced cyclic complex
$(P\bar{\Omega}A/\operatorname{Im} B) = (\bar{A}_{\lambda}^{\otimes n+1}, b)$

Facts: The mixed complexes $\bar{\Omega}A$ or $P\bar{\Omega}A$ gives the socalled reduced Hochschild homology

$$\overline{HH}(A) \overset{\text{def}^n}{=} H(\bar{\Omega}A, b)$$

and the reduced cyclic homology

$$\overline{HC}_n(A) = H_n(\mathcal{B}_{\geq 0}(\bar{\Omega}A), b+B)$$

$$\cong H_n(\bar{A}_{\lambda}^{\otimes n+1}, b)$$

because B is exact.

The reduced complexes are not Morita invariant!

$P\Omega^2 \qquad P\Omega^1 \qquad P\bar{\Omega}^0$

$P\Omega^1 \qquad P\bar{\Omega}^0$

$P\bar{\Omega}^0$

$$0 \to \mathbb{B}_{\geq 0}(\mathbb{C}[0]) \to \mathbb{B}_{\geq 0}(P\Omega) \to \mathbb{B}(P\bar{\Omega}) \to 0$$

gives exact sequence of complexes.

and hence a long exact sequence

$$0 \to HC_{2n+1}(A) \to \overline{HC}_{2n+1}A \to \mathbb{C} \to HC_{2n}A \to$$
$$\to \overline{HC}_{2n}A \to 0$$

Relations in the case where $A = \tilde{\mathcal{A}} = \mathbb{C} \oplus \mathcal{A}$

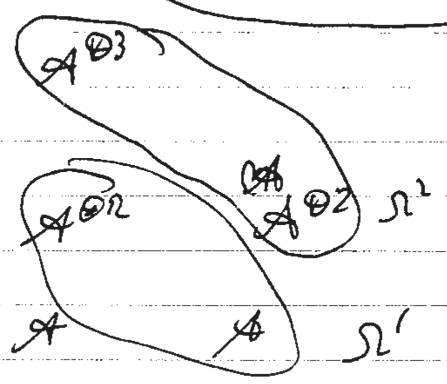
Total complex of { Connes-Tsygan bicomplex of $\mathcal{A}$ }

$\cong \quad \bar{\Omega}A \qquad \mathbb{B}_{\geq 0}(\bar{\Omega}A, b, B)$

$\Omega^n A = \mathcal{A}^{\otimes n+1} \oplus \mathcal{A}^{\otimes n}$ augmented case

$a_0 da_1 \ldots da_n \longleftarrow (a_0, \ldots, a_n)$

$da_1 \ldots da_n \longleftarrow (a_1, \ldots, a_n)$



in Connes-Tsygan bicomplex

$$\therefore \quad HC_n(\mathcal{A}) = \overline{HC}_n(A)$$

$$HH_n(\mathcal{A}) = \overline{HH}_n(A)$$

$$\overline{HC}_n(A) = \mathrm{Ker}\left(HC_n(A) \xrightarrow{\varepsilon} HC_n(\mathbb{C})\right)$$

with a similar description for the Hochschild homology.

The following is obvious

$$\overline{A}_\lambda^{\otimes n} = \mathcal{A}_\lambda^{\otimes n}$$

The interesting point in all of this is the following:

Let $\mathcal{A}$ be unital. We can form complexes giving cyclic homology

$$B_{\geq 0}(\Omega A, b, B)$$

$$B_{\geq 0}(\underset{\tilde{\Omega} A}{\cancel{\bar{\Omega}\tilde{A}}}, b, B) \qquad \Uparrow$$

$\tilde{\Omega}\mathcal{A} = \mathrm{Ker}(\Omega(\tilde{\mathcal{A}}) \to \Omega\mathbb{C} = \mathbb{C}[0])$

$=$ non unital DGA generated by $\mathcal{A}$

If we take the corresponding double complex

$B_{\geq 0}(\tilde{\Omega}\mathcal{A}) =$ Connes Tsygan of $\mathcal{A}$

The point is that there are the same.

Example: Let $A = \mathbb{C}$. As a non-unital algebra one may write $A = \mathbb{C}e$ where $e^2 = e$

$$\mathrm{Hom}_{\substack{\text{nonunital}\\ \text{algebra}}}(\mathbb{C}, R) = \{r \in R : r^2 = r\}$$

$\tilde{A} = \mathbb{C} \oplus \mathbb{C}e \qquad e^2 = e$

$$\begin{tikzcd} \tilde{\Omega} A & e \arrow[r, "B"] & de \arrow[r, "B"] & de^2 \\ & & ede \arrow[ul, "b"] & ede^2 \arrow[l, "b"] \end{tikzcd}$$

$$\Omega A \qquad e \qquad 0 \qquad 0$$

$\tilde{\Omega}\mathcal{A} =$ non unital DG algebra generated by $\mathcal{A}$

$= \mathcal{A} \oplus (\underline{\mathcal{A}^{\otimes 2} \oplus \mathcal{A}}) \oplus (\mathcal{A}^{\otimes 3} \oplus \mathcal{A}^{\otimes 2}) \oplus$

$\qquad a \qquad a_0 da_1 \;\; da \qquad a da_1 da_2 \;\; da_1 da_2$

$= \mathrm{Ker}\{\Omega(\tilde{\mathcal{A}}) \to \Omega(\mathbb{C}) = \mathbb{C}[0]\}$

Let $\mathcal{A} = \mathbb{C}$, considered as a non unital algebra.

$A = \tilde{\mathbb{C}} = \mathbb{C} + \mathbb{C}e \qquad$ where $e^2 = e$

Picture of ΩA

$$\begin{tikzcd}
 & 0 & 1 & 2 & 3 & 4 \\
A & 1 & de \arrow[dl, "b"'] & de^2 \arrow[dl, dash, dotted] & de^3 & de^4 \arrow[dl] \\
e & e & (2e-1)de \arrow[ul, "b"] & (2e-1)de^2 & (2e-1)de^3 & \arrow[ul]
\end{tikzcd}$$

$b(wda) = (-1)^{|w|}(wa-aw)$

so that
$$\left\{\begin{array}{l} e^2=e \Rightarrow de\,e+ede=de \\ \Rightarrow \quad ede = de(1-e) \\ \qquad de.e = (1-e)de \end{array}\right.$$

$ede^2 = de^2e \qquad\qquad de^2 = dede$

$b(de^{2n}) = -[de^{2n-1},e] = -de^{2n-1}e+ede^{2n-1}$

$= -(1-e)de^{2n-1}+ede^{2n-1} = (2e-1)de^{2n-1}$

$b((2e-1)de^{2n}) = -[(2e-1)de^{2n-1},e]$

$= -(2e-1)de^{2n-1}e+e(2e-1)de^{2n-1}$

$= -(2e-1)(1-e)de^{2n-1}+ede^{2n-1} = de^{2n-1}$

$\therefore \quad b: \Omega^{2n}A \xrightarrow{\sim} \Omega^{2n-1}A$ for $n \geq 1$.

$$HH_n(\mathbb{C}+\mathbb{C}e) = \begin{cases} \mathbb{C}+\mathbb{C}e & n=0 \\ 0 & n>0 \end{cases}$$

Hochschild homology adds for direct sums

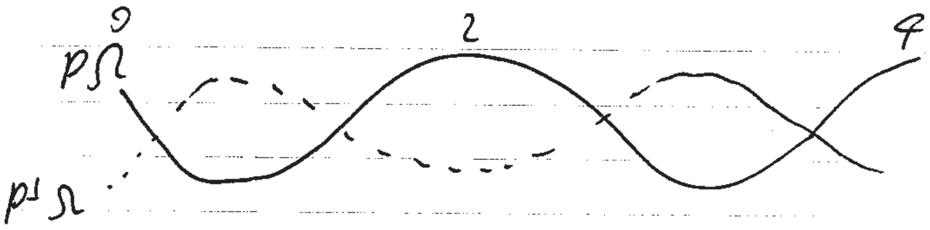
$\mathbb{C}+\mathbb{C}e = \mathbb{C}\oplus\mathbb{C}$

General fact: If A is a separable algebra (product of matrix algebras) then

$$HH_n(A) = \begin{cases} A/[A,A] & n=0 \\ 0 & n>0 \end{cases}$$

$K(de^{2n}) = -de^{2n}$ $\qquad K\ \lambda$ on $d\Omega^n$

$K(de^{2n-1}) = de^{2n-1}$



$K^2 = 1$ in this case and $K = 1$ on $P\Omega$, $K = -1$ on $P^\perp\Omega$.

In general we do not have K of finite order i.e. it has nilpotent parts in the Jordan form.

Consider $\tilde\Omega(\mathcal{A}) \longrightarrow \Omega(\mathcal{A})$ compatible with b, B. One knows that this induces a quasi isomorphism on the $\mathcal{B}_{\geq 0}$ complexes.

$\mathcal{B}_{\geq 0}(\tilde\Omega\mathcal{A})$

Connes Tsygan complex

$$\begin{array}{|ll}
\mathcal{A}^{\otimes 3} & \\
\mathcal{A}^{\otimes 2} & \mathcal{A} \\
\mathcal{A} & \\
\hline
\end{array}$$

$$\begin{tikzcd}
{} \arrow[d] & {} \arrow[d] & {} \arrow[d] \\
\Omega^2 A \arrow[d, "b"] & \Omega^1 A \arrow[l, "B"'] \arrow[d, "b"] & \mathbb{C}e \arrow[l, "B"'] \\
\Omega^1 A \arrow[d, "b"] & \mathbb{C}e \arrow[l, "B"'] & \\
\mathbb{C}e & &
\end{tikzcd}$$

$\mathcal{B}_{\geq 0}(\tilde\Omega\mathbb{C})$

$A = \mathbb{C} + \mathbb{C}e$

$$\begin{tikzcd}
\mathcal{B}_{\geq 0}(\tilde\Omega\mathbb{C}) \arrow[d] \\
\mathcal{B}_{\geq 0}(\Omega\mathbb{C})
\end{tikzcd}$$

$$\begin{array}{|lll}
\downarrow & & \\
0 & 0 & \mathbb{C} \\
0 & \mathbb{C} & \\
\mathbb{C} & & \\
\hline
\end{array}$$

Question: Find the cycle for $b+B$ in $\prod_{n\geq 0}\Omega^{2n}A$ starting with $e \in \Omega^0 A$.

Since b is an isomorphism $\Omega^{2n} \to \Omega^{2n-1}$ such a cycle is unique.

$$\begin{tikzcd}
{} \arrow[d, "b"] \arrow[d, dashed, bend right] \\
{} & {} \arrow[l, "B"']
\end{tikzcd}$$

$$(b+B)\left(e + \sum_{n\geq 1} c_n (2e-1)\, de^{2n\geq 1}\right)$$

$$B\left((2e-1)de^{2n}\right) = \sum_{j=0}^{2n} \kappa^j\, d\left((2e-1)de^{2n}\right)$$
$$= (2n+1)\, 2\, de^{2n+1}$$

so that $\quad c_n + 2(2n-1)c_{n-1} = 0$

giving $\quad c_n = \frac{1}{2}(-1)^n 2^n (2n-1)!!$
$$= \tfrac{1}{2}(-1)^n (2n)!/n!$$

cycle
$$e + \tfrac{1}{2}\sum (-1)^n \frac{(2n)!}{n!}(2e-1)\, de^{2n}$$

Besides ΩA there are ≥ two other algebras generated by A of interest in cyclic theory. These include RA, QA.

$$RA = T(A) \,/\, (\text{ideal generated by } 1_{T(A)} - 1_A)$$
$$= \bigoplus_{n\geq 0} A^{\otimes n}$$

If you choose a splitting of
$$0 \to \mathbb{C} \to A \to \bar{A} \to 0$$
then
$$RA \xrightarrow{\sim} T(\bar{A})$$
so RA is a free algebra.

