Lectures courses by Daniel G Quillen

B. Topics in K-theory and cyclic homology, Hilary Term 1989

68 pages of notes. The lecture notes are concerned with index theory and Fredholm modules for operator families over manifolds. The topics include: cyclic homology classes. Currents on manifolds. Cochains with the b and λ operations. The fundamental class. The double complex for b and b'. The cup producy on cochains. Differential graded algebras. Traces and almost homomorphisms. Bianchi's identity. Index theory on the circle: Toeplitz operators. Index of Fredholm operators. Lefschetz formula. Cyclic 1-cocycles. Abstract GNS construction. Generalized Stinepsring theorem. The GNS algebra. Algebraic analogue of GNS. Cuntz algebra from free products; the superalgebra envelope. Fredholm modules. Gradings. DGA and Ω_A. The Fedosov product. Supertraces and derivations. The β and d double complex. Connections and characteristic classes. The de Rham class. The bar construction with Hochschild differential b. Connes bicomplex. Connes's long exact sequence. Bar construction and Connes S operator. Chain and cochain versions of Connes's bicomplex. Connes cyclic bicomplex. Homotopy and total differentials. Vecor valued traces, vector bundles over manifolds and connections. Closed currents. Duhamel's principle. Superconnections. Graded and ungraded bundles. The index theorem via Getzler calculus.

Editor's remark The lecture notes were taken during lectures at the Mathematical Institute on St Giles in Oxford. There have been subsequent corrections, by whitening out writing errors. The pages are numbered, but there is no general numbering system for theorems and definitions. For the most part, the results are in consecutive order, although in one course the lecturer interrupted the flow to present a self-contained lecture on a topic to be developed further in the subsequent lecture course. The note taker did not record dates of lectures, so it is likely that some lectures were missed in the sequence. The courses typicaly start with common material, then branch out into particular topics. Quillen seldom provided any references during lectures, and the lecture presentation seems simpler than some of the material in the papers.

- D. Quillen, Cyclic cohomology and algebra extensions, *K-Theory* **3**, 205–246.
- D. Quillen, Algebra cochains and cyclic cohomology, *Inst. Hautes Etudes Sci. Publ. Math.* **68** (1988), 139–174.
- J. Cuntz and D. Quillen, Cyclic homology and nonsingularity, *J. Amer. Math. Soc.* **8** (1995), 373–442.

Commonly used notation

k a field, usually of characteristic zero, often the complex numbers

A an associative unital algebra over k, possibly noncommutative

$\bar{A} = A/k$ the algebra reduced by the subspace of multiples of the identity

$\Omega^n A = A \otimes (\bar{A} \otimes \ldots \otimes \bar{A})$

$\omega = a_0 da_1 \ldots da_n$ an element of $\Omega^n A$

$\Omega A = \oplus_{n=0}^{\infty} \Omega^n A$ the universal algebra of abstract differential forms

e an idempotent in A

d the formal differential (on bar complex or tensor algebra)

b Hochschild differential

b', B differentials in the sense of Connes's noncommutative differential geometry

λ a cyclic permutation operator

K the Karoubi operator

$\circ$ the Fedosov product

G the Greens function of abstract Hodge theory

N averaging operator

P the projection in abstract Hodge theory

D an abstract Dirac operator

∇ a connection

I an ideal in A

V vector space

M manifold

E vector bundle over manifold

τ a trace

$T(A) = \oplus_{n=0}^{\infty} A^{\otimes n}$ the universal tensor algebra over A

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Cyclic homology classes

K field of characteristic 0 (usually $\mathbb{C}$)

A algebra over K associative algebra non unital

Trace linear functional. $\tau : A \to k$

$\tau(ab) = \tau(ba)$ $[A,A]$ = sub space spanned by $[a,b] = ab - ba$ $\tau \in (A/[A,A])^*$

$\varphi(a_1, \dots, a_p)$ multilinear functional on A.

$\varphi \in (A^{\otimes p})^*$

$$(b\varphi)(a_0, a_1, \dots, a_p) = \sum_{i=0}^{p} (-1)^i \varphi(a_0, \dots, a_i a_{i+1}, \dots, a_p) + (-1)^p \varphi(a_p a_0, a_1, \dots, a_{p-1})$$

$$(\lambda\varphi)(a_1, \dots, a_p) = (-1)^{p-1} \varphi(a_p, a_p, a_2, \dots, a_{p-1})$$

Definition: A cyclic cocycle on A is a $\varphi(a_0, a_1, \dots, a_n)$ such that $b\varphi = 0$ and $\lambda\varphi = \varphi$

$n = 0$ $\varphi(a)$ $b\varphi(a_0, a_1) = \varphi(a_0 a_1) - \varphi(a_1 a_0)$

$\therefore$ cyclic 0-cocycle is a trace.

Example: $A = C^\infty(M)$ M manifold

A (continuous) trace on A = distribution on M

= 0-dimensional current on M

(Def^n k-dimensional current on M is a continuous linear functional on the space of k-forms on M).

Let $\omega^k \longmapsto \int_\gamma \omega$ denote a k-dimensional current on M.

Associate to it the cochain

$$\varphi_\gamma(a_0,\dots,a_p) = \int_\gamma a_0\,da_1\dots da_p$$

<u>Proposition</u>: $b\varphi_\gamma = 0$ $\quad \lambda\varphi_\gamma = \varphi_\gamma$ if the current γ is closed (i.e. if $\int_\gamma d\omega^{p-1} = 0$ $\forall \omega^{p-1}$).

$$\lambda\varphi_\gamma(a_0,\dots,a_p) = (-1)^{p+1}\varphi_\gamma(a_p,a_0,\dots,a_{p-1})$$

$$= (-1)^{p+1}\int_\gamma a_p\,da_0\dots da_{p-1}$$

$$\varphi_\gamma(a_0,\dots,a_p) = \int_\gamma a_0\,da_1\dots da_p$$

$$a_p\,da_0\dots da_{p-1} = d(a_0\,da_1\dots da_{p-1}\,a_p) + (-1)^{p-1}a_0\,da_1\dots da_p$$

If M is compact and oriented of dimension n, then it has a canonical cyclic n cocycle, namely

$$\int_M a_0\,da_1\dots da_n$$

- called the fundamental class.

Cochains on A. V vector space $C^p(A,V)$
$= \{f: A^p \to V,\ f \text{ multilinear}\} = \mathrm{Hom}(A^{\otimes p},V)$

On $C^\cdot(A,V)$ we have various operations
bf, λf as before. Also, if $f(a_0,\dots,a_p) \in C^p(A,V)$

$$(b'f)(a_0,\dots,a_p) = \sum_{i=0}^{p-1}(-1)^i f(a_0,\dots,a_ia_{i+1},\dots,a_p)$$

$$(Nf)(a_1,\dots,a_p) = \left(\sum_{i=0}^{p-1}\lambda^i\right)f$$

$$(Nf)(a_1,\dots,a_p) = \sum_{i=1}^{p}(-1)^{i(p-1)}f(a_i,\dots,a_p,a_1,\dots,a_{i-1})$$

These operators satisfy identities, which together say that the following diagram is a double complex

$$\begin{tikzcd}
{} & {} & {} & {} \\
C^{p+1}(A,V) \arrow[u, "b"] \arrow[r, "1-\lambda"] & C^p(A,V) \arrow[u, "-b"] \arrow[r, "N"] & {} \arrow[u, "b"] \arrow[r] & {} \\
C^p(A,V) \arrow[u, "b"] \arrow[r, "1-\lambda"] & C^p(A,V) \arrow[u, "-b'"] \arrow[r, "N"] & C^p(A,V) \arrow[u, "b"] \arrow[r, "1-\lambda"] & {} \\
{} \arrow[u] & {} \arrow[u] & {} \arrow[u] &
\end{tikzcd}$$

Identities: $b^2 = 0$ $\quad (b')^2 = 0$

$(1-\lambda)b = b'(1-\lambda)$

$Nb' = bN$

Let R be another algebra, and consider $C^\cdot(A,R)$. Define the cup product of $f \in C^p(A,R)$, $g \in C^q(A,R)$ to be the $p+q$ cochain

$$(f\cdot g)(a_1,\dots,a_{p+q}) = f(a_1,\dots,a_p)\,g(a_{p+1},\dots,a_{p+q})$$

Put $\delta f = -b'f$

e.g. $f \in C^1(A,R)$, $(\delta f)(a_0,a_1) = -b'f(a_0,a_1)$
$= -f(a_0a_1)$

$$\delta(f\cdot g) = \delta f\cdot g + (-1)^{\deg f} f\cdot\delta g$$

$f, g \in C^1(A,R)$

$$\delta(f\cdot g)(a_0,a_1,a_2) = -f(a_0a_1)g(a_2) + f(a_0)g(a_1a_2)$$

$$[(\delta f)\cdot g](a_0,a_1,a_2) \quad - f(a_0a_1)\,g(a_2)$$

$$f\,\delta g\,(a_0,a_1,a_2) = -f(a_0)g(a_1a_2)$$

Example: $f \in C^1(A, R)$ $\quad f : A \to \mathbb{R}$ linear

$$(\delta f + f^2)(a_0, a_1) = -f(a_0 a_1) + f(a_0) f(a_1)$$

$\therefore$ $\delta f + f^2 = 0 \iff f$ is an algebra homomorphism

(suggested by curvature formula)

$C^{\cdot}(A, R)$ with δ is a differential graded algebra. (noncommutative) Let $\tau : R \longrightarrow V$ vector space (e.g. $R/[R, R]$) be a trace on R. Then

Proposition: Put tr_τ :

$C^p_\lambda(A, V)$ = space of multilinear $f(a_0, \dots, a_p)$ valued in V such that $\lambda f = f$

Define $tr_\tau : C^p(A, R) \to C^{p-1}_\lambda(A, V)$

$$tr_\tau f = N\tau f = \tau N f$$

$$tr_\tau f(a_1, \dots, a_p) = \sum_{i=0}^{p-1} \underset{(-1)^{i(p-1)}}{\tau f}(a_{i+1}, \dots, a_p, a_1, \dots, a_i).$$

$$\begin{cases} tr_\tau(f.g) = tr_\tau(g.f)(-1)^{(\deg f)(\deg g)} \\ tr_\tau(\delta f) = -b\, tr_\tau(f) \end{cases}$$ same as $bN = Nb'$

$$\tau(g.f) = \lambda^k_\tau(f.g)$$

$$\therefore N\tau(g.f) = N\lambda^k_\tau(f.g)$$

A, R algebras $\quad I$ ideal in R

$\rho : A \to R$ linear map which is an algebra homomorphism mod I

i.e. $\rho(a_1)\rho(a_2) - \rho(a_1 a_2) \in I \quad \forall a_1, a_2$

$\rho \in C^1(A, R)$

$$(\delta\rho + \rho^2)(a_1, a_2) = \rho(a_1)\rho(a_2) - \rho(a_1 a_2)$$

$\delta\rho + \rho^2 \in C^2(A, I)$

Let $\tau : I^n \to V$ vector space be a linear map such that $\tau([R, I^n]) = 0$

$C^{\cdot}(A, I^n)$ is an ideal in $C^{\cdot}(A, R)$ closed under δ.

$f \in C^p(A, I^n)$ define $tr_\tau(f) = N\tau f \in C^{p-1}_\lambda(A, V)$

For $f \in C(A, I^n)$, $g \in C(A, R)$ we have

$$tr_\tau(fg) = (-1)^{\deg f \deg g} tr_\tau(gf)$$

Review standard characteristic class calculation that $tr(\text{curvature})^n = 0$.

$$\omega = \delta\rho + \rho^2$$

$$\delta\omega = 0 + (\delta\rho)\rho - \rho(\delta\rho)$$

$$= (\delta\rho + \rho^2)\rho - \rho(\delta\rho + \rho^2)$$

$$= \omega\rho - \rho\omega = -[\rho, \omega]$$

Bianchi identity.

$$\delta(\omega^n) = \sum_{i=1}^n \omega^{i-1}(-\rho\omega + \omega\rho)\omega^{n-i} = [-\rho, \omega^n]$$

$$b\, tr_\tau(\omega^n) = -tr_\tau(\delta(\omega^n)) = tr_\tau(\rho\omega^n - \omega^n\rho) = 0$$

$$\omega^n \in C^{2n}(A, I^n)$$

$tr_\tau \omega^n$ is a $2n-1$ cocycle on the algebra A

$$tr_\tau(\omega^n)(a_1, \dots, a_{2n}) = \tau(\omega(a_1, a_2) \dots \omega(a_{2n-1}, a_{2n}))$$

+ cyclic permutations with sign.

is a cyclic $(2n-1)$ cocycle.

Variant: Suppose $\tau([I, I^{n-1}]) = 0$. Then

$\tau([I^p, I^q]) = 0$ for $p + q \geq n$.

Then $tr_\tau(\omega^n)(a_1, \dots, a_{2n}) = n(\tau(\omega(a_1, a_2) \dots \omega(a_{2n-1}, a_{2n}))$

$$- \tau(\omega(a_{2n}, a_1) \dots \omega(a_{2n-2}, a_{2n-1}))$$

Remark : If we copy the standard proof calculation that the de Rham cohomology class of tr (curvature)^n is independent of the choice of connection then we find that the cyclic cohomology class of the above cocycle depends only the actual algebra homomorphism $A \to R/I$.

Example : Take $A = R$. Let $\theta \in C^1(A, A)$ be the identity map of A. Then

$$\delta\theta + \theta^2 = 0$$

Let τ be a trace of A. $\tau : A/[A,A] \to V$

Exercise : Show that $tr_\tau\, \theta^{2n} = 0$ and that $\delta(tr_\tau\, \theta^{2n+1}) = 0$

What is $tr_\tau\, \theta^{2n+1}\ (a_1, \ldots, a_{2n}) =$

Index theory for S^1 - Toeplitz operators

$S^1 = \mathbb{R}/2\pi\mathbb{Z} \quad = \{ \theta \in \mathbb{R}/2\pi\mathbb{Z} \}$

Hilbert space $H = L^2(S^1, d\theta/2\pi)$ is a module over $A = C^\infty(S^1)$. H has an orthonormal basis $e^{in\theta} = z^n \quad (n \in \mathbb{Z})$. Hardy space is the closed linear span of $z^n \quad n \geq 0$.

Let e be the orthogonal projection onto the Hardy space

$$e(z^n) = \begin{cases} 0 & n < 0 \\ z^n & n \geq 0 \end{cases}$$

Toeplitz operator associated to $f \in A$ is defined to be

$$T_f(g) = e(fg) \qquad g \in eH$$

$$\text{or } T_f = efe$$

$$H = eH \oplus (1-e)H$$

Block form of an operator T is

$$T = \begin{pmatrix} eTe & eT(1-e) \\ (1-e)Te & (1-e)T(1-e) \end{pmatrix}$$

Lemma: If $g \in C^\infty(S^1)$ then $[e, g]$ is an operator having a smooth Schwarz kernel - it is a trace class operator. Also $f, g \in C^\infty(S^1)$

$$\text{trace}_H\,(f[e, g]) = \frac{1}{2\pi i}\int_{S^1} f\,dg$$

Proof: $g(\theta) = \sum_n e^{in\theta} c_n = \sum_n e^{in\theta} \int e^{-in\theta'} g(\theta') \frac{d\theta'}{2\pi}$

$$eg = \sum_{n \geq 0} e^{in\theta} \int e^{-in\theta'} g(\theta') \frac{d\theta'}{2\pi}$$

$$= \int \frac{1}{1-e^{i(\theta-\theta')}}\, g(\theta') \frac{d\theta'}{2\pi}$$

$\therefore$ Schwarz kernel of e is $e(\theta, \theta') = 1/1-e^{i(\theta-\theta')}$

$[e, g] = \dfrac{g(\theta) - g(\theta')}{1 - e^{i(\theta-\theta')}}$ Smooth on $S^1 \times S^1$

$$tr_H(f[e,g]) = \int f(\theta) \left[\frac{g(\theta) - g(\theta')}{e^{i(\theta-\theta')} - 1}\right]_{\theta=\theta'} \frac{d\theta}{\pi 2}$$

$$= \int f(\theta)\, g'(\theta)\, d\theta / 2\pi$$

Trace class operators form an ideal $\mathcal{L}^1(H)$ in the algebra $\mathcal{L}(H)$ of bounded operators.

$$R = \mathcal{L}(H) \qquad I = \mathcal{L}^1(H)$$

Then $\rho : f \mapsto T_f \qquad \rho(f) = efe$

$\rho : A \to R$

Then ρ is a homomorphism mod I by the lemma.

$$\rho(f)\rho(g) - \rho(fg) = efege - efge$$

$$= e\, f[e, g]\, e$$

Cyclic cocycle $tr_\tau(\delta\rho + \rho^2)$ τ_H trace defined on the ideal $\mathcal{L}^1(H)$ $\varphi: tr_\tau(\delta\rho+\rho^2)$ is a 1-cocycle

$$\varphi(a_0, a_1) = \tau_H(\rho(a_0)\rho(a_1) - \rho(a_0a_1)) - \tau_H(\rho(a_1)\rho(a_0) - \rho(a_1a_0))$$

Definition: A bounded operator $P: H_1 \to H_2$ between Hilbertspaces is Fredholm if there exists a so-called parametrix operator $Q: H_2 \to H_1$ such that $QP - I$, $PQ - I$ are compact operators.

The index of P = dim ker P − dim coker P

If $QP - I$, $PQ - I$ are trace class then

Proposition: Index $P = \underset{-}{tr_{H_1}}(I-PQ)^n \underset{+}{-} tr_{H_2}(I-QP)^n$

$$\begin{tikzcd} 0 \arrow[r] & K_0 \arrow[r, "P"] \arrow[d, "\varphi"', "id"] & K_1 \arrow[r] \arrow[d, "\varphi"', "id"] & 0 \\ 0 \arrow[r] & K_0 \arrow[r] & K_1 \arrow[r] & 0 \end{tikzcd}$$ complex

Lefschetz: $\sum_i (-1)^i\, tr\,\varphi$ on $H^i(K)$
$= \sum_i (-1)^i\, tr\,\varphi$ on K_i

provided φ's on K_i are trace class

$$\begin{tikzcd} H_1 \arrow[r, "P"] \arrow[d, "I-QP"'] & H_2 \arrow[d, "I-PQ"] \arrow[dl, "Q"'] \\ H_1 \arrow[r, "P"] & H_2 \end{tikzcd}$$ Q homotopy $id \sim 1$

$\mathcal{L}^\infty(H)$ compact operators $\mathcal{L}(H)$ bounded operators
$\mathcal{L}^1(H)$ trace class operators
Toeplitz operator $R = \mathcal{L}(eH) \supset I = \mathcal{L}^1(eH) \xrightarrow{\tau} \mathbb{C}$
$\tau = tr_{eH}$ $\tau([R, I]) = 0$
$\rho(f) = efe$ acting on eH
$\rho: A \to R$ linear

$$(d\rho + \rho^2)(f,g) = \rho(f)\rho(g) - \rho(fg) = efege - efge = ef[e,g]e \in \mathcal{L}^1(eH)$$

$\rho(f)$ is called the Toeplitz operator.
We get a cyclic 1-cocycle on A

$$\varphi = N\tau(\delta\rho + \rho^2)$$

$$\varphi(f,g) = tr_{eH}(\rho(f)\rho(g) - \rho(fg)) - tr_{eH}(\rho(g)\rho(f) - \rho(gf))$$

$$= tr_{eH}(f[e,g])$$

Since i) $tr_H(eT) = tr_H(eTe) = tr_{eH}(eTe)$ for $T \in \mathcal{L}^1(H)$ $(e^2 = e)$
(ii) D derivation, $e^2 = e \Rightarrow De.e + eDe = De$
$\Rightarrow eDe = De.(1-e)$, $(1-e)De = De.e$
e.g. $e[e,g] = [e,g](1-e)$ etc.

$$tr_{eH}(\rho(f)\rho(g) - \rho(fg)) = tr_{eH}(ef[e,g]e) = tr_H(ef[e,g])$$

$$tr_{eH}(\rho(g)\rho(f) - \rho(gf)) = tr_{eH}(egefe - egfe)$$

$$= tr_{eH}(eg(e-1)fe) = tr_H(eg(e-1)f) = tr_H([e,g](e-1)f)$$

$= tr_H((e-1)f[e,g])$ Subtract and we are done. ■

Formula for the cyclic cocycle

$$\varphi(f,g) = tr_H(f[e,g]) = \frac{1}{2\pi i}\int_{S^1} f\,dg$$

Proposition: If $P: H_0 \to H_1$ is a Fredholm operator with parametrix Q (means an inverse modulo the compacts) then one has if $(I-QP)^n$, $(I-PQ)^n \in \mathcal{L}^1(H)$ then

$$\mathrm{Ind}\,P = tr_{H_0}(1-PQ)^n - tr_{H_1}(I-PQ)^n.$$

n some integer ≥ 1.

Apply this for $f \in C^\infty(S^1)$ with $f^{-1} \in C^\infty(S^1)$

$$\rho(f)\rho(f^{-1}) - \rho(1) \in \mathcal{L}^1(eH)$$

∴ $\rho(f)$ is Fredholm and has parametrix $\rho(f^{-1})$ and

$$\begin{aligned}\mathrm{Ind}(\rho(f)) &= tr_{eH}(1-\rho(f^{-1})\rho(f)) - tr_{eH}(1-\rho(f)\rho(f^{-1}))\\ &= -\varphi(f^{-1},f)\\ &= -\frac{1}{2\pi i}\int_{S^1} f^{-1}df = -(\text{winding no. of } f)\end{aligned}$$

To develop this example in at least two ways

1) GNS construction ($\rho: A \to R$, $\rho(a) = eae$)
2) Index - above proposition using K_e, K_1

Let A, B be unital algebras. Let $\rho: A \to B$ be a linear map such that $\rho(1) = 1$

One way to obtain such a ρ is as follows. Let $u: A \to R$ an algebra homomorphism, e be idempotent $e^2 = e$, and let

$B = eRe$ (B is a unital algebra with $I_B = e$, but it is not a unital subalgebra of R).

Take $\rho(a) = eae$ $\rho: A \to B$.

Can any $\rho: A \to B$ $\rho(1) = 1$ be represented in this way?

GNS construction: Consider $M = A \otimes_k B \otimes_k A$ free A bimodule. Define a product on M: $M \times M \to M$

$$(a_1, b_1, c_1)\cdot(a_2, b_2, c_2) = (a_1, b_1\rho(\overset{c_1}{\alpha_1}a_2)b_2, \overset{c_2}{\alpha_2})$$

- really the linear extension - abuse notation

$$(a_1, a_2, \ldots, a_n) = a_1 \otimes a_2 \otimes \ldots \otimes a_n$$

This is associative algebra M - non unital and a A bimodule - these structures are compatible (means that the product map $M \otimes_A M \to M$ is a bimodule.