Universal property of RA

$$\mathrm{Hom}_{alg}(RA, R) = \left\{ \rho : \begin{array}{l} \rho : A \to R \text{ linear} \\ \rho(1_A) = 1_R \end{array} \right.$$

RA comes with a canonical linear map $\rho : A \to RA$ such that $\rho(1) = 1$ $\quad (A = T(A)^{(1)} \subseteq TA \twoheadrightarrow RA)$

There is an obvious homomorphism $RA \twoheadrightarrow A$ – take $\pi : RA \to A$

$$\pi \circ \rho = \mathrm{id}_A \qquad \text{(universal property)}$$

Set $\quad IA = \mathrm{Ker}\{RA \to A)$

The following extension of algebras
$$0 \to IA \longrightarrow RA \to A \to 0$$

is called the universal extension of A.

$$\begin{tikzcd} 0 \arrow[r] & J \arrow[r] & E \arrow[r] & A \arrow[r] \arrow[l, dashed, bend right, "\rho'"'] & 0 \end{tikzcd}$$

$\exists$ section ρ' $\rho'(1) = 1$ as we have vector spaces. Then the universal property gives an extension of A by IA, inducing the extension of A by J.

Fedosov Given a DG algebra $\Omega = \oplus \Omega^n$ we define a $\mathbb{Z}/2$ graded algebra the Fedosov product of Ω to be

$$\omega * \eta = \omega\eta - (-1)^{|\omega|} d\omega d\eta$$

which is an associative product.

The identity is preserved but the grading changes but we still get a $\mathbb{Z}(2)$ graded algebra

stranline

$\deg(d\omega d\eta)| = \deg(\omega\eta)| + 2$

$\mathbb{Z}(2)$ grading. $\Omega = \Omega^{ev} \oplus \Omega^{odd}$

Proposition: $RA \xrightarrow{\sim} \Omega^{ev} A$ equipped with the Fedosov product.

We have the canonical map $\rho: A \to RA$ with curvature of ρ $\omega(a_1, a_2) = \rho(a_1 a_2) - \rho(a_1)\rho(a_2)$

$$\omega : \bar{A} \otimes \bar{A} \to IA$$

$$\Omega^{2n} A \to RA$$

$$a_0 da_1 - da_{2n} \longmapsto \rho(a_0)\omega(a_1, a_2) \dots \omega(a_{2n-1}, a_{2n})$$

well-defined $A \otimes \bar{A}^{2n} \to RA$ giving

$$\bigoplus_{n \geq 0} \Omega^{2n} A \xrightarrow{\sim} RA$$

~~$a_0 da_1 a_2$~~ $a_1, a_2 \mapsto \rho(a_1), \rho(a_2)$

$$a_1 a_2 - da_1 da_2 \longmapsto \rho(a_1)\rho(a_2) = \rho(a_1 a_2) + \omega(a_1, a_2)$$

$\therefore$ $a_1 * a_2 \mapsto \rho(a_1)\rho(a_2)$

which motivates the definition of the Fedosov product.

Defⁿ QA The Cuntz algebra QA is defined to be $A*A$ the free product of A with itself.

$$A \overset{u}{\underset{v}{\rightrightarrows}} R \qquad A*A \to R$$

There are two canonical homomorphisms

$$A \overset{\iota}{\underset{\bar\iota}{\rightrightarrows}} QA \qquad \begin{aligned} pa &= a^+ = \tfrac{1}{2}(\iota a + \bar\iota a) \\ qa &= a^- = \tfrac{1}{2}(\iota a - \bar\iota a) \end{aligned}$$

Proposition: $QA \cong \Omega A$ with the Fedosov product.

$$\Omega^n A \to QA$$
$$a_0 da_1 \dots da_n \longmapsto a_0^+ a_1^- \dots a_n^-$$

$$RA = T(A)/(1_{T(A)} - 1_A) \qquad T(A) = \bigoplus_{n \geq 0} A^{\otimes n}$$

free algebra

$$\rho : A \to T(A) \twoheadrightarrow R(A)$$

The universal property of $R(A)$ together with ρ

$$\begin{tikzcd} 1_A \arrow[dr, "\text{linear map}"] & A \arrow[r] \arrow[dr] & RA \arrow[d, dashed, "\exists!\ \text{hom}"] \\ & 1_R & R \end{tikzcd}$$

Taking $R = A$ and $A \xrightarrow{id} A$ we get a canonical homomorphism $RA \to A$. We set IA to be the kernel of this map.

Curvature $\omega(a_1, a_2) = \rho(a_1 a_2) - \rho(a_1)\rho(a_2)$.

$$\omega : \bar{A}^{\otimes 2} \longrightarrow IA$$

Define: Fedosov product on ΩA by
$$\omega * \eta = \omega\eta - (-1)^{|\omega|} d\omega\, d\eta$$

$(-1)^{|\omega|} = 1$ on Ω^{even}

Defⁿ: $\Phi : \Omega^{even} \to RA$

$$\Omega^{even} = \bigoplus (A \otimes \bar{A}^{\otimes 2n})$$
$$a_0 da_1 \dots da_{2n} \longmapsto \rho(a_0)\omega(a_1, a_2) \dots \omega(a_{2n-1}, a_{2n})$$

Proposition: Φ is an algebra isomorphism when Ω^{even} has the Fedosov product.

Further Φ gives an isomorphism

$$\oplus_{n\geq m}\Omega^{2n}A \xrightarrow{\sim} IA^m$$

Proof: We use the universal property of RA to define a left RA-module structure on $\Omega^{even}A$.

$$a \longmapsto a-(da)d \in \mathrm{End}(\Omega^{ev}A)$$

This linear map induces, by the universal property a homomorphism

$$RA \longrightarrow \mathrm{End}(\Omega^{ev}A)$$

giving a left RA-module structure on $\Omega^{ev}A$ which is unique satisfying

$$\rho(a)\eta = a\eta - da\,d\eta \quad (a\in A, \eta\in\Omega^{ev})$$

Check that Φ is an RA-module homomorphism.

$\rho(a)\,\Phi(a_0da_1\ldots da_{2n}) =$

$\rho(a)\rho(a_0)\,\omega(a_1,a_2)\ldots\omega(a_{2n-1},a_{2n})$

$= (\rho(aa_0)-\omega(a,a_0))(\omega(a_1,a_2)\ldots\omega(a_{2n-1},a_{2n}))$

$= \Phi(aa_0da_1\ldots da_{2n}) - \Phi(da\,da_0da_1\ldots da_{2n})$

$= \Phi\{(a-(da)d)a_0da_1\ldots da_{2n})$

$= \Phi(\rho(a)\cdot a_0da_1\ldots da_{2n})$

Since $R(A)$ is generated the elements $\rho(a)$ one has $x\Phi(\eta) = \Phi(x\eta)$ $(x\in RA)$.

Consequence is that Φ is surjective since it is a module homomorphism so the image is a left ideal, which contains 1 since $\Phi(1)=1$.

Now we consider the map $RA\to\Omega^{ev}$ given by $x\mapsto x\cdot 1$.

Calculate

$\omega(a_1,a_2)\cdot\eta = (\rho(a_1a_2)-\rho(a_1)\rho(a_2))\cdot\eta$

$= a_1a_2\eta - d(a_1a_2)d\eta - (a_1-da_1d)(a_2\eta-da_2d\eta)$

$= a_1a_2\eta - (a_1da_2+(da_1)a_2)d\eta - a_1a_2\eta + a_1da_2d\eta$
$+ da_1d(a_2\eta + da_2d\eta)$

$= da_1 da_2 \eta$

$\omega(a_1, a_2)\cdot\eta = da_1 da_2 \eta$

$\rho(a_0)\,\omega(a_1,a_2) \dots \omega(a_{2n-1}, a_{2n})\cdot\eta$

$= (a_0 - da_0 d)(da_1 da_2 \dots da_{2n-1} da_{2n} \eta)$

$= a_0\, da_1 \dots da_{2n}\eta - da_0 da_1 \dots da_{2n} d\eta$

$\therefore$ Thus $\Phi(\xi)\cdot\eta = \xi * \eta$

$$\Phi : \Omega^{ev} A \twoheadrightarrow RA \xrightarrow[\eta \mapsto \eta\cdot 1]{} \Omega^{ev} A$$

composition is the identity

$$\Phi(\xi)\cdot 1 = \xi * 1 = \xi$$

Hence Φ is an isomorphism with inverse $\eta \mapsto \eta\cdot 1$.

Finally $\Phi(\xi\cdot\eta) = \Phi(\Phi(\xi)\cdot\eta) = \Phi(\xi)\cdot\Phi(\eta)$

since Φ is an RA-module homomorphism.

The above is a standard argument which applies to ΩA, RA, QA etc using the universal property and specific model of the algebra.

<u>Corollary</u>: $gr_{IA}(RA) = \bigoplus_{n\geq 0} IA^n / IA^{n+1}$

$\cong \Omega^{even} A$

with the usual product for forms.

Remark: This proposition implies that there is a canonical vector space splitting of the decreasing I-adic filtration

$$RA \supset IA \supset IA^2 \supset \dots$$

Fact is that one has a canonical increasing filtration

$\mathbb{C} \subseteq \rho A \subseteq (\rho A)^2 \subseteq \dots$

and

$$(\rho A)^{2n+1} \oplus (IA)^{n+1} = RA$$

$$((IA)^n \cap (\rho A)^{2n+1}) \oplus (IA)^{n+1} = IA^n$$

$$\rho(a_0)\,\omega(a_1,a_2) - \omega(a_{2n-1},a_{2n})$$

$$\Phi(\Omega^{2n}A) = (IA)^n \cap (\rho A)^{2n+1}$$

Corollary 2: A linear functional τ on RA is equivalent to a sequence of even cochains

$$\tau_{2n}(a_0, \dots, a_{2n})$$

of normalized even cochains via the formula

$$\tau_{2n}(a_0, \dots, a_{2n}) = \tau(\rho a_0\, \omega(a_1,a_2) - \omega(a_{2n-1},a_{2n}))$$

Question: When is τ a trace on RA?

A DG algebra of cochains:

Let $\underline{A}$ and $\underline{R}$ be algebras and let

$$\Gamma^p = \mathrm{Hom}_{\mathbb{C}}(A^{\otimes p}, R) \qquad \forall p \geq 1$$

$$\Gamma^0 = R\,, \quad \Gamma^p = 0 \quad p < 0$$

and define a product & differential on $\Gamma = \oplus_p \Gamma^p$ by the formulae

$$b'\varphi(a_0, \dots, a_{p+1}) = \sum_{i=1}^{p} (-1)^{i-1} \varphi(\dots, a_i a_{i+1}, \dots)$$

$$(\varphi\varphi')(a_1, \dots, a_{p+p'}) = \varphi(a_1, \dots, a_p)\,\varphi'(a_{p+1}, \dots, a_{p+p'})$$

for $\varphi \in \Gamma^p$, $\varphi' \in \Gamma^{p'}$

b' is a differential for this product on Γ. Thus Γ is a differential graded algebra. These cochains are not normalized and take values in the algebra $\underline{R}$.

Example: Let $\rho : A \to R$ be a linear map. Then the curvature of ρ

$$\omega(a_1,a_2) = \rho(a_1a_2) - \rho(a_1)\rho(a_2)$$

is

$$\omega = b'\rho - \rho^2 \quad \text{in } \Gamma$$

$$\left\{ \begin{array}{ll} \text{cf} & F = dA + A^2 \quad \text{in Yang Mills} \\ & dF = -[A,F] \\ & [d+A, F] = 0 \\ & [d+A, F^n] = 0 \end{array} \right.$$

Bianchi identities

Claim (as a consequence of the DG algebra structure) we have

Bianchi identities $\begin{cases} b'(\omega) = \rho\omega - \omega\rho \\ b'(\omega^n) = \rho\omega^n - \omega^n\rho \end{cases}$

Proof: $b'(\omega) = b'(b'\rho - \rho^2)$

$$= -b'\rho\,\rho + \rho b'\rho = -\omega\rho + \rho\omega$$

For $n \geq 1$ we that $\underline{b'}$ and $Ad(\rho)$ are degree one antiderivations.

eg ~~$b'\rho =$~~

$$\omega(a_1 a_2, a_3) - \omega(a_1, a_2 a_3) =$$
$$= \rho(a_1)\,\omega(a_2, a_3) - \omega(a_1, a_2)\,\rho(a_3).$$

Normalised cochains: $C^n(A) = (\Omega^n A)^* = (A \otimes \bar{A}^{\otimes n})^*$

$$f \equiv f(a_0 da_1 \ldots da_n) \equiv f(a_0, a_1, \ldots, a_n)$$

The operators b, d, K determined by taking the transpose operators on cochains which we denote

by the same letters, except for d^t which we denote $\underline{s}$.
If $f \in C^n$ then

$$(bf)(a_0, \ldots a_{n+1}) = \sum_{i=0}^{n} (-1)^i f(, \ldots a_i a_{i+1} \ldots)$$
$$+ (-1)^{n+1} f(a_{n+1} a_0, a_1, \ldots a_n)$$

$$(sf)(a_0, \ldots, a_{n-1}) = f(1, a_0, \ldots a_{n-1})$$

$$(Kf)(a_0, \ldots, a_n) = (-1)^n f(a_n, a_0, \ldots a_{n-1}) +$$
$$+ (-1)^n f(1, a_n a_0, a_1, \ldots, a_{n-1})$$

"$1-K = bd - db$"
$1-K = bs + sb$

Formula for K^j on C^n of $\underline{A}$ for $0 \leq j \leq n$.
Let b_j denote the operator on unnormalised cochains which gives the sum of the first $\underline{j}$ terms in b $\quad \sum_{i=0}^{j-1}$.