Form the semidirect product of this algebra

$$R = A \oplus A \otimes B \otimes A$$ GNS algebra associated to ρ GNS(ρ)

$\hat{u}: A \to R$ embedding of A in R.

$$\hat{e} = (1,1,1) \text{ in } M \quad \hat{e}^2 = (1,1,1)\cdot(1,1,1) = (1, 1\rho(1.1)1, 1) = (1,1,1)$$

$$\hat{e}a\hat{e} = (1,1,1)a(1,1,1) = (1,1,1)(a,1,1) = (1,\rho(a),1) \in 1 \otimes B \otimes 1$$

$$\begin{aligned}\hat{e}(a,b,\alpha)\hat{e} &= (1,1,1)(a,b,\alpha)(1,1,1)\\ &= (1,\rho(a)b,\alpha)(1,1,1) = (1,\rho(a)b\rho(\alpha),1) \in 1 \otimes B \otimes 1\end{aligned}$$

$$\hat{e}R\hat{e} = 1 \otimes B \otimes 1 \cong B$$

$$a \mapsto \hat{e}a\hat{e} = 1 \otimes \rho(a) \otimes 1 \leftrightarrow \rho(a)$$

Proposition 1: Universal property of GNS(ρ). Given S unital and $u: A \to S$ algebra homomorphism and $v: B \to S$ a linear map satisfying

$$v(b_1)u(a_n)v(b_2) = u(b_1\rho(a)b_2)$$

$$\iff \begin{cases} v(b_1b_2) = v(b_1)v(b_2) \\ v(1)u(a)v(1) = v(\rho(a)) \end{cases}$$

Then there is a unique Y homomorphism from $GNS(\rho) \overset{w}{\to} S$ alg.

carrying $(\hat{a}, \hat{v})$ to (u, v)

$\hat{v}(b) = (1, b, 1)$

Proof: $w(a) = u(a)$

$w(a, b, \alpha) = u(a)v(b)u(\alpha)$

Check that this works. ■

Proposition 2: Take (B, ρ) to be universal for linear maps of A to another algebra such that $\rho(1) = 1$.

$T(A)$ tensor algebra on A of A as vector space

$$T(A) = \bigoplus_{n \geq 0} A^{\otimes n}$$

$B = T(A) / \text{ideal generated by } 1 - 1_A$

$1 \in A^{\otimes 0} = k \qquad 1_A \in A^{\otimes 1}$

$B \cong T(W) \qquad \text{if } A = k \oplus W$

Gelfand Naimark Segal construction

$$\text{GNS} \begin{array}{l} \nearrow \\ \searrow \end{array}$$

Generalised Stinespring theorem

Given $\rho: A \to B \quad \rho(1) = 1$

Question: Can you find a $A \otimes B^{op}$ module E together with B^{op} module maps

$$B \overset{i}{\hookrightarrow} E \overset{i^*}{\to} B$$

such that $i^*i = id_B \qquad i^*ai(b) = \rho(a)b$?

GNS A $*$ algebra $\quad B = \mathbb{C}$

$\rho: A \to \mathbb{C}$ positive (state)

GNS construction produces a Hilbert space E with A action ($*$ representation of A) i unit vector v in E

$i^* \quad \langle v|$.

Generalised Stinespring theorem says that for ρ a completely positive map of C^* algebras one can find such (E, i, i^*) with E a Hilbert B module.

Given any such (E, i, i^*) there is a natural action of $GNS(\rho)$ on E given by

$$\underset{\underset{M}{\uparrow}}{(a, b, \alpha)}\underset{\underset{E}{\uparrow}}{(\xi)} = aibi^*(\alpha\xi))$$

– this motivates the previous definition.

One can show that such triples (E, i, i^*) are in 1-1 correspondence with $A \otimes B^{op}$ module factorizations of $\tilde{\rho}$

$$A \otimes B \xrightarrow{\tilde{\rho}} \text{Hom}(A, B)$$

$$\tilde{\rho}(a \otimes b)(\alpha) = \rho(\alpha a)b$$

$$\begin{tikzcd} A \otimes B \arrow[rr, "\tilde{\rho}"] \arrow[dr, dotted, "\tilde{i}"'] & & \text{Hom}(A, B) \\ & E \arrow[ur, dotted, "\tilde{i}^*"'] & \end{tikzcd}$$

The smallest such triple is $E = \text{Im}(\tilde{\rho})$.

Algebraic analogue of the GNS construction – actual

GNS E is the Hilbert completion of $\mathrm{Im}(\tilde{\rho})$.

Blackadar – reference K theory of operator algebras

Proposition 1: Given (R, e, u, v) where R is an algebra, e is an idempotent in R $u : A \to R$ an algebra homomorphism $v : B \to eRe$ algebra homomorphism,$^{+}$ then there is a unique algebra homomorphism $\phi : GNS(\rho) \to R$ such that it carries $\hat{e}, \hat{u}, \hat{v}$ to e, u, v resp.

$+$ satisfying $v(\rho(a)) = eu(a)e$

Proof: $\phi(a) = u(a)$

$\phi(a, b, \alpha) = u(a) v(b) u(a)$

– this is necessary. Hence ϕ is unique. Now check that this works. ∎

Various algebras defined by universal mapping properties

free product $A * B$ $\quad \mathrm{Hom}_{alg}(A*B, R)$

$= \mathrm{Hom}_{alg}(A, R) \times \mathrm{Hom}_{alg}(B, R)$

$=$ algebra generated by a single idempotent e

$= k \oplus k\hat{e} \quad \hat{e}^2 = e$

$\mathrm{Hom}_{alg}(k \oplus k\hat{e}, R) = \{ e \in R : e^2 = e \}$

$T(A) \quad \oplus A^{\otimes n}$

$T(V) = \oplus_{n \geq 0} V^{\otimes n}$ $\quad V$ vector space

$\mathrm{Hom}_{alg}(T(V), R) = \mathrm{Hom}_k(V, R)$

$B = T(A) / (1 - 1_A)$

$\hat{\rho} : A \underset{incl.}{\hookrightarrow} T(A) \underset{proj.}{\twoheadrightarrow} B$

$\mathrm{Hom}_{alg}(B, R) = \{ \rho \in \mathrm{Hom}_k(A, B) \mid \rho(1) = 1 \}$

$u \mapsto u\hat{\rho}$

Proposition 2: Take $B = T(A) / (1 - 1_A)$

$\hat{\rho} =$ canonical map : $A \to B$.

Then $C = GNS(\hat{\rho})$ is canonically isomorphic to $A * (k \oplus k\hat{e})$

Proof: An algebra homomorphism $C \to R$ is the same as (e, u, v) as in the first proposition Propn 1.

An algebra homomorphism

$$A * (k \oplus ke) \to R$$

is the same as (e, u) where $u : A \to R$ algebra hom and $e^2 = e$ in R.

$V : B \to eRe$ canonical

By forgetting v, get $\wedge$ algebra homomorphism $A * (k \oplus ke) \to C$ via $(\hat{u}, \hat{e})$.

To show this is an isomorphism must show how to construct a unique v given (u, e).

Set $(u : A \to R,\ e \in R)$ Let $\rho(a) = eu(a)e$. Then $\rho : A \to eRe$ with $\rho(1) = e$. By the universal property of B and $\hat{\rho}$ there is a unique algebra homomorphism $v : B \to eRe$ such that $v(\hat{\rho}(a)) = \rho(a) =$ $= eu(a)e$. ∎

Cuntz algebra associated with A is $A * A$. Two canonical homomorphisms $A \rightrightarrows A * A$

$QA = A * A$. $\quad a \to a, \tilde{a}$

There is a unique involution ~ on QA which interchanges the two ~~factors~~ homomorphisms.

Recall a superalgebra is by definition an algebra R with an action of $\mathbb{Z}/2\mathbb{Z}$ denoted ~~a~~ $r \mapsto \varepsilon r$

$R = R^+ \oplus R^-$ split into ±1 eigenspaces

$\mathrm{Hom}_{alg}(A, R) = \mathrm{Hom}_{superalg}(QA, R)$

QA is the superalgebra envelope of alg A.

Cuntz – Zekri algebra $\mathcal{E}(A) = (A*A) \tilde{\otimes} k[\mathbb{Z}/2\mathbb{Z}]$

(tensor with the group algebra). Let F generate $\mathbb{Z}/2\mathbb{Z}$

$k[\mathbb{Z}/2\mathbb{Z}] = k \oplus kF \qquad F^2 = 1$

$\mathcal{E}(A) = (A*A) \oplus (A*A)F$

multiplication comes from $F\omega F = \tilde{\omega} \quad \omega \in A*A$

First note (i) $A * k[\mathbb{Z}/2\mathbb{Z}] = (A*A) \tilde{\otimes} k[\mathbb{Z}/2\mathbb{Z}]$

Same alg to any alg R
homomorphisms

$\mathrm{Hom}_{alg}(A * k[F], R) = \{(u, F) : \underset{F^2=1}{u : A \to R \text{ alg. hom}}\}$

Given such a (u, F) get another algebra hom. $FuF : A \to R$. Then we get $A*A \to R$ unique superalgebra map extending u. This extends then to the cross product because the $\mathbb{Z}/2\mathbb{Z}$ action on R is inner $(A*A) \tilde{\otimes} k[\mathbb{Z}/2\mathbb{Z}]$

(ii) idempotents are in 1-1 correspondence with involutions

$e \mapsto F = e - (1-e) = 2e-1 \qquad e = \frac{1+F}{2}$

$k \oplus k\hat{e} = k[\mathbb{Z}/2\mathbb{Z}]$

Therefore $GNS(\hat{\rho} : A \to T(A)/(1-1_A)) =$ Cuntz –

Zekri algebra $(A*A) \tilde{\otimes} k[\mathbb{Z}/2\mathbb{Z}]$.

Definition : Suppose A $*$ module. Then a Fredholm module over A is a representation of A on a separable Hilbert space H together with an involution F on H $(F = F^*,\ F^2 = I)$ such that $[F, a] \in \mathcal{L}^\infty(H)$ (compact operators) for all $a \in A$.

Ex/ $H = L^2(S^1) \qquad A = C^\infty(S^1) \qquad F = 2e-1$

e projection on Hardy space

$[F, f] = 2[e, f]$ – even trace class

A graded Fredholm module ~~is~~ over A is a Fredholm module (H, F) together with a $\mathbb{Z}/2\mathbb{Z}$ grading on H. $H = H^+ \oplus H^- \qquad \varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

such that $\begin{cases} \varepsilon a = a\varepsilon & \forall a \in A \\ \varepsilon F = -\varepsilon F \end{cases}$

General class of examples when $A = C^\infty(M)$ M - compact Riemannian manifold are given by the Dirac operators.

In the above Dirac $D = \frac{1}{i} d/d\theta + c$

$F = D/(D^2)^{1/2}$

(need to have zero kernel – hence we perturb the Dirac operator $\frac{1}{i} d/d\theta$)

Remark: A Fredholm module over A is the same thing as a $*$ representation on some Hilbert space of the Cuntz - Zekri algebra $A * \mathbb{C}[F] = (A*A) \tilde{\otimes} \mathbb{C}[F]$ such that the ideal $\mathrm{Ker}\{A * \mathbb{C}[F] \to A \otimes \mathbb{C}(F) = A*A\}$ = ideal generated by $\{[F, a] : a \in A\}$ such that the ideal is represented by compact operators.

A graded Fredholm module is equivalent to a $*$ representation of the Cuntz algebra $QA = A * A$ such that the ideal $\mathrm{Ker}\,(A * A \to A)$ is represented by compact operators.

Graded case: For ε as above $F = \begin{pmatrix} 0 & T^{-1} \\ T & 0 \end{pmatrix}$

$F^* = F \Rightarrow T$ is unitary $T : H^+ \xrightarrow{\sim} H^-$

Can assume $H^- = H^+$ and $F = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$

Then as $\varepsilon a = a \varepsilon$

$$a = \begin{pmatrix} ua & 0 \\ 0 & va \end{pmatrix}$$

where u, v are $*$ homomorphisms $u, v : A \to \mathcal{L}(H^+)$

$A * A \to \mathcal{L}(H^+)$

Compactness condition $[F, a] = \begin{pmatrix} 0 & va - ua \\ ua - va & 0 \end{pmatrix}$

$\therefore$ $[F, a]$ compact $\forall a$

$\Longleftrightarrow$ ua, va are congruent mod $\mathcal{L}^\infty(H^+)$

$\Longleftrightarrow$ the map $A * A \to \mathcal{L}(H)$ carries $\mathrm{Ker}(A * A \to A)$ to compact operators.

Programme is to study various algebras associated to A including Ω_A, $A * A$, $T_r(A) = T(A)/((-1_A)$ and their traces.