Claim

$$K^j f = \lambda^j (1 - b_j s) f \qquad f \in C^n$$
$$0 \leq j \leq n$$

Proof: $(\lambda^{-i}k^if)(a_0,\dots,a_n) = (K^jf)(a_j,\dots,a_n,a_0,\dots a_{j-1})(-1)^{jn}$

$= (-1)^{jn}(K^jf)(a_j\,da_{j+1}\dots da_n\,da_0\dots da_{j-1})$

$= (-1)^{jn} f(K^i a_j \dots da_{j-1})$

$= (-1)^{jn+j(n-1)} f(da_0 \dots da_{j-1}\,a_j\,da_{j+1} \dots da_n)$

$= f(a_0\,da_1 \dots da_n) - \sum_{i=0}^{j-1}(-1)^i f(da_0 \dots d(a_ia_{i+1}) \dots da_n)$

$= f(a_0,\dots,a_n) - \sum_{i=0}^{j-1}(-1)^i f(a_0,\dots,a_ia_{i+1},\dots)$

$= f - b_j s f$

Double complex of cochains

$$\begin{tikzcd}
{} & {} & {} \\
C^2 \arrow[u] \arrow[r, "B"] & C^1 \arrow[u] \arrow[r, "B"] & C^0 \arrow[u] \\
C^1 \arrow[u, "b"] \arrow[r] & C^0 \arrow[u, "b"'] & \\
C^0 \arrow[u] & &
\end{tikzcd}$$

H^i of this $= HC^i(A)$

$HC^i(A) = HC_i(A)^*$

Think of $*$ as $\mathrm{Hom}(?,V)$. Then
$$HC^i(A,V) = \mathrm{Hom}(HC_i(A),V)$$

Yoneda's lemma allows us to recover $HC_i(A)$ from $HC^i(A,V)$. (Substitute for taking the double dual).

$A = RA/IA$ universal extension

$\rho : A \to RA$ $\quad \omega = b'\rho - \rho^2 : \bar{A}^{\otimes 2} \to IA$

$RA \quad \bigoplus_{n\geq 0} A \otimes \bar{A}^{\otimes 2n} \xrightarrow{\sim} RA$

given by $(\rho\omega^n)(a_0,\dots,a_{2n}) = \rho(a_0)\omega(a_1,a_2)\dots\omega(a_{2n-1},a_{2n})$

Further we have that
$$\bigoplus_{n\geq m} A \otimes \bar{A}^{\otimes 2n} \xrightarrow{\sim} IA^m$$

Definition: Let I be an ideal in an algebra R and let $\underline{\tau}$ be in $(I^m)^*$. Then $\underline{\tau}$ is a trace if $\tau(rx - xr) = 0 \ \forall r\in R, x \in I^m$

Also, $\underline{\tau}$ is an I-adic trace if $\tau(xy - yx) = 0$ $(x\in I^j, y \in I^k$ where $j+k=m)$

To describe traces on IA^m in terms
Since $IA^m = \bigoplus_{n\geq m} (A \otimes \bar{A}^{2n})$ a linear
functional $\tau \in (IA^m)^*$ is equivalent to a family of cochains $\tau_{2n} \in C^{2n}$ $n \geq m$

$$\tau_{2n} = \tau(\rho\omega^n)$$

$$(a_0, \dots, a_{2n}) \longmapsto \tau(\rho(a_0)\omega(a_1,a_2)\dots\omega(a_{2n-1},a_{2n}))$$

We need the following

$$*(b\tau_{2n} - (1+\kappa)S\tau_{2n+2})(a_0,\dots,a_{2n+1})$$
$$= \tau([\rho(a_0)\omega^n(a_1,\dots,a_{2n}), \rho a_{2n+1}])$$

Proof: $b\tau_{2n} = b'\tau_{2n}$ + crossover term.

$$\tau'_{2n} = b'\tau\rho\omega^n = \tau(b'(\rho\omega^n))$$

Recall that $b'\rho = \rho^2 + \omega$

$$b'\omega^n = \rho\omega^n - \omega^n\rho$$

$$b'(\rho\omega^n) = (\rho^2+\omega)\omega^n - \rho(\rho\omega^n - \omega^n\rho)$$
$$= \omega^{n+1} + \rho\omega^n\rho$$

So $b'\tau_{2n} = \tau(\omega^{n+1}) + \tau(\rho\omega^n\rho)$

Crossover term in $b\tau(\rho\omega^n)(a_0,\dots,a_{2n+1})$ is

$$(-1)^{2n+1}\tau(\rho\omega^n)(a_{2n+1}, a_0, a_1, \dots, a_{2n}) =$$
$$= -\tau(\rho(a_{2n+1}a_0)\omega^n(a_1,\dots,a_{2n}))$$
$$= -\tau(\omega(a_{2n+1},a_0)\omega^n(a_1,\dots,a_{2n})) \qquad b'\rho = \rho^2+\omega$$
$$-\tau\, \rho(a_{2n+1})\rho(a_0)\omega^n(a_1,\dots,a_{2n}))$$

$$(b'\tau_{2n})(a_0,\dots,a_{2n+1}) = \tau(\omega^{n+1}(a_0,\dots,a_{2n+1}))$$
$$+\tau(\rho(a_0)\omega^n(a_1,\dots,a_{2n})\rho(a_{2n+1}))$$

Adding

$$b\tau_{2n}(\quad) = \tau(\omega^{n+1}(a_0,\dots,a_{2n+1})) -$$
$$\tau(\omega^n(a_{2n+1},a_0,\dots,a_{2n})) +$$

$$+ \tau([\rho a_0 \omega^n(a_1, \dots, a_{2n}), \rho a_{2n+1}])$$

Now $\quad \delta \tau_{2n+1} = \delta \tau \rho \omega^{n+1} = \tau \omega^{n+1}$

$\quad\quad K s = \lambda s$

<u>Proposition:</u> For $\tau \in (IA^m)^*$ TFAE

1) τ is a trace $\quad \tau[RA, IA^m] = 0$

2) $b\tau_{2n} = (1+K) s \tau_{2n+2}$ for $n \geq m$

3) $\begin{cases} b\tau_{2n} = \frac{1}{n+1} B \tau_{2n+2} \\ \text{and} \quad K^2 \tau_{2n+2} = \tau_{2n+2} \quad \text{for } n \geq m. \end{cases}$

If $\tau_{2n}^{\#} = \frac{(-1)^n}{n!} \tau_{2n}$ then (3) becomes

$$b \tau_{2n}^{\#} + B \tau_{2n+2}^{\#} = 0$$

Proof: (i) ⇒ (2) immediately from (*).

Conversely, (2) ⇒ (i) because (*) and (2) give that

for $n \geq m$ $\quad \tau[\rho(a_0)\omega^n(a_1, \dots, a_{2n}), \rho(a_{2n+1})] = 0$

and

$\rho(a_0)\omega^n(a_1, \dots, a_{2n})$ span IA^m

and $\rho(a_{2n+1})$ generate RA

$\tau(vn) \overset{?}{=} \tau(nv) \quad\quad \tau(v_1 v_2 n) = \tau(n_* v_2 v_1)$

– enough to move generators of the algebra one at a time.

(2) ⇒ (3) $\quad\quad b\tau_{2n} = (1+K) s \tau_{2n+2}$

so $\quad sb\tau_{2n} = (1+K) s^2 \tau_{2n+2} = 0$

since K commutes with b, s.

$$\begin{aligned} (1-K^2)\tau_{2n+2} &= (1+K)(\cancel{1-K})\,\tau_{2n+2} \\ &= (1+K)(bs+sb)\,\tau_{2n+2} \\ &= b(1+K)s\,\tau_{2n+2} \\ &= bb\tau_{2n} = 0 \end{aligned}$$

Recall $\quad B\tau_{2n+2} = \sum_{j=0}^{2n+2} K^j s \tau_{2n+2}$

$= (n+1)(1+K) s \tau_{2n+2} \quad\quad$ since $K^2 \tau_{2n+2} = \tau_{2n+2}$

Hence (2) ⇒ (3).

The converse simply reverses the argument of (2) ⇒ (3).

$$RA \supset IA \supset \dots \supset IA^m \supset$$

τ linear function on IA^m

$\quad \tau_{2n} = \tau(\rho\omega^n) \quad\quad n \geq m$

Definition: Let $I \subset R$ be an ideal, τ a linear function on I^m. Then τ is called an I-adic trace on I^m when

$$\tau\,[I^i, I^j] = 0 \quad \text{for } i+j=m$$

<u>Proposition</u>: TFAE for a linear function on IA^m

1) τ is an IA-adic trace
2) $b\,\tau_{2n} = \frac{1}{n+1} B\,\tau_{2n+2}$, $K^2\tau_{2n} = \tau_{2n}$ $(n \geq m)$

Proof: We need the following identity.

$$* \quad (K^2\tau_{2n})(a_0, \dots, a_{2n}) = \tau(\omega(a_{2n-1}, a_{2n})\rho(a_0)\omega^{n-1}(a_1, \dots, a_{2n-2}))$$

or equivalently

$$(1-K^2)\tau_{2n}(a_0, \dots, a_{2n}) = \tau\,[\rho(a_0)\omega^{n-1}(a_1, \dots, a_{2n-2}), \omega(a_{2n-1}, a_{2n})]$$

Recall

$$(K^j f_n)(\ ) = \lambda^j(1-b_j s) f_n \qquad 0 \leq j \leq n.$$

$$K^2\tau_{2n} = \lambda^2(1-b_2 s)\tau(\rho\omega^n)$$

$$= \lambda^2(\tau(\rho\omega^n) - b_2\tau(\omega^n))$$

$$= \lambda^2(\tau(\rho\omega^n) - \tau(b'(\omega)\omega^{n-1}))$$

and $b'\omega = \rho\omega - \omega\rho$

Hence we get

$$\lambda^2(\tau(\rho\omega^n - \rho\omega^n + \omega\rho\omega^{n-1}))$$

$$= \lambda^2\tau(\omega\rho\omega^{n-1}) \qquad \text{giving the identity.}$$

(1) $\Rightarrow$ (2) Only have to prove, invoking the previous proposition, that $K^2\tau_{2m} = \tau_{2m}$ which follows from the identity (*).

2) $\Rightarrow$ 1) By previous proposition we know that τ is a trace on IA^m and

$$\tau\,[IA^{m-1}, x] = 0$$

when x is of the form $\omega(a_1, a_2)$.