Ω_A - algebra of noncommutative differential forms.

$a \mapsto \bar{a}$ $\bar{A} = A/k$. differential graded algebras $R = \bigoplus_{n \in \mathbb{Z}} R^n$

$d : R^n \to R^{n+1}$

<u>Proposition</u>: There exists a DGA Ω_A, which is unique up to canonical isomorphism such that

$\Omega_A^0 = A$ $\Omega_A^n = 0$ $n < 0$ and such that

$A \otimes \bar{A}^{\otimes n} \to \Omega_A^n$ $(a_0, \bar{a}_1, \ldots, \bar{a}_n) \mapsto a_0 da_1 \ldots da_n$ is a vector space isomorphism for all $n \geq 0$.

Moreover given any (R, u) where R is DGA and $u : A \to R^0$ is a homomorphism there is a unique DGA homomorphism $\Omega_A \to R$ which extends u.

Proof: Put $K =$ the complex $K^n = \begin{cases} A \otimes \bar{A}^{\otimes n} & n \geq 0 \\ 0 & n < 0 \end{cases}$

$$\partial(a_0, \bar{a}_1, \ldots, \bar{a}_n) = (1, \bar{a}_0, \bar{a}_1, \ldots, \bar{a}_n)$$

<u>Lemma</u>: Let R be a DGA and $u : A \to R^0$ a homomorphism and $\tilde{u} : K \to R$ be given by

$\tilde{u}(a_0, \bar{a}_1, \ldots, \bar{a}_n) = u(a_0)\, du(a_1) \ldots du(a_n)$

If R is the smallest DG subalgebra of itself containing $u(A)$, then $\tilde{u}$ is surjective. (Use Leibniz!)

Proof: Let $L = \mathrm{Im}\,\tilde{u}$ $\tilde{u}$ is a map of complexes and so L is a subcomplex of R.

Consider $\{x \in R : xL \subseteq L\}$. This is a DGA subalgebra of R and it contains $u(A)$, so by hypothesis this set is all of R. Hence L is a left ideal in R containing I and so $L = R$. ■

Existence of Ω_A: K The k-linear operators on K form a DGA $\mathrm{End}(K)$ with differential

$$d\omega = \delta \circ \omega - (-1)^{\deg \omega}\, \omega \circ \delta$$

$i : A \to \mathrm{End}(K)$ $\quad$ $ia(a_0, \bar{a}_1, \ldots, \bar{a}_n) = (aa_0, \bar{a}_1, \ldots, \bar{a}_n)$

Put $\Omega =$ the smallest DG subalgebra of $\mathrm{End}(K)$ containing $i(A)$.

Have two maps $\Phi = \tilde{i} : K \to \Omega$

$\Psi(a_0, \bar{a}_1, \dots, \bar{a}_n) = i(a_0)\, di(a_1) \dots di(a_n)$ surjective

$\Phi : \Omega \subset \operatorname{End}(K) \xrightarrow{\circ 1} K$ $\qquad$ K is a left

$\qquad \omega \longmapsto \omega \circ 1$ $\qquad$ Ω module

$$\Phi(\Psi(a_0, \dots, \bar{a}_n)) = \Phi(\,i(a_0)\, d\,i(a_1) \dots d\,i(a_n) \circ 1)$$
$$= w_0 [\partial, i(a_1)] \cdots [\partial, ia_n] 1$$

$$[\partial, i(a_n)] \circ 1 = \partial a_n = (1, \bar{a}_n)$$
$$= (a_0, \bar{a}_1, \dots, \bar{a}_n)$$

$\Phi \circ \Psi = id$ $\qquad$ Ψ surjective $\quad \therefore \quad \Phi, \Psi$ are isomorphisms.

$$[\partial, i(a_{n-1})](1, \bar{a}_n) = \partial\, i a_{n-1} (1, a_n)$$
$$= \partial(a_{n-1}, \bar{a}_n) = (1, \bar{a}_{n-1}, \bar{a}_n)$$

So Ω with Ω^0_A identified by i with A is the desired D.G.A.

<u>Lemma 2</u>: Suppose (R, u) such that $\tilde{u} : K \to A$ is an isomorphism. Then given any (R', u') there is a unique DGA homomorphism $v : R \to R'$ such that $vu = u'$.

Proof: Form $R \times R'$ and let S be the DG subalgebra generated by the image of $\qquad u_0 = (u, u') : A \to R^0 \times R^{0'}$

projections
$$\begin{tikzcd} R \times R' \arrow[r] \arrow[dr] & R \\ & R' \end{tikzcd}$$

induce DGA hom. $\quad S \xrightarrow{p_1} R \quad S \xrightarrow{p_2} R'$.

Claim p_1 is an isomorphism.

$$\begin{tikzcd} & K \arrow[dl, "\tilde{u}_0"'] \arrow[d, "\tilde{u}\ \simeq"] \\ S \arrow[r, "p_1"'] & R \end{tikzcd}$$

$\tilde{u}, \tilde{u}_0$ surj.

$\tilde{u}$ iso

Take $v = p_2 p_1^{-1} : R \to R'$ ■

$\Omega_A = \oplus_{n \geq 0} \Omega^n_A \qquad \Omega^n_A \cong A \otimes \bar{A}^{\otimes n}$

$$a_0 da_1 \dots da_n \longleftrightarrow (a_0, \bar{a}_1, \dots, \bar{a}_n)$$

$$(a_0 da_1 \dots da_n)\alpha = a_0 da_1 \dots d(a_n \alpha)$$
$$- a_0 da_1 \dots d(a_{n-1} a_n) d\alpha$$
$$+ a_0 da_1 \dots d(a_{n-2} a_{n-1}) da_n d\alpha$$
$$\vdots$$
$$(-1)^{n-1} a_0 d(a_1 a_2) da_3 \dots da_n d\alpha$$
$$+ (-1)^n a_0 a_1 da_2 \dots da_n d\alpha$$

$A * A$ defined by a universal property: There are two canonical homomorphisms $i, \tilde{i} : A \to A * A$ such that given two homomorphisms $u, u' : A \to B$, B another algebra there is a unique hom $w : A * A \to B$ $wi = u$, $w\tilde{i} = u'$.

In particular we get an automorphism of order 2 $w : z \mapsto \tilde{z}$ of $A * A$ such that $wi = \tilde{i}$ and $w\tilde{i} = i$. $A * A$ is a superalgebra (by definition of superalgebra).

Let $\quad a^+ = (ia + \tilde{i}a)/2 \qquad a^- = (ia - \tilde{i}a)/2$ be the even and odd parts of ia.

$$ia = a^+ + a^- \qquad \tilde{i}a = a^+ - a^-$$
$$(aa_0)^+ = a^+ a_0^+ + a^- a_0^-$$
$$(aa_0)^- = a^- a_0^+ + a^+ a_0^-$$

Using these one sees that the elements of the form
$$a_0^+ a_1^- \dots a_n^- \qquad (n \geq 0)$$
span the algebra $A * A$. Let V be the span. V contains iA and $\tilde{i}A$. $\qquad a^+ V \subset V$

$a^+ a_0^+ a_1^- \dots a_n^- = (aa_0)^+ a_1^- \dots a_n^- - (. a^- a_0^- a_1^- \dots a_n^-$

$a^- V \subset V$

Hence V is a left ideal which contains 1 and so is the whole algebra $A * A$.

Define $\Psi : \Omega_A \to A * A$

$$\Psi(a_0 da_1 \dots da_n) = a_0^+ a_1^- \dots a_n^-$$

<u>Proposition</u>: Ψ is a vector space isomorphism and with respect to Ψ the multiplication in $A * A$ becomes the $*$ product

$$\omega * \eta = \omega\eta - (-1)^{\deg \omega} d\omega\, d\eta$$

Proof: We have seen that it is surjective. We define an left module structure over $A * A$ on $\Omega_A = \Omega$.

$\Omega_A = \Omega_A^{\text{even}} \oplus \Omega_A^{\text{odd}}$ $\mathbb{Z}/(2)$ grading.

$\mathrm{End}_k(\Omega)$ is a superalgebra.

$$\begin{aligned} a \longmapsto & (1+d)(\text{left mult. by } a)(1+d)^{-1} \\ &= (1+d)a(1-d) = a + da - ad - dad \\ &= a + [d,a] + [d,a]d \\ &= a + (\text{left mult. by } da) + (\text{left mult by } da)d \\ &= a + da - (da)d \end{aligned}$$

is a homomorphism from A to $\mathrm{End}(\Omega)$. So it extends uniquely to a superalgebra homomorphism from $A * A$ into $\mathrm{End}_k(\Omega)$, such that

$$i a \mapsto a + da - (da)d$$
$$\bar{i} a \mapsto a - da - (da)d$$
$$a^+ \mapsto a - (da)d \qquad a^- \mapsto (da)$$

$\therefore$ Ω becomes an $A * A$ module with the rules

$a^+ . \eta = (a - (da)d)\eta = a\eta - da.d\eta$

$a^- . \eta = (da)\eta$

Check

$$\Psi\omega . \eta = \omega * \eta$$

$\omega = a_0 da_1 \dots da_n \qquad \Psi(\omega) = a_0^+ a_1^- \dots a_n^-$

$$\begin{aligned} \Psi\omega . \eta &= a_0^+ a_1^- \dots a_n^- . \eta \\ &= (a_0 - (da_0)d) da_1 \dots da_n . \eta \\ &= a_0 da_1 \dots d_n \eta - (-1)^n da_0 da_1 \dots da_n d\eta \\ &= \omega\eta - (-1)^n d\omega\, d\eta \end{aligned}$$

$\eta = 1 \Rightarrow \Psi\omega . 1 = \omega \Rightarrow \Psi$ injective

$\Psi : \Omega \to A * A$ is an isomorphism.

$$\begin{aligned} \Psi^{-1}(\Psi\omega . \Psi\eta) &= (\Psi\omega . \Psi\eta) . 1 \\ &= \Psi\omega . (\Psi\eta . 1) \\ &= \Psi\omega . (\eta) = \omega * \eta \quad \blacksquare \end{aligned}$$

Remarks: 1) $\mathbb{Z}/2$ grading on $A * A$ corresponds to even-odd grading on Ω_A.

2) $i a \longleftrightarrow a + da$

$\bar{i} a \longleftrightarrow a - da$

3) $J = \mathrm{Ker}\{A * A \xrightarrow{\text{fold}} A\} \longleftrightarrow \Omega_A^{>0}$

$= $ ideal generated by a^-

$J^n \longleftrightarrow \Omega_A^{\geq n}$

4) $gr^J(A * A) = \bigoplus_{n \geq 0} J^n / J^{n+1} \underset{\substack{\text{canonical} \\ \text{alg isomorphism.}}}{\cong} \Omega_A$

Consider a supertrace τ on $A * A$.

Given τ define cochains

$$\varphi_n(a_0, \dots, a_n) = \tau(a_0^+ a_1^- \dots a_n^-)$$

$\varphi_n(a_0, a_1, \ldots, a_n) = 0$ if $a_i = 1$ for some $i \geq 1$
- normalised here.

$\varphi_n(1, a_1, \ldots, a_n) = \tau(a_1^- \ldots a_n^-)$ cyclic cochain

i.e. $\varphi(1, a_n, a_1, \ldots a_{n-1}) = (-1)^{n-1} \varphi_n(1, a_1, \ldots, a_n)$

$$\begin{aligned}(b\varphi_n)(a_0, a_1, \ldots, a_n, a_{n+1}) = {} & \tau((a_0 a_1)^+ a_2^- \ldots a_{n+1}^-) \\ & - \tau(a_0^+ (a_1 a_2)^- \ldots a_{n+1}^-) \\ & + \ldots \\ & + (-1)^n \tau(a_0^+ a_1^- \ldots (a_n a_{n+1})^-) \\ & + (-1)^n \tau((a_{n+1} a_0)^+ a_1^- \ldots a_n^-)\end{aligned}$$

$$\begin{aligned}(a_0 a_1)^+ &= a_0^+ a_1^+ + a_0^- a_1^- \\ (a_1 a_2)^- &= a_1^+ a_2^- + a_1^- a_2^+ \\ (a_2 a_3)^- &= a_2^+ a_3^- + a_2^- a_3^+\end{aligned}$$

$$\begin{aligned}\therefore (b\varphi_n)(a_0, a_1, \ldots, a_{n+1}) &= \tau(a_0^- a_1^- \ldots a_{n+1}^-) \\ &\quad + (-1)^{n+1} \tau(a_{n+1}^- a_0^- \ldots a_n^-) \\ &= 2\tau(a_0^- \ldots a_{n+1}^-)\end{aligned}$$

$\therefore (b\varphi_n)(a_0, a_1, \ldots, a_{n+1}) = 2\varphi_{n+2}(1, a_0, \ldots, a_{n+1})$

Definition: B on normalised cochains $\varphi(a_0, \ldots, a_n)$

$$(B\varphi)(a_1, \ldots, a_n) = \sum_{i=1}^{n} (-1)^{i(n-1)} \varphi(1, a_{i+1}, \ldots, a_n, a_1, \ldots, a_i)$$

$$(B\varphi_{n+2})(a_0, \ldots, a_{n+1}) = (n+2)\varphi_{n+2}(1, a_0, \ldots, a_{n+1})$$

R algebra

$R \xrightarrow{d} \Omega^1_R \xrightarrow{d} \Omega^2_R$

Definition: Let M be an R bimodule. Then a derivation $D : R \to M$ is a k linear map such that

$$D(xy) = x(Dy) + (Dx)y$$

<u>Proposition</u>

$$R^{\otimes 4} \xrightarrow{b'} R^{\otimes 3} \xrightarrow{b'} R^{\otimes 2} \xrightarrow{b'} R \to 0 \qquad (+)$$

$$b'(x_1, \ldots, x_n) = \sum (-1)^{i-1} (x_1, \ldots, x_i x_{i+1}, \ldots, x_n)$$

$(b')^2 = 0$

$s(x_1, \ldots, x_n) = (1, x_1, \ldots, x_n)$

$b's + sb' = \text{id}$ homotopy operator.

b' is a R-bimodule map; s is a right R-module map. The sequence $(+)$ is exact.

Let $\Omega^1_R = \operatorname{Ker}(R^{\otimes 2} \xrightarrow{b'} R) = \operatorname{Im}(R^{\otimes 3} \xrightarrow{b'} R^{\otimes 2})$

$= \operatorname{Coker}(R^{\otimes 4} \xrightarrow{b'} R^{\otimes 3})$.

Ω^1_R is an R-bimodule

$b'(x, y, z, w) = (xy, z, w) - (x, yz, w) + (x, y, zw)$

There is a map $\partial : R \to \Omega^1_R$ given by sending y to class of $1 \otimes y \otimes 1$ in $R^{\otimes 3} \bmod b' R^{\otimes 4}$

One has $\partial(yz) = (1, yz, 1) \equiv (y, z, 1) + (1, y, z)$ $\bmod b' R^{\otimes 4}$

$= y\partial z + (\partial y)z$

Hence $\partial : R \to \Omega^1_R$ is a derivation.

<u>Proposition</u>: onto $\operatorname{Hom}_{R \otimes R}(\Omega^1_R, M) \to \operatorname{Der}(R, M)$

$u \mapsto u \circ \partial$

i.e. For any D there is a unique u such that

$$\begin{tikzcd} R \arrow[r, "D"] \arrow[dr, "\partial"'] & M \\ & \Omega^1_R \arrow[u, "u"'] \end{tikzcd}$$
commutes.

Proof: Obvious if one takes $\Omega^1_R = R^{\otimes 3}/b'R^{\otimes 4}$ $\blacksquare$

$$0 \to \Omega^1_R \xrightarrow{j} R\otimes R \xrightarrow{b'} R \to 0 \quad \text{is exact}$$
$\uparrow$ induced by $b' : R^{\otimes 3} \to R^{\otimes 2}$

$j : x\partial y z \mapsto (xy, z) - (x, yz)$

j is the bimodule map which corresponds to the derivation $y \mapsto y\otimes 1 - 1\otimes y$

Example: Suppose R is free. $R = T(V)$. Claim that $R\otimes V\otimes R \xrightarrow{\sim} \Omega^1_R$

$(x, v, y) \mapsto x(\partial v)y$

Proof: This is equivalent to 'any ~~derivation~~ linear map from V to a bimodule M extends to a derivation D of the tensor algebra $T(V) \to M$'

$v_1 \dots v_n = (v_1, \dots, v_n) \in V^{\otimes n}$

$D(v_1 \dots v_n) = \sum_i v_1 \dots (Dv_i) v_{i+1} \dots v_n$

$\lambda : V \to M$

$D(v_1 \dots v_n) = \sum_i v_1 \dots (\lambda v_i) v_{i+1} \dots v_n$

Formula for D corresponding to λ

Definition: If M is an R bimodule then let

$M\otimes_R = M/[R,M]$ $\quad [R,m]$ = subspace spanned by $[r,m] = rm - mr$

$\Omega^1_R\otimes_R$

Proposition: $\mathrm{Hom}_k(\Omega^1_R\otimes_R, W) = \{\varphi : R^{\otimes 2} \to W \mid b\varphi = 0\}$ where $(b\varphi)(x,y,z) = \varphi(xy,z)$

$- \varphi(x, yz) + \varphi(zx, y)$.

Proof:

$$\begin{tikzcd} R^{\otimes 4} \arrow[r, "b'"] \arrow[d] & R^{\otimes 3} \arrow[r] \arrow[d] & \Omega^1_R \arrow[r] \arrow[d] & 0 & \text{exact} \\ R^{\otimes 4}/[R,R^{\otimes 4}] \arrow[r] \arrow[d, equal] & R^{\otimes 3}/[R,R^{\otimes 3}] \arrow[r] \arrow[d, equal] & \Omega^1_R\otimes R \arrow[r] & 0 & \text{exact} \\ R^{\otimes 3} & R^{\otimes 2} & & & \end{tikzcd}$$

$R^{\otimes 3} \xrightarrow{\sim} R^{\otimes 4}/[R, R^{\otimes 4}]$ $\quad (x,y,z) \mapsto (x,y,z,1)$

map back is $\quad (x,y,z,w) \mapsto (wx,y,z)$

$(x,y,z) \mapsto (x,y,z,1) \longmapsto (xy,z,1) - (x,yz,1) + (x,y,z)$

$\Omega^1_R\otimes_R = \mathrm{Coker}\{R^{\otimes 3} \xrightarrow{b} R^{\otimes 2}\}$ $\blacksquare$

R, R' $\Omega^1_R \otimes R$

Terminology: $\varphi : R^{\otimes 2} \to W$ such that $b\varphi = 0$ will be called a Hochschild 1-cocycle with values in W.

$R\otimes R \to \Omega^1_R\otimes_R$

$(x,y) \mapsto x\partial y$

the universal Hochschild one-cocycle

Remark: Everything above extends to DGA, with the supersign conventions

Ω^1_R is a DG bimodule over R.

$R \xrightarrow{\partial} \Omega^1_R$ is compatible with d.

$\Omega^1_R\otimes_R$ is a complex

Two canonical maps

$R \xrightarrow{\partial} \Omega^1_R\otimes_R \qquad x \mapsto \partial x$

$$R \xrightarrow{\partial} \Omega^1_R \otimes R \xrightarrow{\beta} R \xrightarrow{\partial} \Omega^1_R \otimes R$$

$$\beta(x\partial y\, z) = \beta(zx\partial y) = [(zx), y]$$

$\beta \circ \partial = 0 \qquad \partial \circ \beta = 0$

Supersign conventions

$$[x, y] = xy - (-1)^{\deg x}(-1)^{\deg y}\, yx \quad \text{etc.}$$

<u>Key Lemma</u> Let R be a DGA $\quad R^n \to R^{n+1} \to \ldots$

We have a double complex

$$\ldots \xrightarrow{\beta} R \xrightarrow{\partial} \Omega^1_R \otimes_R \xrightarrow{\beta} R \xrightarrow{\partial} \ldots$$

$$\begin{tikzcd}
\arrow[r] & R^{n+2} \arrow[r, "\partial"] & (\Omega^1_R \otimes R)^{n+2} \arrow[r, "\beta"] & R^{n+2} \\
\arrow[r] & R^{n+1} \arrow[r, "\partial"] \arrow[u, "d"] & (\Omega^1_R \otimes_R)^{n+1} \arrow[r, "\beta"] \arrow[u, "d"] & R^{n+1} \\
\arrow[r] & R^n \arrow[r, "\partial"] \arrow[u, "d"] & (\Omega^1_R \otimes_R)^n \arrow[r, "\beta"] \arrow[u, "d"] & R^n
\end{tikzcd}$$

Let $\rho \in R^1 \qquad \gamma = d\rho + \rho^2 \in R^2$

Form $\gamma^n \in R^{2n} \qquad \frac{\gamma^n}{n!}\partial\rho \in R(\Omega^1_R \otimes_R)^{2n+1}$

Then $d(\gamma^n) = \beta(\gamma^n \partial\rho)$

$d(\gamma^{n+1}\partial\rho) = (n+1)\,\partial\,\gamma^n$

$\partial(\gamma^{n+1}) = (n+1)\gamma^n\partial\gamma =$

$d(\gamma^n\partial\rho) = \gamma^n\partial\gamma$

Proof: $d\gamma^n = -[\rho, \gamma^n] \qquad$ Bianchi $d\gamma = -[\rho, \gamma]$

$= -\rho\gamma^n + \gamma^n\rho$

$\beta(\gamma^n\partial\rho) = [\gamma^n, \rho] = \gamma^n\rho - \rho\gamma^n$

$d(\gamma^n\partial\rho) = d\gamma^n\,\partial\rho + \gamma^n d\partial\rho$

$= (-\rho\gamma^n + \gamma^n\rho)\partial\rho + \gamma^n\partial d\rho$

$\gamma^n\partial\gamma = \gamma^n\partial(d\rho + \rho^2)$

$= \gamma^n(\partial d\rho) + \gamma^n\partial(\rho^2)$

$= \gamma^n(\partial d\rho) + \gamma^n\rho\partial\rho + \rho\gamma^n\partial\rho\,(-1)$

Construction of $\Omega^1_R \otimes_R$ continued later.

Review ∇, ∇^2, NS Theorem for vector bundles)

M manifold $\quad E$ complex vector bundle $\quad V \cong \mathbb{C}^n$

$\tilde{V}$ = trivial bundle over $M \qquad M \times V \to M$

$\Gamma(M, E)$ smooth sections $\qquad \Gamma(M, \tilde{V}) = C^\infty(M, V)$

A connection on a bundle E is an operator

$$\Gamma(M, E) \xrightarrow{\nabla} \Gamma(M, T^* \otimes E)$$

$$\nabla(fs) = f\nabla s + df.s$$

If $E = \tilde{V}$ then $\nabla = d + \theta \qquad \theta \in \Omega^1_M \otimes End(V)$

$\nabla s = ds + \theta s \qquad$ matrix 1-form.

Put $\Omega(M, E) = \Gamma(M, \Lambda^p T^* \otimes E) \qquad E$-valued p forms.

∇ extends uniquely to an operator

$$\Omega^0(M, E) \xrightarrow{\nabla} \Omega^1(M, E) \to \Omega^2(M, E)$$

$\omega \in \Omega(M) \qquad \nabla(\omega\eta) = d\omega\,\eta + (-1)^{\deg\omega}\omega\nabla\eta$

$\eta \in \Omega(M, E)$

$\nabla^2 \neq 0$ in general but $\nabla^2(\omega\eta) = \omega\nabla^2\eta$

E.g. $\quad \nabla^2 = (d+\theta)^2 = d^2 + d\theta + \theta d + \theta^2$

$= (\text{multip}^n \text{ by } d\theta) + \theta^2$

$\nabla^2 \in \Omega^2(M, End\,E)$ is called the curvature connection.