In general, if $I = RXR \subset R$ is an ideal generated by a subset X then

$$[I^{m-1}, I] = [I^{m-1}, RXR]$$

$$\subset [I^{m-1}RX, R] + [RI^{m-1}R, X] + [XRI^{m-1}, R]$$

Circular bracket identity

$$[y, x_1 \dots x_k] = \sum_{j=1}^{k} [x_{j+1} \dots x_k y x_1 \dots x_{j-1}, x_j]$$

$[I^{m-1}, I] \subseteq [I^m, R] + [I^{m-1}, X]$

$\therefore \quad \tau([I^m, R]) = 0 = \tau([I^{m-1}, X])$

$\Rightarrow \quad \tau\,[I^{m-1}, I] = 0$

$\Rightarrow \quad \tau$ is an I-adic trace
(by the circular bracket identity argument)

$\tau\,[\underline{I}^{m-1}, I] = 0 \Leftrightarrow \quad \tau$ is an I-adic trace.
Since
$$[I^{m+k}, I^k] \subseteq \sum_{j=1}^{k} [I^{k-j} I^{m-k} I^{j-1}, I]$$
$$\subseteq \quad [I^{m-1}, I]$$

Consequences

There is an action of $\mathbb{Z}/2m\mathbb{Z}$ on the space of traces on IA^m given by $\underline{K}$ acting on the cochains.

$$\begin{cases} b\,\tau_{2n} = \frac{1}{n+1} B\,\tau_{2n+2} & n \geq m \\ K^2 \tau_{2n} = \tau_{2n} & n > m \end{cases}$$

$\underline{K}$ commutes with B, b and $\underline{K}$ so that the I-adic traces are the fixed points under K^2. Also note

$$\begin{cases} K^n = 1 + K^{-1} s b & \text{on } \underline{n}\text{-cochains} \\ K^{n+1} = 1 - bs \end{cases}$$

$K^{2m}\tau_{2m} - \tau_{2m} = K^{-1} s b\, \tau_{2m} = K^{-1} s \frac{1}{n+1} B \tau_{2m+2} = 0$
so that $K^{2m} = I$ on traces

We can split any trace τ on IA^m into a

$$\tau = \underbrace{\frac{1}{m}\sum K^{2i}\,\tau}_{IA\text{ adic trace}} + (1-K^2)\,\varsigma_{K^2}\,\tau$$

trace on IA^m / IA^{m+1} which average to zero under K^2.

Fact: $\quad \left\{ \tau_{2m} \;\middle|\; \begin{matrix} b\,\tau_{2m} = 0 \\ P_{K^2}\tau_{2m} = 0 \end{matrix} \right\}$

Recall that on the complement of P $Im(P^{\perp})$ on cochains we have

$$\begin{matrix} \operatorname{Ker} s & \xrightarrow[\sim]{b} & \operatorname{Ker} b \\ \cup & & \| \\ \operatorname{Im} s & & \operatorname{Im} b \end{matrix}$$

Proposition: Traces τ on IA^m / IA^{m+1} which average to zero under K^2 (i.e. are orthogonal to the IA-traces) are in one-one correspondence with elements of $\varphi \in (1-\lambda^2)(\bar{A}^{\otimes 2m})^*$ via the rule

$$\tau_{2m} = b\varphi$$

Cyclic cohomology: $C^n(A) = (A \otimes \bar{A}^{\otimes n})^*$

Form double complex

$$\begin{tikzcd}
C^2 \arrow[r, "B"'] & {} \\
C^1 \arrow[u, "b"'] \arrow[r] & C^0 \\
C^0 \arrow[u, "b"']
\end{tikzcd}$$

The total cohomology of this double complex is the cyclic cohomology $HC^n(A)$.

negative theory

positive theory

$$\begin{tikzcd}
{} \arrow[r] & C^2 \\
{} \arrow[u] \arrow[r] & C^1 \arrow[u] \\
C \arrow[u] \arrow[r] & C^0 \arrow[u] \\
C \arrow[u]
\end{tikzcd}$$

trace τ on IA^m determines an element of $HC^{2m-1}(A)$ via the double complex.

Theorem One has canonical isomorphisms

$$HC^{2m-1}(A) = \frac{\{IA\text{-adic traces on } IA^m\}}{\{\text{traces on } RA\}}$$

$$\overline{H}C^{2m-1}(A) = \frac{\{IA\text{-adic traces on } IA^m\}}{\{\text{reduced traces on } RA\}}$$

A trace τ on R is reduced if $\tau(1) = 0$.

Need ~~HC~~ to describe HC^i via double complexes appropriate to families $\{\tau_{2n} ; n \geq m\}$.

$$C^n(A) = C^n = (\Omega^n A)^* = \{f(a_0, \dots, a_n) : f = 0 \text{ if } a_i = 1 \text{ some } i\}$$

$C^0 = A^*$ $\quad \overline{C}^0 = (\overline{A})^*$ reduced linear f's.

Let $\hat{K}^n = \prod_{p+q=n} K^{pq} = \prod_p C^{n-2p}$

be the total complex of cochains with arbitrary support.

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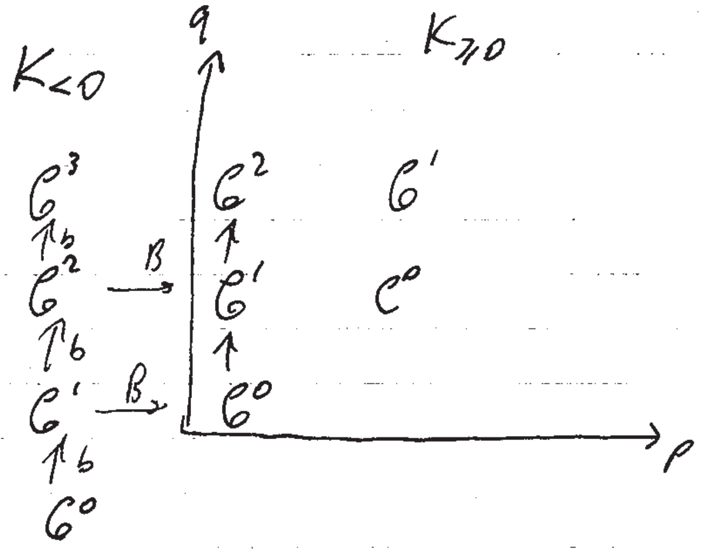
$K^{pq} = C^{q-p}$



Also let $\quad K_f^n = \bigoplus_{p+q=n} K^{pq} = \bigoplus_{p+q=n} C^{q-p}$

be the total complex of cochains with finite support.

$$\overline{K}^{pq} = \begin{cases} C^{q-p} & q \neq p \\ \overline{C}^0 & q = p \end{cases}$$

defined similarly.

Cohomology formulas:

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$$\overset{K}{}$$

$$0 \to K_{p\geq 0} \to K \to K_{p\not\geq 0} \to 0$$

gives an exact sequence of complexes.

$$H^i(K_{\geq 0}) = HC^i(A) \quad \text{by def}^n!$$

$$H^i(\overline{K}_{\geq 0}) = \overline{HC}^i(A) \quad \text{by def}^n$$

$$(HC^i(A) = (HC_i(A))^*).$$

$$H^i(K_{\geq 0}) = H^i(PK_{\geq 0}) = HC^i(A)$$

where P is the spectral projection associated to Karoubi operator and the eigenvalue 1.

$$P^\perp = 1 - P = b(Gs) + (Gs)b$$

so that all the columns in $P^\perp K$ are contractible.

Recall $\underline{B}$ is exact on $P\mathcal{C}$ except in degree zero, and $\underline{B}$ is exact on $P\overline{\mathcal{C}}$.

$$H^i(K) = H^i(PK) = \begin{cases} \mathbb{C} & i \text{ even} \\ 0 & i \text{ odd} \end{cases}$$

$$\begin{tikzcd}
{} & {} & {} \\
P\mathcal{C} \arrow[u, "b"] \arrow[r, "B"] & P\mathcal{C}' \arrow[u] \arrow[r, "B"] & P\mathcal{C}^0 \arrow[u] \\
P\mathcal{C} \arrow[u, "b"] \arrow[r, "D"] & P\mathcal{C}^0 \arrow[u, "b"] & \\
P\mathcal{C} \arrow[u] & &
\end{tikzcd}$$

and $H^i(\overline{K}) = H^i(P\overline{K}) = 0 \quad \forall i$

Now look at

$$0 \to \overline{K}_{\geq 0} \to \overline{K} \to \overline{K}_{<0} \to 0$$

$$\forall i \quad H^{i-1}(P\overline{K}_{<0}) \xrightarrow{\simeq} H^i(P\overline{K}_{\geq 0}) = \overline{HC}^i(A)$$

$$H^{2m}(PK_{<0}) \xrightarrow{\delta} H^{2m+1}(PK_{\geq 0}) = HC^{2m+1}(A)$$

$$\mathbb{C} \to H^{2m}(PK_{<0}) \to H^{2m+1}(PK_{\geq 0}) \to \underbrace{H^{2m+1}(PK)}_{0}$$

<u>Proof of the Theorem</u> (Reduced case)

Let V be the space of even cochains $\{f_{2n}\,;\, n \geq m\}$ which are fixed under K^2 and which satisfy the cocycle condition $bf_{2n} + Bf_{2n+2} = 0$, for $n \geq m$.

Let W be the similar space of $\{f_{2n} : n \geq 0\}$; same conditions

$$\left.\begin{array}{c} K^2 f_{2n} = f_{2n} \\ bf_{2n} + Bf_{2n+2} = 0 \end{array}\right\} n \geq 0$$

$$f_0(1) = 0$$

We have the restriction map $W \xrightarrow{r} V$.

$$V \cong \text{IA-adic traces on } IA^m$$

$$\left(\frac{(-1)^n}{n!}\tau(\rho\omega^n)\right)_{n \geq m} \longleftarrow\!\shortmid\ \tau$$

from our definitions of IA-adic traces.

We also have

$$W \cong \text{reduced traces on } RA$$

To prove $V/rW \cong HC^{2m-1}(A)$

Clearly

$K^2 f = f \qquad f_{2n} \quad n \geq m$

Consider the action of the Karoubi operator on V, W. We have $K^2 = 1$ for this action. Hence we can decompose into eigenspaces ± 1

$$V = V_+ \oplus V_-$$
$$W = W_+ \oplus W_-$$

which are compatible so that

$$V/rW = V_+/rW_+ \oplus V_-/rW_-$$

But V_+ = cocycles of degree $2m-2$ in $P\overline{K}_{<0}$

and

W_+ = cocycles of degree $2m$ in $\overline{PK}$

Point is that if $bf_{2n} + Bf_{2n+2} = 0$ then K is of finite order on f_{2n}

$$(K^{n(n+1)} = 1 - bB)$$

Conclude that $V^+/rW^+ = \overline{HC}^{2m-1}A$

Finally, we check that $V_-/rW_- = 0$.

$bs + sb = 1 - K = 2$ one'space $K = -1$.

If $f \in V_-$, then

$$sbf_{2n} = -sBf_{2n+2} = 0$$

$$f = \tfrac{1}{2} bs f_{2n} \qquad n \geq m$$

so

$$Bf_{2m} = \tfrac{1}{2} Bbs f_{2n} = 0$$

so we extend chains by zero (

$$f_{2n} = \begin{cases} f_{2n} & n \geq m \\ 0 & n < m \end{cases}$$

to give an element of W_-.

The same argument works in the non-reduced case.