Characteristic classes are defined by substituting ∇^2

into invariant polynomial functions on matrices.

$\det \nabla^2 \quad tr(\Lambda^q \nabla^2)$

e.g. $tr(\nabla^2)^n \quad \in \Omega^{2n}(M)$

<u>Proposition:</u> (i) $tr(\nabla^2)^n$ is closed (ii) Its de Rham cohomology class is independent of the choice of connection.

Proof: (i) Local matter, can assume that $E = \tilde{V}$.

Then $\nabla = d + \Theta \qquad \nabla^2 = (d\Theta) + \Theta^2$

For matrix forms α one has

$$d \, tr\, \alpha = tr\, [\nabla, \alpha]$$

Because $d\, tr\, \alpha = tr\, d\alpha = tr[d, \alpha]$

$= tr\,[d, \alpha] + tr\,[\Theta, \alpha] = tr\,[d+\Theta, \alpha] = tr\,[\nabla, \alpha]$

$d\, tr\,(\nabla^{2n}) = tr\,[\nabla, \nabla^{2n}] = 0$

(ii) homotopy: Let ∇_t be a one-parameter family of connections.

$\frac{\partial}{\partial t} tr\,(\nabla_t^2)^n \qquad\qquad \partial/\partial t\, \nabla_t^2 = \nabla_t \dot{\nabla}_t + \dot{\nabla}_t \nabla_t = [\nabla_t, \dot{\nabla}_t]$

$= tr \sum_{i=1}^n (\nabla_t^2)^i (\nabla_t^2)^{\cdot} (\nabla_t^2)^{n-i-1}$

$= tr \sum_{i=1}^n (\nabla_t^2)^i [\nabla_t, \dot{\nabla}_t] (\nabla_t^2)^{n-i-1}$

$= tr\,([\nabla_t, \sum_{i=1}^n (\nabla_t^2)^i \dot{\nabla}_t (\nabla_t^2)^{n-i-1}])$

since $[\nabla_t, \nabla_t^2] = 0$.

$= d\, tr\,(\sum_{i=1}^n (\nabla_t^2)^i \dot{\nabla}_t (\nabla_t^2)^{n-i-1})$

$= dn\, tr\,((\nabla_t^2)^{n-1} \dot{\nabla}_t)$

$tr\,(\nabla_1^{2n}) - tr\, \nabla_0^{2n} = d \int_0^1 n\, tr((\nabla_t)^{2n-1} \dot{\nabla}_t)\, dt$

$\therefore tr\, \nabla_1^{2n}, tr\, \nabla_0^{2n}$ are cohomologous.

Any two connections are joined by a linear path.

$\nabla_t = (1-t) \nabla_0 + t \nabla_1$ ∎

Grassmannian connections and Narasimhan–Ramanan Thm

$\tilde{V}$ has an obvious connection $\nabla = d$

Suppose $\tilde{V} = E \oplus E'$ $\qquad$ $E \underset{i}{\overset{i^*}{\leftrightarrows}} V \underset{j}{\overset{j^*}{\rightleftarrows}} E'$

are the embeddings (resp.* projections)

Then E, E' have induced connections (the Grassmannian connections)

$\nabla = i^* d \cdot i \qquad \nabla' = j^* d \cdot j$

$\nabla^2 = i^* d\, i\, i^* d\, i = i^* d\, (1 - j j^*)\, d\, i$

$= -(i^* d\, j)(j^* d\, i)$

$i^* d\, j$ is linear over $C^\infty(M)$ as $i^* j = 0$

$i^* d j \in \Omega^1(M, Hom(E', E))$.

$j^* d i \in \Omega^1(M, Hom(E, E'))$.

$(\nabla')^2, \nabla^2$ – product of two tensors (Gauss's discovery)

What does this say about $tr\, \nabla^{2n}$ etc.?

Change notation. Suppose we have a vector bundle with a flat connection D and a splitting (involution) F (endomorphism of bundle of square 1).

$$D = \frac{D + FDF}{2} + \frac{D - FDF}{2}$$

$$D = \nabla + \alpha$$

∇ is a connection which commutes with F. α is a 1-form anti-commuting with F.

$D = i^* D_i + j^* D_j$ in the above example.

$\alpha = i^* D_j + j^* D_i$

$0 = D^2 = \nabla^2 + \nabla\alpha + \alpha\nabla + \alpha^2 = \nabla^2 + \alpha^2 + [\nabla,\alpha]$

$\nabla^2+\alpha^2$ commutes with F, $[\nabla,\alpha]$ anticommutes with F.

∴ $\nabla^2+\alpha^2 = 0$, $[\nabla,\alpha] = 0$

$d\,\mathrm{tr}\,(\nabla^2)^n = \mathrm{tr}\,[\nabla,(\nabla^2)^n] = 0$

∴ $\mathrm{tr}(\nabla^2)^n$ is closed.

$\mathrm{tr}\,(\nabla^2)^n = \mathrm{tr}\,(-\alpha^2)^n = (-1)^n\,\mathrm{tr}(\alpha^{2n}) = 0$

Since $\mathrm{tr}\,\alpha^{2n} = \mathrm{tr}\,(\alpha.\alpha^{2n-1}) = -\mathrm{tr}\,(\alpha^{2n-1}\alpha) = -\mathrm{tr}(\alpha^{2n})$

Not surprising since $\mathrm{tr}\,\nabla^{2n} = \mathrm{tr}_E\,(\nabla_E^{2n}) + \mathrm{tr}_{E'}(\nabla_{E'}^{2n})$

Instead we consider $\mathrm{tr}\,(F\nabla^{2n}) = 2\,\mathrm{tr}_E(\nabla_E^{2n})^E$

$(-1)^n\,\mathrm{tr}\,(F\alpha^{2n})$ is closed

$d\,\mathrm{tr}\,F\alpha^{2n} = \mathrm{tr}\,(d(F\alpha^{2n})) = \mathrm{tr}\,[\nabla, F\alpha^{2n}]$

$[\nabla,F]=0$, $[\nabla,\alpha]=0$

Proposition: In this situation (D flat, F an involution $D = \frac{1}{2}(D+FDF) + \frac{1}{2}(D-FDF)$ the forms $\mathrm{tr}\,(F\alpha^{2n})$ are closed and their de Rham class doesn't change as F is varied smoothly.

Proof: Only need to check the homotopy formula.

Suppose we have a family $F = F_t$ of involutions on the bundle.

$\partial_t(\mathrm{tr}\ F\alpha^{2n}) = \mathrm{tr}(\dot F\alpha^{2n}) + \sum_1^{2n}\mathrm{tr}(F\alpha^i\dot\alpha\alpha^{2n-1-i})$

$= 0 + (2n)\,\mathrm{tr}\,(F\dot\alpha\,\alpha^{2n-1})$

$\dot\alpha = \left(\frac{D-FDF}{2}\right)^{\cdot}$ depends only on the part of $\dot\alpha$ anticommuting with α

$= \frac{-\dot F DF - FD\dot F}{2} = \frac{-\dot F(\nabla+\alpha)F - F(\nabla+\alpha)\dot F}{2}$

$= -\frac{1}{2}\underset{\text{anticommutes with } F}{(\dot F\nabla F + F\nabla\dot F)} - \frac{1}{2}\underset{\text{commutes with } F}{(\dot F\alpha F + F\alpha\dot F)}$

$= -\frac{1}{2}[\nabla, F\dot F]$

$\partial_t\ \mathrm{tr}\,(F\alpha^{2n}) = (-n)\,d\,\mathrm{tr}\,(\dot F\alpha^{2n-1})$

Tensor coalgebra: coalgebra C $\quad C \xrightarrow{\Delta} C\otimes C$

$C \xrightarrow{\eta} k$ $\quad\Delta$ associative

Let A be a vector space, $C = T(A) = \bigoplus_{n\geq 0} A^{\otimes n}$

It is a coalgebra with $\Delta: C \to C\otimes C$

$\Delta(a_1,\dots,a_n) = \sum_{i=0}^n (a_1,\dots,a_i)\otimes(a_{i+1},\dots,a_n)$

$$C \xrightarrow{\Delta} C^{\otimes 2} \xrightarrow{\Delta\otimes \mathrm{Id}-\mathrm{Id}\otimes\Delta} C^{\otimes 3} \qquad \text{compos. is } 0$$

$$R \xleftarrow{\mu} R^{\otimes 2} \xleftarrow{\mu\otimes I - I\otimes\mu} R^{\otimes 3}$$

Lemma: $0 \to A\otimes C \xrightarrow{I} C^{\otimes 2} \xrightarrow{J} C^{\otimes 3}$

$I(a_0\otimes(a_1,\dots,a_n)) = \sum_{0\leq j\leq k\leq n}(a_{k+1},\dots,a_n,a_0,a_1,\dots,a_j)\otimes(a_{j+1},\dots,a_k)$

$J = 1\otimes\Delta - \Delta\otimes 1 + \cancel{\sum_i \delta\otimes}\ \sigma(1\otimes\Delta)$

σ denotes the forward shift cyclic permutation.

$\sigma: C^{\otimes 3} \to C^{\otimes 3}$

$$C^{\otimes 2} \xrightarrow{1\otimes\Delta} C^{\otimes 3} \xrightarrow{\sigma} C^{\otimes 3}$$

$(x,y,z) \longmapsto (z,x,y)$

The above sequence is exact.

Proof: Motivation $R = T(V)$ tensor algebra on V

One has the exact sequence

(1) $R^{\otimes 4} \xrightarrow{b'} R^{\otimes 3} \xrightarrow{p} R\otimes V\otimes R$

$b'(xy, z, w) = (xy, z, w) - (x, yz, w) + (x, y, zw)$

$p(x \otimes v_1, \dots, v_n \otimes y) = \sum_{i=1}^{n} (x v_1 \dots v_{i-1}) \otimes v_i \otimes (v_{i+1} \dots v_n y)$

$\mathrm{Hom}_{R\text{ bimodules}}(\mathrm{Coker}\, b', M) = \mathrm{Der}(R, M)$ any alg R

and for $R = T(V)$ $\quad \underset{D \in}{\mathrm{Der}(R, M)} = \mathrm{Hom}(V, M)$

$D(v_1 \dots v_n) = \sum_{i=1}^{n} v_1 \dots v_{i-1} Dv_i \, v_{i+1} \dots v_n$

$\mathrm{Hom}(V, M) = \mathrm{Hom}_{R\text{ bimodules}}(R \otimes V \otimes R, M)$

Apply $M \mapsto M/[R, M] = M \otimes_R$ to (1) and we get

(2)
$$\begin{tikzcd}
R^{\otimes 4} \arrow[r, "b'"] \arrow[d] & R^{\otimes 3} \arrow[r, "p"] \arrow[d] & R \otimes V \otimes R \arrow[r] \arrow[d] & 0 \\
R^{\otimes 3} \arrow[r, "b"] & R^{\otimes 2} \arrow[r, "\bar{p}"] & R \otimes V \arrow[r] & 0
\end{tikzcd}$$

$(x, y, z, w) \mapsto (wx, y, z)$ - maps down

$b(x, y, z) = (xy, z) - (x, yz) + (zx, y)$

$\bar{p}(x \otimes v_1 \dots v_n) = \sum_{i=0}^{n} v_{i+1} \dots v_n x v_1 \dots v_{i-1} \otimes v_i$

(3)
$$\begin{tikzcd}
R^{\otimes 3} \arrow[r, "b"] \arrow[d, "s"', "\sigma"] & R^{\otimes 2} \arrow[r, "\bar{p}"] \arrow[d, "s"', "\sigma"] & R \otimes V \arrow[d, "s"', "\sigma"] & \\
R^{\otimes 3} \arrow[r, "\tilde{b}"] & \cancel{V \otimes} R^{\otimes 2} \arrow[r, "\tilde{p}"] & V \otimes R \arrow[r] & 0
\end{tikzcd}$$
exact

[illegible]

$(x, y, z) \xmapsto{\sigma} (z, x, y) \xmapsto{b} (zx, y) - (z, xy) + (yz, x)$
$\xmapsto{\sigma} (y, zx) - (xy, z) + (x, yz)$

$\tilde{b} = \sigma b \sigma$

$\tilde{b} = (1 \otimes \mu)\sigma^{-1} - \mu \otimes 1 + 1 \otimes \mu$

$\tilde{p}(v_1 \dots v_n \otimes x) = \sum_{i=1}^{n} v_i \otimes v_{i+1} \dots v_n x v_1 \dots v_{i-1}$

Conclusion is that (3) is exact.