Quillen

$$HC^{2m-1}(A) \overset{\sim}{\leftleftarrows} \frac{\{IA\text{-adic traces on } IA^m\}}{\{\text{traces on } RA\}}$$

The map from IA-adic traces to odd cyclic cohomology is due to Connes, in the following form. (originally)

τ on IA^m $\qquad \tau_{2m} = \tau(\rho\omega^m)$

$$C^{2n} \xrightarrow{B} C^{2n-1}$$

$$B\tau_{2n} = \sum_{j=0}^{2n-1} K^j S\tau_{2n} = \sum_0^{2n-1} \lambda^j \tau\,\omega^n$$

$$= n(1+\lambda)\tau(\omega^n)$$

$\psi := (1+\lambda)\tau(\omega^n)$ is a reduced cyclic $(2n-1)$ cocycle.

$$\varphi(a_0,\dots a_{2n-1}) = \tau(\omega(a_0,a_1)\dots\omega(a_{2n-3},a_{2n-2}))$$

$$-\ \tau(\omega(a_{2n-1},a_0)\dots\omega(a_{2n-3},a_{2n-2}))$$

These cocycles give classes in $HC^{2n-1}(A)$ $n\geq 1$, m

$$\overline{HC}^{2m-1}(A) \xrightarrow{S} \overline{HC}^{2m+1}(A) \xrightarrow{S}$$

Lecture on the analogue of the de Rham complex in non-commutative geometry

M smooth manifold

$$\Omega(M): \quad \Omega^0 M \xrightarrow{d} \Omega^1 M \xrightarrow{d} \Omega^2 M$$

De Rham's theorem: $H^i(\Omega M) = H^i(M, \mathbb{C})$

The de Rham complex can be constructed for any commutative algebra $\underline{A}$ over $\mathbb{C}$ as follows.

$$0 \to I \to A \otimes A \xrightarrow{m} A \to 0$$

Kähler differentials $\Omega^1_A = I/I^2$

$d: A \to \Omega^1 A$

defined by $da = a \otimes 1 - 1 \otimes a \pmod{I}$

$$\Omega^i A = \Lambda^i_A \Omega^1_A$$

where Λ_A denotes the exterior tensor

power as an $\underline{A}$-module. $\underline{A}$ is unital

$$A \cong \Omega^0_A \xrightarrow{d} \Omega^1_A \xrightarrow{d} \Omega^2_A \longrightarrow \dots$$

Suppose $\underline{A}$ is finitely generated commutative algebra over $\mathbb{C}$. We can associate to $\underline{A}$ an affine algebraic variety

$$M(A) = \mathrm{Hom}_{\mathbb{C}\text{-alg}}(A, \mathbb{C})$$

(maximal ideal space)

$$A = \mathbb{C}[X_1, \dots X_n]/(f_1, \dots f_k)$$

$$M(A) \cong \{z \in \mathbb{C}^n : f_i(z_1, \dots z_n) = 0 \ \forall i\}$$

<u>Theorem</u> (Lefschetz, Atiyah-Hodge, Grothendieck) Assume $\underline{A}$ has no nilpotent elements and that $M(A)$ is nonsingular. Then

$$H^i(\Omega_A) \cong H^i(M(A), \mathbb{C}).$$

[de Rham ⇒ smooth fns; Stein-Grauert ⇒ holomorphic differentials; result here is that algebraic differentials are enough]

[$M(A) = M(A/\text{nilpotents})$]

Theorem (Zariski - Grothendieck) $\underline{A}$ has no nilpotent elements and $M(A)$ nonsingular if and only if $\underline{A}$ has the lifting property with respect to nilpotent extensions i.e. given $A = R/I$ an extension of commutative algebras where $I^n = 0$ for some $\underline{n}$ then there is a lifting homomorphism $A \to R$.

Definition: Such an algebra $\underline{A}$ is called smooth.

Theorem If $A = R/I$ where $\underline{R}$ is smooth, then
$$H^i(M(A)) = H^i(\varprojlim \Omega_R / I^n \Omega_R)$$

Noncommutative algebras

Periodic cyclic homology (only $\mathbb{Z}/2$ graded) of $\underline{A}$ denoted $HP_i(A)$ $i = 0, 1$ is defined to be the homology of the $\mathbb{Z}(2)$ graded complex

$$\begin{tikzcd} \hat{\Omega}^{even} A \arrow[r, shift left, "b+B"] & \hat{\Omega}^{odd} A \arrow[l, shift left, "b+B"] \end{tikzcd}$$

$$\Omega A = \bigoplus_{n \geq 0} \Omega^n A \qquad \Omega^n A = A \otimes \bar{A}^{\otimes n}$$

$$\hat{\Omega} A = \prod \Omega^n A \qquad b, B \text{ as before}$$

$$(b+B)^2 = \underset{0}{b^2} + \underbrace{bB + Bb}_{0} + \underset{0}{B^2} = 0$$

Theorem If $\underline{A}$ is smooth and commutative then

1) $HH_i(A) \overset{def}{=} H_i(\Omega A, b) = \Omega^i_A$

2)
$$\begin{tikzcd} \hat{\Omega}^{ev} A \arrow[r, shift right, "b+B"'] & \hat{\Omega}^{odd} \arrow[l, shift right, "b+B"'] \end{tikzcd}$$

$$\begin{tikzcd} \Omega^{ev}_A \arrow[r, shift left, "d"] \arrow[d, "\mu"'] & \Omega^{odd}_A \arrow[l, shift left, "d"] \arrow[d] \\ \Omega^{ev}_A \arrow[r, shift left] & \Omega^{odd}_A \arrow[l, shift left, "d"] \end{tikzcd}$$

$$\mu(a_0 da_1 \ldots da_n) = \frac{1}{n!} a_0 da_1 \ldots da_n$$

The map $\underset{\sim}{\mu}$ is a quasi-isomorphism, hence

$$B(a_0 da_1 \dots da_n) = \sum_{i=0}^{n} (-1)^{in} da_i \dots da_n da_0 \dots da_{i-1}$$

$$HP_{\pm}(A) = H^{\pm}(\Omega_A) = H^{\pm}(M(A))$$

- only get the $\mathbb{Z}/2$ graded version

Question: What is the analogue in the non-commutative setting of smooth commutative algebras and what is the analogue of Ω_A for $\underline{A}$ smooth?

Definition: An algebra $\underline{A}$ will be called quasi-free if the following conditions are satisfied

1) Lifting property relative to nilpotent extensions.

$$\begin{tikzcd} & R \arrow[d] \\ A \arrow[ur, dashed, "\exists"] \arrow[r] & R/I \end{tikzcd} \qquad I^n = 0$$

2) $H^2(A, M) = 0$ for any A-bimodule M

3) $\Omega^1 A$ is projective as an A-bimodule

Examples: free algebras, free group algebras
separable algebra $M_n \mathbb{C}$

A quasi free $\to$ $M_n A$ quasi free

This class is not closed under tensor products.

Note A nonsingular ordinary manifold of dimension greater than or equal to two is singular in the non commutative setting.

Theorem: If $\underline{A}$ is quasi free, then

(i) $HH_i(A) = \begin{cases} 0 & i \geq 2 \\ \cdot & \end{cases}$

2) $HP_0(A) = \operatorname{Ker}\{B : HC_0(A) \to HH_1(A)\}$.

$HP_1(A) = HC_1(A)$

from the Connes exact sequence.

3)
$$\begin{tikzcd}
\hat{\Omega}^{\overset{+}{\cancel{\text{even}}}} A \arrow[r, shift left, "b+B"] \arrow[d, "\mu"'] & \hat{\Omega}^{-} A \arrow[l, shift left, "b+B"] \arrow[d, "\mu"] \\
A \arrow[r, shift right, "d"'] & \Omega^1 A/[A,\underset{\Omega^1 A}{\cancel{B}}\,A] \arrow[l, shift right, "b"']
\end{tikzcd}$$
ⓧ

$d = B$ in degree zero

Here $\underset{\sim}{\mu}$ is a quasi isomorphism.

Definition: Define $X(A)$ to be the $\mathbb{Z}/2$ graded complex ⓧ.

Now $X(A)$ gives $HP_i(A) = H(X(A))$ for $\underline{A}$ quasi-free.

- analogue of Lefschetz-Atiyah-Hodge-Grothendieck

If $A = R/I$ where $\underline{R}$ is quasi-free then

$HP_i(A) = \cancel{H}_i \varprojlim_i X(R/I^n)$

- analogue of Zariski Grothendieck.

Proof: There are two steps 1) Calculate that it holds in the case of the universal extension

$A = RA/IA$

2) $\varprojlim_n X(R/I^n)$ is ~~independent~~ independent of R

Homotopy property for $X(\cdot)$ for quasi-free algebras.

[If Y is a vector field on M there are Lie derivatives $\mathcal{L}_Y = d\iota_Y + \iota_Y d$ Cartan homotopy property]

$X(R)$: $R \underset{d}{\overset{b}{\rightleftarrows}} \Omega^1 R_\natural$ $\qquad \Omega^1 R_\natural = \Omega^1 R/[R,\Omega^1 R]$

$b(xdy) = [x,y]$

$[R,\Omega^1 R] = b\,\Omega^2 R$

$b(\omega dx) = (-1)^{|\omega|}[\omega,x]$ in general.

Take $R = RA$. Then $\Omega^1(RA) \leftarrow RA \otimes \bar{A} \otimes RA$

$x\,d\rho(a)\,y \longleftarrow\!\shortmid\, x \otimes a \otimes y$

RA is a free algebra

Consider ~~algebra~~ . liftings of $RA \to RA \oplus M$ where M is an RA bimodule. These are the same as derivations $RA \to M$ or equivalently bimodule maps $\Omega^1 RA \longrightarrow M$ by the universal property of Ω^1. On the other hand, they are the same as a linear lifting $A \to RA \oplus M$, equivalently a linear map $\bar{A} \to M$, equivalently a bimodule map $RA \otimes \bar{A} \otimes RA \longrightarrow M$. This gives the above isomorphism.

Hence $\therefore$ $(\Omega^1(RA))_\natural \leftarrow RA \otimes \bar{A}$

$x\,d\rho(a) \qquad x \otimes a$

since in the commutator quotient space

$x\,d\rho(a)y \sim y x\,d\rho(a)$.

$$\begin{tikzcd}
RA \arrow[r, shift right, "d"'] & \Omega(RA)_\natural \arrow[l, shift right, "b"'] \\
\bigoplus_{n\geq 0}(A\otimes\bar{A}^{2n}) \arrow[u, "\cong"', "\rho\omega^n"] & \bigoplus_{n\geq 0} A\otimes\bar{A}^{\otimes 2n+1} \arrow[u, "\cong", "\rho\omega^n d\rho"']
\end{tikzcd}$$

To compute b, d in $X(RA)$ relative to this description we pass to the cochain

Pass to linear functionals. Let $\tau \in (RA)^*$ ie equivalent to cochains $\tau(\rho\omega^n) = \tau_{2n}$ $n \geq 0$

$T \in (\Omega^1(RA)_\natural)^*$ equivalently $T(\rho\omega^n d\rho) = T_{2n+1}$

Formula (*)

$$(*)\begin{cases} (\tau b)_{2n+1} = b\tau_{2n} - (1+K)s\,\tau_{2n+2} \\ (Td)_{2n} = -n P_2 b T_{2n-1} + B\,T_{2n+1} \end{cases}$$

where

P_2 is the spectral projection onto the generalized eigenspace of the eigenvalue 1 of the operator K^2

In the present case one has $\underline{K}$ of finite order

so

$$P_2 b T_{2n-1} = \frac{1}{n}\sum_{j=0}^{n-1} K^{2j}\, b\, T_{2n-1}$$

Now $\tau b = 0 \iff \tau$ is a trace

so we have already proved the first of these formulae in the analysis of traces on the universal extension RA.

It is repeated here to facilitate the proof of the second part

$$b^{*}\tau_{2n} = b'\tau_{2n} + \text{cross-over}$$

$$b'\tau_{2n} = \tau(b'\rho w^{n}) = \tau\{(\rho^{2}+w)w^{n} - \rho(\rho w^{n} - w^{n}\rho$$

by the Bianchi identities $= \tau\{w^{n+1} + \rho w^{n}\rho\}$

Cross over term is

$$\lambda\tau(b'(\rho)w^{n}) = \lambda\tau\{(\rho^{2}+w^{3})w^{n}\}$$

$$\therefore\ b\tau_{2n} = (1+\lambda)\ \tau(w^{n+1}) + \tau(\rho w^{n}\rho) + \lambda\tau(\rho^{2}w^{n}$$

$$(\tau\overset{*}{b})_{2n+1} = \tau\{\overset{*}{b}(\rho w^{n}d\rho)\}$$

b^{*} is the b in the X complex $b^{x}(xdy) = [x,y]$

$$= \tau(\rho w^{n}\rho) + \lambda\tau(\rho\rho w^{n})$$

so the first formula becomes clear, since

$$S\tau_{2n+2} = S\tau(\rho w^{n+1}) = \tau(w^{n+1})$$

$$b'T_{2n-1} = T\{b'(\rho w^{n-1}d\rho)\}$$

$$= T\{(\rho^{2}+w)w^{n-1}d\rho - \rho(\rho w^{n-1} - w^{n-1}\rho) - \rho w^{n-1}d(\rho^{2}+w)\}$$

$$= T(w^{n}d\rho - \rho w^{n-1}dw) - T(\rho w^{n-1}d\rho.\rho)$$

Since

$$d(\rho^{2}+w) = \rho d\rho + d\rho.\rho + dw$$

$$= T(w^{n}d\rho - \rho w^{n-1}dw) - \lambda T(\rho^{2}w^{n-1}d\rho)$$

cross-over term is

$$\lambda\ T(b'(\rho)w^{n-1}d\rho) = \lambda T(w^{n}d\rho) + \lambda T(\rho^{2}w^{n-1}d\rho)$$

add to give

$$bT(\rho w^{n-1}d\rho) = (1+\lambda)T(w^{n}d\rho) - T(\rho w^{n-1}dw)$$

Calculate

$$K^{2j}T(\rho w^{n-1}dw) = \lambda^{2j}(1-b_{2j}s)T(\rho w^{n-1}dw)$$

$$= \lambda^{2j}\ T((\rho w^{j} - b(w^{j}))w^{n-1-j}dw)$$

$$= \lambda^{2j}\ T(w^{j}\rho\ w^{n-1-j}dw)$$

Now $\underline{T}$ is a trace

$$= T(\rho w^{n-1-j}dw\,w^{j})$$

$$(Td)_{2n} = T(d^{X}(\rho w^n))$$

$$= \sum_{j=0}^{n-1} T(\rho w^{n-1-j} dw\, w^j) + T(d\rho w^n)$$

$$= \sum_{j=0}^{n-1} K^{2j} T(\rho w^{n-1} dw) + T(d\rho w^n)$$

$$= \sum_{j=0}^{n-1} K^{2j}\{-bT(\rho w^{n-1} d\rho) + (1+\lambda) T(w^n d\rho)\} + \cancel{\lambda} T(w^n d\rho)$$

$$= \underbrace{-\sum_{j=0}^{n-1} K^{2j} b\, T_{2n-1}}_{n P_2 b T_{2n-1}} + \underbrace{\sum_{j=0}^{2n} K^j s\, T_{2n+1}}_{B T_{2n+1}}$$

Let $X(RA)^*$: $(RA)^*$ $(\Omega'(RA)_\natural)^*$

This is identified with even and odd cochains of arbitrary support and with differentials given by ⊛. Decompose

These differentials commute with $\underline{K}$ so we can decompose into subcomplexes

Put $X(RA)^* = X^*$

$$X^* = PX^* \oplus P^\perp P_2 X^* \oplus P_2^\perp X^*$$

Re. $X^* = (K^k = 1$ generalized eigenspace$) \oplus \{K = -1$ ge'space$\} \oplus (K \neq \pm 1)$.

<u>Theorem</u> 1) PX^* is isomorphic to $(P\Omega)^*$ with differential $b + B$.

2) $P^\perp(P_2X^*)$ one has the operator $b^X = 0$ and d^X is bijective.

3) On $P_2^\perp X^*$ one has $d^X = 0$ and b^X is bijective.

$$\begin{tikzcd} X(RA) & RA \arrow[r, shift right, "\bar d"'] \arrow[d, "\cong"] & \Omega'(RA)_\natural \arrow[l, shift right, "\bar b"'] \arrow[d, "\cong"] \\ \Omega(A) & \Omega^+A \arrow[r, shift right, "b+B"'] & \Omega^+A \arrow[l, shift right] \end{tikzcd}$$

(doesn't commute!)

$\bar b(xdy) = [x, y]$

$\bar d(x) = dx \mod [\ ,\ ]$.

$$RA \cong \bigoplus_{n\geq 0} A \otimes \bar A^{2n};\ \Omega'(RA)_\natural \cong \bigoplus_{n\geq 0} A \otimes \bar A^{2n+1}$$

$$\begin{cases} \tau \in (RA)^* & \tau_{2n} = \tau(\rho\omega^n) \\ T \in (\Omega^1(RA)_\natural)^* & T_{2n+1} = \tau(\rho\omega^n d\rho) \\ (\tau\bar b)_{2n+1} = b\tau_{2n} - (1+K)S\tau_{2n+2} & \\ (T\bar d)_{2n} = -nP_2 bT_{2n-1} + BT_{2n+1} & \end{cases}$$

$\Big\{$ P_2 is the spectral projection onto the generalized eigenspace for K^2 and eigenvalue 1.

P 1-spectral projection for K

$$\begin{cases} \underset{K=\pm1}{P_2} = \underset{K=1}{\underset{\sim}{P}} + \underbrace{P^\perp P_2}_{K=-1} \end{cases}$$

Let's consider the complex $X(RA)^*$ (call this X^*) and the decomposition

$$X^* = \underset{\substack{K=+1 \\ \mathrm{Ker}(1-K)^2}}{PX^*} \oplus \underset{K=-1}{(P_m^\perp P_2)\cancel{X}^*} \oplus \underset{K\neq\pm1}{P_2^\perp X^*}$$

Theorem 1) PX^* is isomorphic to $P(\Omega^*)$, where $\Omega^* = (\Omega A)^*_\natural$, equipped with the differential $b+B$

2) $P^\perp P_2 X^*$ has $\bar d^t$ bijective and $\cancel{\bar b}^t = 0$

3) $P_2^\perp X^*$ has $\bar b^t$ bijective and $\bar d^t = 0$.

Proof: 1) On $P_2 X^*$ $(K = \pm 1)$ the differentials can be written

$$\begin{cases} (T\bar d)_{2n} = -nbT_{2n-1} + BT_{2n+1} \\ (\tau\bar b)_{2n+1} = b\tau_{2n} - \frac{1}{n+1}B\tau_{2n+2} \end{cases}$$

Rescale our cochains: $\tau^\#_{2n} = \frac{(-1)^n}{n!}\tau_{2n}$

$T^\#_{2n+1} = \left((-1)^n/n!\right) T_{2n+1}$

and you get

$$(\tau\bar b)^\#_{2n+1} = b\tau^\#_{2n} + B\tau^\#_{2n+2}$$

2) On $P^\perp P_2 X^*$ the differentials are

$$\begin{cases} (\bar b\tau)^\#_{2n+1} = b\tau^\#_{2n} + B\cancel{\tau^\#_{2n+2}} = 0 \\ (T\bar d)^\#_{2n} = b\bar T^\#_{2n-1} + \cancel{B} = 0 \end{cases}$$

Exercise: Show that b on the $K=-1$ eigenspace is an isomorphism from odd to even cochains with inverse $\frac{1}{2}S$.

Note $\quad 1-K = sb+bs \quad$ so $\quad 2 = sb+bs$

$$\begin{tikzcd} P^{\perp}\Omega = bP^{\perp}\Omega \arrow[r, bend right, "\cong\ d"'] & \oplus\, dP^{\perp}\Omega \arrow[l, bend right, "\cong\ b"'] \end{tikzcd}$$

$$(P^{\perp}d\Omega)^{n} \cong (1-\lambda)\bar{A}^{\otimes n}$$

3) $P_2^{\perp}X^*$ where $K \neq \pm 1$

$$\begin{cases} (T\bar{d})_{2n} = 0 \\ (T\bar{b})_{2n+1} = bT_{2n} - (1+K)sT_{n+2} \end{cases}$$

$$(b-(1+K)s)^2 = b^2 - (1+K)(bs+sb) - s^2$$
$$= 0 - (1-K^2) - 0$$

which is invertible on the image of $P_2^{\perp}$.

$\therefore$ $b-(1+K)s$ is bijective from even cochains to odd cochains and also from odd to even.

<u>Corollary</u>: $H^i(X(RA)^*) = \begin{cases} \mathbb{C} & i=0 \\ 0 & i=1 \end{cases}$

Because $\quad H^*(P\Omega_1^*,\ b+B) = \begin{cases} \mathbb{C} \\ 0 \end{cases}$

Instead of $X(RA)^*$ let us consider

$\varinjlim_m X(RA/IA^m)^*$ $\quad$ (= cochains of finite support

In general $\quad X(R/I^{n+1}) \qquad R/I^{n+1} \to \Omega(R/I^{n+1})_\natural$

$$\Omega^1(R/I) = \Omega^1R/[I,\Omega^1R] + \Omega^1(R)I + dI$$

$$\Omega^1(R/I^{n})_\natural = \Omega^1R/[R,\Omega^1R] + I\Omega^1R + dI$$

Upshot:

$$(\Omega^1R/I^{n+1}\Omega^1R)_\natural \twoheadrightarrow \Omega^1(R/I^{n+1})_\natural \twoheadrightarrow (\Omega^1R/I^{n}\Omega^1R)_\natural$$

$(RA/IA^{m+1})^* =$ cochains (τ_{2n}) $(\tau_{2n} = 0 \;\; n\geq$

$(\Omega^1 RA / IA^m \Omega^1 RA)_\natural \simeq$ cochains (τ_{2n+1}) $(\tau_{2n+1} = 0$ $n \geq m$

<u>Theorem</u> We let $X(RA)^*_c = \varinjlim_m X(RA/IA^m)$

c - continuous.

Then X^*_c has a decomposition into subcomplexes

$$X^*_c = PX^*_c \oplus P^\perp P_2 X^* \oplus \underset{\text{in } \operatorname{Im} P}{P_2^\perp X^*_c}$$

where

1) $PX^*_c \cong$ finite support cochains with $b+B$

2) $P^\perp P_2 X^*_c$ has $\bar{b}^t = 0$, $\bar{d}^t \cong$

3) $P_2^\perp X^*_c$ has $\bar{b}^t \cong{}'$, $\bar{d}^t = 0$

Corollary: $H^i(X^*_c(RA)) = \varinjlim_n HC^{i+2n}(A)$.

$\to K_{n+1} \to K_n \to K_{n-1} \to$ inverse system of complexes

$$H_i(\varprojlim_n K_n) \to \varprojlim_n H_i(K_n)$$

- a canonical map, generally not an isomorphism.
Milnor exact sequence. It is understood by
exact sequence involving $\varprojlim^1 H_{i+1}(K_n)$

$$0 \longrightarrow \varprojlim^1 H_{i+1}(K_n) \longrightarrow H_i(\varprojlim_n K_n) \longrightarrow$$

$$\longrightarrow \lim_n H_i(K_n) \longrightarrow 0$$

Hence it is better to take $H_i(\varprojlim_n K_n)$.

$$\hat{X}(RA) = \varprojlim X(RA/IA^n)$$

$$\begin{tikzcd}
\widehat{RA} \arrow[r, shift left] & \widehat{\Omega^1 RA_\natural} \arrow[l, shift left] \\
\prod(A\otimes\bar{A}^{\otimes 2n}) \arrow[u, "\iota"] & \prod(A\otimes\bar{A}^{(2n+1)}) \arrow[u, "\iota"']
\end{tikzcd}$$

completed complex for the I-adic topology.

"Chain" description of $X(RA)$ itself.

$$\begin{tikzcd}
RA \arrow[r, shift right, "\bar{d}"'] \arrow[d, phantom, "\shortparallel"] & \Omega^1 RA_\natural \arrow[l, shift right, "\bar{b}"'] \\
\Omega^{ev} A & \Omega^{odd} A
\end{tikzcd}$$

$$\bar{b}(\rho\omega^n d\rho) = b(\rho\omega^n) - (1+K)s(\rho\omega^{n+1})$$

$$\bar{d}(\rho\omega^n) = -n P_2 b(\rho\omega^{n-1}d\rho) + B(\rho\omega^n d\rho)$$

$\bar{b}$, $\bar{d}$ act on the values of the cochains, b, s, K etc act on the arguments.

Definition: Algebra $\underline{R}$ is quasifree if $\Omega^1 R$ is a projective bimodule over $\underline{R}$.

Equivalent conditions

1) $\Omega^1 R$ is a projective R-bimodule

2) Every ~~nilpotent~~ square-zero extension of $\underline{R}$ has a lifting

$$\begin{tikzcd}
0 \arrow[r] & J \arrow[r] & S \arrow[r] & R \arrow[r] \arrow[l, bend right] & 0
\end{tikzcd}$$

$J^2 = 0$

3) Every nilpotent extension of $\underline{R}$ has a lifting

There are the noncommutative analogues of nonsingular varieties.

<u>Theorem</u> If $A = R/I$ where $\underline{R}$ is quasi-free then

$$\hat{X}(R, I) \overset{def^n}{=} \varprojlim_n X(R/I^n)$$

$$= \left(\varprojlim R/I^n \underset{\bar{d}}{\overset{\bar{b}}{\rightleftarrows}} \varprojlim (\Omega^1 R / I^n \Omega^1 R)_\natural \right)$$

computes $HP(A)$.