Now to prove lemma we may assume dim $A < \infty$. (take inductive limits !)

Suffices to prove that the dual sequence is exact.

(split up into spaces of the same degree - then take the dual as vector spaces). Claim you then get (3) above.

Take $V = A^*$

$C_n^* = (A^{\otimes n})^* = V^{\otimes n}$

Have to check that

$I \mapsto \tilde{p}$ $\quad$ i.e. $I^t = \tilde{p}$
$J^t = \tilde{b}$

$\tilde{b} = (1 \otimes \mu)\sigma^{-1} - \mu \otimes 1 + 1 \otimes \mu$

$(\tilde{b})^t = \sigma(1 \otimes \Delta) - \Delta \otimes 1 + 1 \otimes \Delta = J$

Check $I^t = \tilde{p}$

$\langle I(a_0 \otimes (a_1 \dots a_n)), (v_0 \dots v_p) \otimes (v_{p+1}, \dots, v_n) \rangle$

$\overset{?}{=} \langle a_0 \otimes (a_1, \dots, a_n), \underset{\tilde{p}}{\tilde{p}}((v_1, \dots, v_p) \otimes (v_{p+1}, \dots, v_n)) \rangle \quad \bar{R} \otimes R$

lhs $\langle \sum_{0 \le j \le h \le n} (a_{h+1}, \dots, a_n, a_0, \dots, a_j) \otimes (a_{j+1}, \dots, a_h), v_0 \dots v_p \otimes v_{p+1} \dots v_n \rangle$

$\langle a_0 \otimes (a_1 \ldots a_n), \sum_{i=0}^{p} v_i \otimes v_{i+1} \ldots v_p v_{p+1} \ldots v_n v_0 \ldots v_{i-1} \rangle$

$= \sum_{i=0}^{p} \langle a_0, v_i \rangle \langle a_1, v_{i+1} \rangle \ldots \langle a_{p-i}, v_p \rangle \ldots \langle a_{n-i}, v_n \rangle \ldots \langle a_n, v_{i-1} \rangle$

$=$ ~~$\sum_{i=0}^{p} \langle a_{n-i+1}, \ldots$~~

$= \sum_i \langle (a_{n-i+1}, \ldots a_n, a_0, \ldots a_{p-1}) \otimes (a_{p-i+1}, \ldots a_{n-i}), (v_0, \ldots v_p) \otimes (v_{p+1} \ldots v_n) \rangle$

In the earlier expression we have 0 unless $k-j=n-p$. Changing the notation accordingly shows we have the pairing. $\blacksquare$

Bar construction of algebras: Let A be a (non-unital) algebra $\tilde{A} = k \oplus A$. The bar construction of $\tilde{A}$ is the DG coalgebra C
$= T(A[1])$ (A located in degree 1 i.e. $C_n = A^{\otimes n}$)

$\Delta(a_0, \ldots a_n)$ as before with differential b'. This means that $\Delta : C \to C \otimes C$ is a map of complexes.

Apply the lemma to C with standard signs. This gives an exact sequence

$$(*) \quad 0 \to A[1] \otimes C \xrightarrow{I} C \otimes C \xrightarrow{J} C^{\otimes 3}$$

J as before

$$I(a_0 \otimes (a_1, \ldots a_n)) = \sum_{0 \le j \le k \le n} (a_{k+1}, \ldots, a_n, a_0, a_1, \ldots a_j) \otimes (a_{j+1}, \ldots a_k)(-1)^{n(k+1)}$$

$J = 1 \otimes \Delta - \Delta \otimes 1 + \sigma^{-1}(1 \otimes \Delta)$ is a map of complexes

Hence there is a unique differential on $A[1]$

<u>Theorem</u> $(*)$ is an exact sequence of complexes, where $A[1] \otimes C$ is equipped with the Hochschild differential b.

Proof: Use the map $C^{\otimes 2} \xrightarrow{\tau \otimes 1} A[1] \otimes C$ where τ projection onto $A^{\otimes 1}$.

$(\tau \otimes 1) I = id \quad \therefore \quad \tau \otimes 1$ is an injection

$I(a_0 \otimes (a_1 \ldots a_n)) =$

$\uparrow$ apply $(b' \otimes I + I \otimes b')$

$$= \sum (-1)^{n(k+1)} b'(a_{k+1}, \ldots a_n, a_0, \ldots a_j) \otimes (a_{j+1}, \ldots a_n)$$
$$(-1)^{j+1+n(k+1)} (a_{k+1}, \ldots, a_j) \otimes b'(a_{j+1}, \ldots a_k)$$

Now apply $\tau \otimes 1$. The lower term only contributes when $j=0$ & $k=n$ and the contribution is

$- a_0 \otimes b'(a_1, \ldots a_n)$

The upper contributes when $k=n-1$, $j=0$ or $k=n$ $j=0$.

$(-1)^n b'(a_n, a_0) \otimes (a_1, \ldots, a_{n-1})$
$+ \overset{b}{b'}(a_0, a_1, \ldots) \otimes (a_1, \ldots a_n)$

A non unital algebra

C bar construction on $\bar{A} = k \oplus A$

$$\to A^{\otimes 3} \xrightarrow{b'} A^{\otimes 2} \xrightarrow{b'} A \xrightarrow{b'} k$$

(C, b') is a DG coalgebra

Theorem (*) $0 \to A[1] \otimes C \xrightarrow{I} C^{\otimes 2} \xrightarrow{J} C^{\otimes 3}$

$$(A[1]\otimes C)_{n+1} = A \otimes C_{n} = A^{\otimes n+1}$$

$$I(a_0,\dots,a_n) = \sum_{0\le j\le k\le n} (-1)^{n(k+1)} (a_{k+1},\dots,a_n,a_0,\dots,a_j) \otimes (a_{j+1},\dots,a_k)$$

$$J = -\Delta\otimes 1 + 1\otimes\Delta + \sigma(1\otimes\Delta)$$

(i) (*) is exact, (ii) I is a map of complexes when $A[1]\otimes C$ is given the differential b.

$$Ib = (b'\otimes 1 + 1\otimes b')I$$

Terminology - $A\otimes C$ with the differential b will be called a Hochschild complex. If $1\in A$ then $X_t H_*(A\otimes C, b) = H_*(A,A)$

$$H_*(C) = \begin{cases} k & * = 0 \\ 0 & n \neq 0 \end{cases}$$

$$A\otimes C \xrightarrow{I} C\otimes C \xrightarrow{1\otimes\eta} C\otimes k = C \qquad \eta \text{ counit } C\to k$$

$$(1\otimes\eta) I(a_0,\dots,a_n) = \sum_{0\le j\le k\le n} (-1)^{(j+1)n} (a_{j+1},\dots,a_n,a_0,\dots,a_j)$$

$$= N(a_0,\dots,a_n)$$

$$N = \sum_{i=0}^{n} \lambda^i \quad \text{on } A^{\otimes n+1}$$

Conclude that $N : A\otimes C \to C$ satisfies $b'N = Nb$

$C \xrightarrow{\Delta-\sigma\Delta} C\otimes C$ dual to $R\otimes R \to R$
$(x,y) \mapsto [x,y]$

Verify that $J(\Delta - \sigma\Delta) = 0$ (follows since $b^2=0$).

By exactness there is a factorization

$$\begin{tikzcd} & C \arrow[dl, dashed, "u"'] \arrow[d, "\Delta-\sigma\Delta"] \\ A[1]\otimes C \arrow[r] & C\otimes C \end{tikzcd}$$

where u is some map of complexes.

$$u = (\tau\otimes 1)(\Delta - \sigma\Delta)$$

$\tau : C \to A[1] = A^{\otimes 1}$ projection

Exercise - check that $u : C \to A[1]\otimes C$ is given by $(a_1,\dots,a_n) \mapsto (a_1,\dots,a_n) - (-1)^{n-1}(a_n,a_1,\dots,a_{n-1})$

i.e. $u = 1-\lambda$ on $A^{\otimes n}$.

Conclude that $b(1-\lambda) = (1-\lambda)b'$

Therefore we have constructed the Connes bicomplex (cyclic bicomplex)

$$\begin{tikzcd} & \arrow[d, "b"] & \\ A^{\otimes 3} & A^{\otimes 3} \arrow[l, "N"'] \arrow[d, "b"] & A^{\otimes 3} \arrow[l, "1-\lambda"'] \arrow[d, "b'"] \\ A^{\otimes 2} & A^{\otimes 2} \arrow[l, "N"'] \arrow[d, "b"] & A^{\otimes 2} \arrow[l, "1-\lambda"'] \arrow[d, "b'"] \\ A & A \arrow[l, "N"'] & A \arrow[l, "1-\lambda"'] & A \arrow[l] \end{tikzcd}$$

doubly periodic in both directions.

Def^n: Cyclic complex of A is $CC(A) = \ker\{N : A\otimes C \to C\}$

$$0 \leftarrow CC(A) \leftarrow A\otimes C \xleftarrow{1-\lambda} C \leftarrow CC(A) \leftarrow 0$$

Connes long exact sequence

$$HC_n(A) \longrightarrow H_{n+1}(A,A) \to HC_{n+1}(A) \to HC_{n-1}(A)$$

Cochains: C, A is as before, L algebra (e.g. $B(\ell^2)$).
Put $C(A,L) = \mathrm{Hom}_k(C, L)$. This is an algebra with the product defined by: Let $f, g \in C(A,L)$. Then $f.g = \mu_L (f \otimes g) \Delta$

$$C \xrightarrow{\Delta} C \otimes C \xrightarrow{f \otimes g} L \otimes L \xrightarrow{\mu} L$$

μ multiplication in L

It is a DG algebra with the differential

$$\delta f = d_L \circ f - (-1)^{\deg f} f \circ d_C \qquad d_L = 0$$
$$= -(-1)^{\deg f} f \circ b'$$

Different signs from before.

$f \in C^p(A,L)$ $g \in C^q(A,L)$

$$(f.g)(a_1, \dots, a_{p+q}) = \mu_L\, f \otimes g \left(\sum_{i=0}^{p+q} (a_1, \dots a_i) \otimes (a_{i+1}, \dots, a_n) \right)$$
$$= (-1)^{pq} f(a_1, \dots, a_p)\, g(a_{p+1}, \dots, a_{p+q})$$

$$\delta f(a_1, \dots, a_{p+1}) = (-1)^{p+q} f(b'(a_1, \dots a_{p+1}))$$
$$= (-1)^{p+1}$$

To define a pairing on the bar cochains with values in Hochschild cochains. Let $f \in C(A, L)$, $h \in C(A,M)$ where M is an L bimodule. Let $\tau : M/[L,M] \to k$ (or more generally, into some vector space).