Proof: 1) Two parts of the proof: check it for the universal extension $A = RA/IA$.

2) Homotopy property of X complex for quasi-free algebras

Given $R \xrightarrow{\theta_t} R'$ differentiable family of homomorphisms

inducing $\theta_{t*} : X(R) \longrightarrow X(R')$

then up to chain homotopy $X(R) \to X(R')$ is independent of $\underline{t}$.

In differential geometry there is the analogue of the de Rham cohomology.

$$\partial_t(\theta_{t*}) = \overset{*}{d}h + hd^*$$

i.e. homotopy of the d^*.

Last term stated the formula for $\underline{h}$.

X - analogue of de Rham complex

In the commutative case the product of nonsingular things is nonsingular - not so in the noncommutative case.

<u>Connections</u>: Let R be an algebra and let $\underline{E}$ be a right $\underline{R}$-module. A connection ∇ on $\underline{E}$ (Connes' definitions) is an operator

$$\nabla : E \longrightarrow E \otimes_R \Omega^1 R$$

satisfying Leibniz

$$\nabla(\xi x) = \nabla\xi . x + \xi dx$$

for $\xi \in E$ and $\underline{x} \in R$.

$(\nabla\xi . x = \nabla\xi \otimes x$ etc$)$.

Given a connection ∇, it extends to a degree +1 operator on $E \otimes_R \Omega^* R$

satisfying

$$\nabla(\alpha\omega) = (\nabla\alpha)\omega + (-1)^{|\alpha|}\alpha d\omega$$

for $\underline{\alpha} \in E \otimes_R \Omega^* R$, $\omega \in \Omega R$.

One has

$$\nabla^2(\alpha\omega) = (\nabla^2\alpha)\omega$$

So ∇^2 is determined by

$$\nabla^2 : E \to E \otimes_R \Omega^2 R$$

which is called the curvature of ∇.

If E is finitely generated and projective right R module (analogue of a vector bundle) then one can define

$$tr_E (\nabla^2)^n \in \Omega^{2n}/[\Omega,\Omega]^{2n}$$

(Chern character classes)

$$(*) \quad 0 \to \Omega^1 R \to R \otimes R \xrightarrow{m} R \to 0$$

$$dx \longmapsto x \otimes 1 - 1 \otimes x$$

$$x \otimes y \longmapsto xy$$

(*) splits either as a left or a right module sequence.

Hence it is exact when tensored by something.

One has an exact sequence of right R-modules

$$\begin{tikzcd}
0 \arrow[r] & E \otimes_R \Omega' R \arrow[r] \arrow[drr, "i"] & E \otimes_R (R \otimes R) \arrow[r] \arrow[d, phantom, "\wr"] & E \otimes_R R \arrow[r] \arrow[d, phantom, "\wr"] & 0 \\
 & & E \otimes R \arrow[r, "m"] & E \arrow[r] & 0 \\
 & & \xi \otimes n \arrow[r, mapsto] & \xi n &
\end{tikzcd}$$

$$i(\xi \otimes dn) = \xi n \otimes 1 - \xi \otimes n$$

<u>Proposition</u>: A connection on $\underline{E}$ is in one-to-one correspondence with a splitting of this exact sequence.

i.e. a right-module map
$l : E \to E \otimes R$ such that $ml = id$

Proof: $0 \longrightarrow E \otimes_R \Omega' R \xrightarrow{i} E \otimes R \xrightarrow{m} E \longrightarrow 0$.

We have an equivalence between linear maps $\nabla : E \to E \otimes_R \Omega' R$ and linear maps l which are sections of m
$l : E \to E \otimes R$; $ml = id$

$$i(\nabla \xi) = \xi \otimes 1 - l(\xi)$$

$$i\,(\nabla(\xi n) - (\nabla \xi) n - \xi\, dn)$$

$$= (\xi n \otimes 1 - l(\xi n)) - (\xi \otimes 1 - l(\xi))n - (\xi n \otimes 1 - \xi \otimes n)$$

$$= l(\xi) n - l(\xi n)$$

i.e. ∇ is a connection if and only if $\underset{\sim}{l}$ is a right module map.

<u>Corollary</u>: A right module has a connection if and only if it is projective.

Now suppose $\underline{E}$ is a bimodule over $\underline{R}$. Define a connection on the bimodule $\underline{E}$ to be a connection on $\underline{E}$ as a right module and such that

$$\nabla(n \xi) = n \nabla \xi$$

These are equivalent to bimodule liftings $E \xrightarrow{l} E \otimes R$ such that $ml = id$. They exist if and only if $\underline{E}$ is projective as a bimodule. Such an E has $H_i(R, E) = 0$

for $i > 0$.

<u>Proposition</u>: Let ∇ be a connection on the bimodule E and extend ∇ to $E \otimes_R \Omega R$. Then one has

$$b\nabla + \nabla b = 1 \quad \text{in degrees} > 0$$

Proof: ~~$b(\xi dx.$~~ $E \otimes \Omega^n R$ is spanned by $\xi dx_1 \ldots dx_n$ $(\xi \in E;\ x_1, \ldots, x_n \in R)$ and b is defined by

$$b(\alpha dx) = (-1)^{|\alpha|}(\alpha x - x\alpha)$$

$$\nabla b(\alpha dx) = \nabla(-1)^{|\alpha|}(\alpha x - x\alpha)$$
$$= (-1)^{|\alpha|}((\nabla\alpha)x + (-1)^{|\alpha|}\alpha dx - x\nabla\alpha)$$

$$b\nabla(\alpha dx) = b(\cancel{\nabla\alpha}\, dx + (-1)^{|\alpha|}\alpha d^2x)$$
$$= (-1)^{|\alpha+1|}(\nabla\alpha . x - x\nabla x)$$

$$\therefore \quad b\nabla + \nabla b = 1 \quad (\text{degree} > 0).$$

Hence one sees that the Hochschild complex is contractible in high degrees.

<u>Remark</u>: There is a connection on $\Omega^1 R$ if and only if R is quasi-free. A connection is equivalent to a lifting $l : \Omega^1 R \to \Omega^1 R \otimes R$ $ml = id$.

<u>Examples of connections</u>:

$$\nabla : \Omega^1 R \longrightarrow \Omega^1 R \otimes_R \Omega^1 R = \Omega^2 R$$

satisfying

$$\begin{aligned} \nabla(x\xi) &= x\nabla\xi \\ \nabla(\xi x) &= \nabla\xi . x + \xi dx \end{aligned} \quad \begin{cases} x \in R \\ \xi \in \Omega^1 R \end{cases}$$

$$\Omega^1 R \cong R \otimes \bar{R}$$
$$x dy \longleftarrow x \otimes y$$

∇ is ~~the same~~ determined by $\varphi : \bar{R} \to \Omega^2 R$ satisfying $\varphi = \nabla d$

$$\varphi(xy) = x\varphi(y) + \varphi(x)y + dxdy$$

<u>Example</u>: $R = T(V)$ free algebra on V

$$\Omega^1 R = R \otimes V \otimes R$$
$$x dv y \longleftarrow x \otimes v \otimes y$$

There is a canonical connection ∇ determined by the requirement that $\nabla v = 0 \quad \forall v \in V$.

$$\begin{tikzcd}
0 \arrow[r] & \Omega^2 R \arrow[r, hook, "i"] & \Omega^1 R \otimes R \arrow[r, "m"] \arrow[dll, bend right, "\nabla"'] & \Omega^1 R \arrow[r] \arrow[dl, bend right, "\ell"'] & 0 \\
R\otimes V\otimes R\otimes V\otimes R \arrow[r] & R\otimes V\otimes R\otimes R \arrow[r] & R\otimes V\otimes R \arrow[r] & 0 &
\end{tikzcd}$$

$$dv\otimes y \longmapsto dvy$$

$$dv\otimes 1 \xleftarrow{\ell} dv$$

$$(R\otimes V\otimes R)\otimes R$$

$$0 \longrightarrow \Omega^2 R \longrightarrow \Omega^1 R\otimes R \longrightarrow \Omega^1 R \to 0$$

$$\begin{tikzcd}
 & dvy\otimes 1 \arrow[r, mapsto] \arrow[dl] \arrow[d, phantom, "\|"] & dvy \arrow[dl, leftrightarrow, "\ell"] \\
dvdy \arrow[dr] & dv\otimes y \arrow[d, phantom, "+"] & \\
 & dv(y\otimes 1 - y\otimes 1) &
\end{tikzcd}$$

$$\nabla(xdvy) = x(\nabla(dv)y + dvdy) = xdv$$

$$(\nabla d)(v_1 \dots v_n) = \nabla \sum v_1 \dots v_{j-1} dv_j\, dv_{j+1} \dots v_n$$
$$= \sum_{j=1}^{n-1} v_1 \dots v_{j-1}\, dv_j\, d(v_{j+1} \dots v_n)$$

<u>Fedosov's Construction</u> Let

$$\Omega^0 \to \Omega^1 \to \Omega^2 \to$$

be a differential graded algebra. Define the Fedosov product on Ω to be

$$x \circ y = xy + c(-1)^{|x|} dxdy$$

where $\underline{c}$ is a fixed constant.

Example: $\Omega = \Omega A \qquad c = (-1)$

Then $(\Omega A), \circ) \cong QA$, in particular

$(\Omega^+ A, \circ) \cong RA$

$$a_0 da_1 \dots da_{2n} \longleftrightarrow \rho(a_0)\omega^n(a_1, \dots a_{2n})$$

as an algebra isomorphism.

Consider Ω a commutative differential graded algebra. Let $R = (\Omega^+, \circ)$

Ω^+ - even part of Ω

X(R)

$$\begin{tikzcd} R \arrow[r, shift right, "\bar{d}"'] \arrow[d, equal, "="'] & \Omega' R_q \arrow[l, shift right, "\bar{b}"'] \arrow[d] \\ \Omega^+ \arrow[r, shift right, "Nd"'] & \Omega^- \arrow[l, shift right, "2cd"'] \end{tikzcd} \qquad \begin{tikzcd} xd^R y \arrow[d, mapsto] \\ \Phi(x,y) \end{tikzcd}$$

where $\Phi(x,y) = xdy + |y| d(xy)$

$$\Phi(x,y) = (1+|y|) xdy + |y| dx.y$$

Claim is that 1) Φ does define a map on $\Omega' R_q$ and that the diagram commutes.

1) means that $b\Phi = 0$ i.e.

$$\bar{\Phi}(xoy, z) \pm \bar{\Phi}(x, yoz) + \bar{\Phi}(zox, y) = 0$$

One calculates

$$\Phi(zox, y) = (1+|y|)(zox)dy + |y| d(zox)y$$

$$= (1+|y|)(zx + cdzdx)dy + |y|(dzxy + zdxy)$$

$$\Phi(xoy, z) = (1+|z|)(xy + cdxdy)dz + |z|(dxyz + xdyz)$$

$$\Phi(x, yoz) \pm \Phi(x, yz) + c\,\Phi(x, dydz)$$

$$= (1+|y|+|z|)x(dyz + ydz) + (|y|+|z|)dx.yz + + c(1+|y|+(+|z|)dxdydz$$

$$\bar{b}(xd^R y) = [x,y]^o \quad \text{so}$$

$$\begin{tikzcd} {[x,y]^o} \arrow[d, equal] & xd^R y \arrow[l, mapsto] \arrow[d, mapsto] \\ 2c\,dxdy & xdy + |y| d(xy) \arrow[l, mapsto, "2cd"'] \end{tikzcd}$$

But $[x,y]^o = xoy - yox$
$= xy + c\,dxdy - yx - c\,dydx$
$= 2c\,dxdy$ since x, y even

so the diagram commutes this way round.

Define N by $Ny = |y| y$ on homogeneous elements.

$$\begin{tikzcd} y \arrow[r, mapsto, "\bar{d}"] \arrow[d, equal] & d^k y \arrow[d, mapsto, "\alpha"] \\ y \arrow[r, mapsto] & dy + e(y)dy = N(dy) \end{tikzcd}$$

and so this also commutes.

~~We~~ The approach to periodic cyclic (co)homology we have been developing is based on the idea of representing ~~cyclic~~ $\left\{\begin{array}{l}\text{even classes as traces}\\ \text{odd} \quad - - - \quad \text{cyclic one cocycles}\end{array}\right.$ on a nilpotent extension of the algebra

RA/IA^N $\quad\quad$ $\Omega(RA)/IA^N)_\natural$

↑
nilpotent extension of ~~A~~ $\underline{A}$

Recall Connes Theorem: $\underline{A}$ commutative algebra smooth then

$$HP_{\pm}(A) = H^{\pm}(\Omega_A)$$

Thus closed currents on $\underline{A}$ (means linear

functionals on Ω_A which kill $d\Omega_A$) determine periodic cyclic cohomology and every class is represented this way.

$$\begin{tikzcd} RA \arrow[r, shift left, "\bar{d}"'] & \Omega^1(RA)_\natural \arrow[l, shift left, "\bar{b}"'] \\ \Omega^+A \arrow[u, "\rho w^n"', "\cong"] \arrow[r, shift right, "-N_{K^2}b + B"'] & \Omega^+A \arrow[u, "\cong"'] \arrow[l, shift right, "b-(1+K)d"'] \end{tikzcd}$$

If $\xi \in \Omega^{2n}A$ then

$$N_{K^2}\, b\xi = \sum_{j=0}^{n-1} K^{2j}\, b\xi$$

If $\tau \in (RA)^*$ then $(\tau\bar{b})_{2n+1} = b\tau_{2n} - (1-K)S\tau_{2n+2}$

If $\underline{A}$ is commutative one can define

$$\begin{tikzcd} \Omega^+A \arrow[r, shift left] \arrow[d, "\mu"'] & \Omega^-A \arrow[l, shift left] \arrow[d, "\mu"] \\ \Omega^+_A & \Omega^-_A \arrow[l, shift right, "-2d"'] \arrow[l, shift left, "Nd"] \end{tikzcd}$$

$\mu : \Omega^+A \to \Omega^+_A$ is the canonical DG algebra homomorphism

Claim that this commutes.

$$b(\omega da) = (-1)^{|\omega|}(\omega a - a\omega)$$

Define b on Ω_A to be zero. Then $\underline{\mu}$ is compatible with $\underline{d}, \underline{b}$

$$1 - K = bde\ db = 0$$

so that the Karoubi operator is the identity

$$-N_{K^2} b + B = B = Nd$$
$$b - (bea)d = -2d.$$

Poisson structures on manifolds $\underline{M}$

More general than symplectic structure.

A Poisson structure on $\underline{M}$ implies that there is an operator $\underline{b}$ of degree (-1) on the manifold $\underline{M}$ such that $(\Omega(M), b, d)$ is a mixed complex

$$\begin{cases} b^2 = 0 \\ [b, d] = 0 \end{cases}$$

If $\{f, g\}$ is the Poisson bracket, then

Definition: A Poisson structure on M is a bilinear operation on functions
$$(f, g) \mapsto \{f, g\}$$
on $C^\infty(M)$ so that it is 1) a biderivation
2) $\{\ ,\ \}$ makes $C^\infty(M)$ into a Lie algebra
(1) – derivation with one of variables fixed.

Example: $(f, g) \longmapsto 0$

e.g. $f(x, \xi)$ $\qquad f \in C^\infty(TM)$

$$\{f, g\} = \sum_i \left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial \xi_i} - \frac{\partial f}{\partial \xi_i}\frac{\partial g}{\partial x_i}\right)$$

$$\{fh, g\} = f\{h, g\} + h\{f, g\} \text{ etc.}$$

$\{f, g\}(x)$ depends only on $df(x)$, $dg(x)$ so $\{\ ,\ \}$ is determined by an ~~smoo~~ element $\underset{\sim}{w}$ of $C^\infty(M, \Lambda^2 T)$.

ιw is an operator of degree (-2) on ΩM where ι denotes interior product.

We define
$$b = [\iota(w), d]$$
to give an operator of degree (-1), which

anticommutes with $\underline{d}$. $bd + db = 0$.
By calculation
$$b^2 = 0 \iff \text{Jacobi identity holds.}$$

Weyl Algebras V vector space of finite dimension with a skew symmetric form on V
$$\omega : \Lambda^2 V \to \mathbb{C} \qquad \omega \in (\Lambda^2 V)^*$$
Let $A_\omega = T(V)/([v_1, v_2] - \omega(v_1, v_2))$ be defined by analogy with the Clifford algebra
$$C_q = T(V)/(v^2 - q(v))$$
$$q : S^2 V \to \mathbb{C}$$

Examples of Weyl algebras: Let $V = \mathbb{C}p \oplus \mathbb{C}q$ and $\omega(p,q) = 1$
$$A_\omega = T(p,q)/(pq - qp = 1)$$
$$\cong \mathbb{C}[x, \partial_x]$$
differential operators with polynomial co

Generalization: Let V be as before, S another alg
$$\omega : \Lambda^2 V \to \text{center of } S$$
$$A_{(S,\omega)} = S \otimes T(V) / ([v_1, v_2] = \omega(v_1,$$

Examples: 1) (Universal case) $S^{univ} = S(\Lambda^2 V)$ – the symmetric algebra on $\Lambda^2 V$.
Then there is a canonical map homomorphism
$$A_{(S(\Lambda^2 V), \omega^{univ})} \longrightarrow A_{(S,\omega)}$$
for any (S, ω).

1) Take $S = (\Lambda V)^*$, $\underline{\omega}$ to be the obvious inclusion $\Lambda^2 V \subseteq \Lambda V$
$$A_{\Lambda V} = \Lambda V \otimes T(V) / ([v_1, v_2] = v_1 \wedge v_2$$
Claim that $A_{\Lambda V} = \Omega_{S(V)}$ with Fedosov product.
$$\cong \underbrace{S(V)}_{\text{generated by } \underline{v}} \otimes \underbrace{\Lambda(V)}_{\text{generated by } d\underline{v}}$$

In general A_ω is isomorphic (as a vector space) to $S(V)$
In the example if $A_\omega = T(p,q)/([p,q] = 1)$
Standard recipe for an isomorphism
$$S(p,q) \cong \mathbb{C}[x, \partial_x]$$

One writes $q^m p^n \longleftrightarrow x^m \partial_x^n$

This is not the canonical isomorphism!

The canonical one is due to Weyl and

$$(c_1 q + c_2 p)^n \longleftrightarrow (c_1 x + c_2 \partial_x)^n$$

for any $c_1, c_2 \in \mathbb{C}$, $n \in \mathbb{N}$.

$$qp \longmapsto \tfrac{1}{2}(x\partial_x + \partial_x x)$$

The Weyl algebra is a twisted polynomial algebra.

Fedosov product

$$\xi \circ \eta = \xi\eta + c\, d\xi d\eta$$

Now $\Lambda V = \Lambda[dv ; v \in V]$ is an obvious subalgeb of $(\Omega_{S(V)}, \circ)$ (closed forms).

Map generators

$$A_{\Lambda V} = \Lambda(V) \otimes T(V)/\sim \;\longrightarrow \Omega_{S(V)}$$

$$\begin{tikzcd} V \arrow[r, mapsto] & dv \in \Omega \end{tikzcd}$$

$$\begin{tikzcd} V \arrow[r, mapsto] & V \in \Omega \end{tikzcd}$$

$$[V_1, V_2] \longmapsto V_1 \circ V_2 - V_2 \circ V_1$$
$$= v_1 v_2 + c\, dv_1 dv_2 - V_2 V_1 - c d$$
$$= 2c\, dv_1 dv_2$$

Hence if we take $c = 1/2$ one has the relations $[v_1, v_2] = dv_1 dv_2$ satisfied.

<u>Index Theorem on $\mathbb{R}^n$</u>

q_i — position co-ordinates

p_i — momentum co-ordinates

$$\left.\begin{aligned} q_i &— \text{multiplication by } x_i \\ p_i &— h/i\, \partial x_i \end{aligned}\right\} \text{on } L^2(\mathbb{R}^n)$$

$$\mathrm{Tr}_{L^2}(e^{-1/2 p^2 + V(q)}) \qquad V - \text{potential}$$

How is this quantum-mechanical quantity related to the classical

$$\sim \int_{\text{cotangent space}} \left(\frac{dq\,dp}{2\pi h}\right)^n e^{-1/2 p^2 + V(q)}$$

asymptotic as $h \to 0$.

Want to assign in a natural way Schwartz class functions on the cotangent space co-ordinates (q,p) to operators on the Hilbert space $L^2(\mathbb{R}^n)$ which are of trace class, depending upon the parameter $\underline{h}$ such that the above asymptotic formula holds.

Theory of pseudo-differential operators.

Let V be the linear functions on the cotangent space, W the cotangent space

$$W^* = V$$

$\mathcal{S}(W)$ = Schwartz functions on W

Natural symplectic form on W

$$\omega : \Lambda^2 W \to \mathbb{C}$$

Weyl algebra

$$A_\omega = \text{twisted form of } S(V)$$
$$= \text{polynomial functions on } W$$

Analogous twisted algebra

$$\mathcal{S}(W)_\omega$$

$f \in \mathcal{S}(W)$ $\qquad f = \int_{v \in V} \hat{f}(v) e^{iv} dv$

$$g = \int \hat{g}(v) e^{iv} dv$$

To multiply these we define $f \circ_\omega g$ to be the extension of

$$e^{iv_1} \circ e^{iv_2} = e^{i(v_1+v_2)} e^{i\omega(v_1,v_2)}$$

$$[v_1, v_2] = \omega(v_1, v_2)$$

Uniqueness of Stone-von Neumann representation on Hilbert space of canonical anticommutation relations.

This is done for every $\underline{h}$

$$[v_1, v_2] = h\omega(v_1, v_2)$$

$$e^{iv_1} \circ e^{iv_2} = e^{i(v_1+v_2)} e^{ih\omega(v_1,v_2)}$$

to get a family of algebras parametrized by $\underline{h}$.

$$\mathcal{S}(W) = \mathcal{S}(W)_{h\omega} \xrightarrow{\text{Trace}}$$

$$\xrightarrow[\text{Quantization}]{}$$

Have different algebra structures on the Schwartz space

$$\begin{tikzcd}\mathcal{L}(W) \arrow[r] \arrow[rr, bend right, "\int"'] & \mathcal{L}(W)_{\hbar w} \arrow[r, "\text{Trace}"] & \mathbb{C}\end{tikzcd}$$

Claim is that the Trace is given by

$$\int_W f = \hat{f}(0)$$

There is a K-class for the identity of $\mathcal{L}(W)$.
Idempotent matrix
Well-defined map

$$\text{K theory of } \mathcal{L}(W) \to \text{K theory of } \mathcal{L}(W)_{\hbar w} \xrightarrow{\text{index}} \mathbb{C}$$

$$HP^*(\mathcal{L}(W)) = HP^*(\mathcal{L}(W)_{\hbar w})$$

Homotopy invariant of algebra – means that we can let $\underline{h} \to 0$.

Comes down to an algebraic question:

Just like we formed $A_{\Lambda V}$ and $A_{(S,w)}$ we can form

$$\mathcal{L}(W)_{(S,w)} = S \otimes \mathcal{L}(W) / \text{relations}$$

$$e^{iv_1} * e^{iv_2} = e^{i(v_1+v_2)} \underset{\substack{\wedge \\ S}}{e^{iw(v_1,v_2)}}$$

(eg $\underline{S}$ nilpotent).
Formal algebraic problem: Deal with $S = (\Lambda^2 V)^*$ as the ultimate thing to work with.
$S(V)$ replaced by $\mathcal{L}$

$$\begin{tikzcd}K_0(\mathcal{L}(V)) \arrow[r, "\text{analytically}"] & \mathbb{C} & \\ K_0((S(V))_{\Lambda^2 V}) \arrow[u, "\text{nilpotent ext}"] \arrow[ur, "\text{trace on } S(W)_{\Lambda^2 V}"'] & & \\ & & \text{Fedosov algebra of forms which are Schwartz class on } W \arrow[u]\end{tikzcd}$$

Avoids the need to go to h^n powers in asymptotic expansions.