Then we define a Hochschild cochain – an element of $\underline{\mathrm{Hom}_k(A \otimes C, k)}$ to be denoted $\tau(\delta f h)$

e.g. if $\rho \in C^1(A, L)$ then $(\delta\rho + \rho^2)(a_1, a_2) = \rho(a_1 a_2) - \rho(a_1)\rho(a_2)$

by the composite

$$A \otimes C \xrightarrow{I} C \otimes C \xrightarrow{f \otimes h} L \otimes M \xrightarrow{\mu} M \xrightarrow{\tau} k$$

[$\mathrm{Hom}_k(A \otimes C, k)$ is a complex with differential $\delta\varphi = \mp \varphi b$]
$-(-1)^{\deg \varphi}$

Proposition: (1) We have

$$\delta \tau(\partial f h)$$
$$= \tau(\partial(\delta f) h) + (-1)^{\deg f} \tau(\partial f\, \delta h)$$

(2) Let $f, g \in C(A,L)$ $h \in C(A,M)$, then

$$\tau(\partial(fg) h) = \tau(\partial f\, gh) + (-1)^{\deg f(\deg g + \deg h)} \tau(\partial g\, hf).$$

Proof: (2) follows from $JI = 0$ ((1) is straightforward using $\delta(u) = [d, u]$ etc.)

$$\tau(\partial(fg)h) = ?$$

$$\begin{tikzcd}
A \otimes C \arrow[r, "I"] & C \otimes C \arrow[r, "fg \otimes h"] \arrow[d, "\Delta \otimes 1"] & L \otimes M \arrow[r, "\tau\mu"] & k \\
& C \otimes C \otimes C \arrow[r, "f \otimes g \otimes h"] & L \otimes L \otimes M \arrow[u, "\mu \otimes 1"] &
\end{tikzcd}$$

$$\tau(\delta(fg)h) = \tau\mu\, (\mu \otimes 1)(f \otimes g \otimes h)(\Delta \otimes I)\, I$$

$$\tau(\delta f) gh = \tau\mu\, (1 \otimes \mu)(f \otimes g \otimes h)(1 \otimes \Delta) I$$

$$\tau(\partial g\, hf) = \tau\mu\, (1 \otimes \mu)(g \otimes h \otimes f)(1 \otimes \Delta) I$$
$$\qquad\qquad \uparrow \sigma^{-1}\sigma \qquad \uparrow \sigma^{-1}\sigma$$

$$= \tau\mu\, (1 \otimes \mu) \sigma^{-1} (f \otimes g \otimes h) \sigma\, (1 \otimes \Delta) I$$

$$L \otimes L \otimes M \xrightarrow{\mu \otimes 1} L \otimes M \xrightarrow{\mu} M \to k$$

$L\otimes L\otimes M \xrightarrow{1\otimes\mu} L\otimes M \xrightarrow{\mu} M \to k$

$L\otimes M\otimes L \xrightarrow{1\otimes\mu} L\otimes M \xrightarrow{\mu} M$

~~om~~: $L\otimes L\otimes M \xrightarrow{\sigma^{-1}} L\otimes M\otimes L \xrightarrow{1\otimes\mu} L\otimes M \xrightarrow{\mu} M$

Condition of τ implies that $\tau[L,M]=0$ – so all these maps $\tau\mu(\mu\otimes 1)$ $\tau\mu(1\otimes\mu)$ $\tau\mu(1\otimes\mu)\sigma^{-1}$ are the same.

$\tau\mu(1\otimes\mu)\sigma^{-1}(x,y,m) = \tau\mu(y,m,x)$
$= \tau(ymx)$
$\tau\mu(\mu\otimes 1)(x,y,m) = \tau(xym) \quad = \tau(ymx)$

Now take the alternating sum and we find that
$JI = 0$

New Notation: A non unital algebra, $A = k\oplus A$
bar construction of $\bar{A}$ denoted B is the DG co-algebra $B_n = A^{\otimes n}$
$\Delta(a_1,\dots,a_n) = \sum_{i=0}^{n}(a_1,\dots,a_i)\otimes(a_{i+1},\dots,a_n)$
and with $d = b'$.

If L is an algebra then we can form the DG algebra of bar cochains on A values in L.
$C^n(A,L) = \mathrm{Hom}_k(B_n, L)$
Product $f.g = \mu(f\otimes g)\Delta$
$f^p g^q(a_1,\dots,a_{p+q}) = (-1)^{pq} f(a_1,\dots,a_p)\, g(a_{p+1},\dots,a_{p+q})$
Differential $\delta f = -(-1)^{\deg f} f\circ b'$

Inside $C(A,L)$ we can do curvature calculations:
$\rho\in C^1(A,L) \qquad \omega = \delta\rho + \rho^2$
$\omega(a_1,a_2) = \rho(a_1 a_2) - \rho(a_1)\rho(a_2)$
Bianchi $(\delta + \mathrm{ad}\rho)(\omega) = 0 \Rightarrow \delta\omega^n = -\mathrm{ad}\rho\,\omega^n$
$= -\rho\omega^n + \omega^n\rho$

Recall that if τ is a trace on L then one gets a trace on $C(A,L)$
$tr_\tau(\gamma) = N\tau\gamma \in C_\lambda(A)$ cyclic cochains
$\delta\, tr_\tau(\omega^n) = 0$

Goal: to define Connes' S operator and to show the $S[tr_\tau(\frac{\omega^n}{n!})] = [tr_\tau[\omega^{n+1}/(n+1)!]]$.

If $R = C(A,L)$ we need $\Omega^1_R\otimes_R = \Omega^1_R/[R,\Omega^1_R]$

Differentials of bar cochains:

$$\begin{tikzcd}
R^{\otimes 4} \arrow[r] & R^{\otimes 3} \arrow[r, "b'"] \arrow[d] & R^{\otimes 2} \arrow[r, "b'"] & R \\
& \Omega^1_R \arrow[ru] & &
\end{tikzcd}$$

$$\begin{tikzcd}
B \arrow[r, "\Delta"] & B^{\otimes 2} \arrow[r, "\Delta\otimes 1 - 1\otimes\Delta"] \arrow[d, "p"] & B^{\otimes 3} \arrow[r] & B^{\otimes 4} \\
& B\otimes A[1]\otimes B \arrow[ru, bend right, "I"'] & &
\end{tikzcd}$$

exact because B has a counit

$I((a_1,\dots,a_p)\otimes a_{p+1}\otimes(a_{p+2},\dots,a_n))$
$= \sum_{\substack{0\le j\le p \\ p+1\le k\le k}} (a_1,\dots,a_j)\otimes(a_{j+1}\ \ a_k)\otimes(a_{k+1},\dots,a_n)$

$B^{\otimes 3} \xrightarrow{\Delta\otimes 1\otimes 1 - 1\otimes\Delta\otimes 1 + 1\otimes 1\otimes\Delta} B^{\otimes 4}$

$$P((a_1,\dots,a_p)\otimes(a_{p+1},\dots,a_n)) = (a_1,\dots,a_{p-1})\otimes a_p\otimes(a_{p+1},\dots,a_n)$$
$$-(a_1,\dots,a_p)\otimes a_{p+1}\otimes(a_{p+2},\dots,a_n)$$

Verify:

$$IP = \Delta\otimes 1 - 1\otimes\Delta$$

I is injective, P is surjective

$\exists$ induced differential on $B\otimes A[1]\otimes B$ such that I and P are maps of complexes.

There is a retraction $B^{\otimes 3}\to B\otimes A[1]\otimes B$

$1\otimes(\text{proj. on } A[1])\otimes 1$ – this gives the differential

$$d((a_1,\dots,a_p)\otimes a_{p+1}\otimes(a_{p+2},\dots,a_n))$$
$$= b'(a_1,\dots,a_p)\otimes a_{p+1}\otimes(a_{p+2},\dots,a_n)$$
$$+(-1)^p(a_1,\dots,a_{p-1})\otimes a_p a_{p+1}\otimes(a_{p+2},\dots,a_n)$$
$$+(-1)^p(a_1,\dots,a_p)\otimes a_{p+1}a_{p+2}\otimes(a_{p+3},\dots,a_n)$$
$$+(-1)^{p+1}(a_1,\dots,a_p)\otimes a_{p+1}\otimes b'(a_{p+2},\dots,a_n)$$

The above denotes the analogue of Ω'_R for the bar construction.

If $f,g,h\in C(A,C)$, then $f\partial g h\in \mathrm{Hom}_k(B\otimes A[1]\otimes B, C)$ as the composite

$$B\otimes A[1]\otimes B \xhookrightarrow{I} B^{\otimes 3}\xrightarrow{f\otimes g\otimes h} C^{\otimes 3}\xrightarrow{\mu} C$$

$\mathrm{Hom}_k(B\otimes A[1]\otimes B, C)$ bimodule over $C(A,C)$

$g\mapsto \partial g$

Need the analogue of $\Omega'_R\otimes_R$

Def^n: Let M be a C-bicomodule. $M\xrightarrow{\Delta_\ell} C\otimes M$, $M\xrightarrow{\Delta_r} M\otimes C$ which commute.

$M\boxtimes^C = \mathrm{Ker}\{M\xrightarrow{\Delta_\ell-\sigma\Delta_r} C\otimes M\}$ analogue of $M\otimes_R = M/[R,M]$ σ – flip

left exact factor of M.

Claim: $((C\otimes V\otimes C)\boxtimes^C$ free bicomodule $\xleftarrow[\sigma(1\otimes\Delta)]{\sim} V\otimes C$

Proof:
$$\begin{tikzcd} 0 \arrow[r] & V\otimes C \arrow[r, "\sigma(1\otimes\Delta)"] & C\otimes V\otimes C \arrow[r, "\Delta\otimes1\otimes1-\sigma(1\otimes1\otimes\Delta)"] \arrow[l, bend left, "\eta\otimes1\otimes1"] & C\otimes C\otimes V\otimes C \arrow[l, bend left, "1\otimes\eta\otimes1\otimes1"] \end{tikzcd}$$
is exact

Verify that the bottom are a homotopy of the top.

$$\begin{tikzcd} & 0 \arrow[r] & B\otimes A[1]\otimes B \arrow[r, "I"] & B\otimes B\otimes B \arrow[r, "\Delta\otimes1\otimes1-\dots+"] & B^{\otimes 4} \\ B\otimes B \arrow[urr, "P"] & & & & \\ & 0 \arrow[r] & A[1]\otimes B \arrow[r, "\bar{I}"] \arrow[uu, "\sigma(1\otimes\Delta)"] & B\otimes B \arrow[r, "1\otimes\Delta-\Delta\otimes1-\sigma(1\otimes\Delta)"'] \arrow[uu, "\sigma(1\otimes\Delta)"] & B^{\otimes 3} \arrow[uu, "\sigma(1\otimes1\otimes\Delta)"] \\ B \arrow[uu, "\sigma\Delta"] \arrow[urr, "\bar{P}"'] & & & & \end{tikzcd}$$

exact because the top is

$$\bar{I}(a_0\otimes(a_1,\dots,a_n)) = \sum_{0\le i\le k\le n}(-1)^{n(k+1)}(a_{k+1},\dots,a_n,a_0,\dots,a_0)\otimes(a_{i+1},\dots,a_k)$$

$$\bar{P}(a_0\otimes(a_1,\dots,a_n)) =$$

?

$$= -a_0\otimes(a_1,\dots,a_n)+(-1)^{n-1}a_n\otimes(a_1,\dots,a_{n-1}$$

Induced differential on $A[1]\otimes B$ is

$$d(a_0\otimes(a_1,\dots,a_n)) = a_0a_1\otimes(a_2,\dots,a_n) - a_0\otimes b'(a_1,\dots,a_n) + (-1)^n(a_na_0)\otimes(a_1,\dots,a_n)$$

If one identifies $A\otimes A^{\otimes n} = A^{\otimes n+1}$ then

$d=b$, $\quad \bar{p} = -1+\lambda$

Analogue of the sequence

$$\xrightarrow{\beta} \bar{R} \xrightarrow{\partial} \Omega^1_R\otimes_R \xrightarrow{\beta} \bar{R}$$

$\beta(xdyz) = (zx)y - y(zx)$

will be a sequence of complexes as follows

$C(A) = \mathrm{Hom}_k(B,k)$ bar cochains

$C(A,A^*) = \mathrm{Hom}_k(A[1]\otimes B, k)$

$(A[1]\otimes B, b)$ called the cyclic bar construction

$C(A,A^*)$ cyclic bar cochains

$$\begin{tikzcd}
C^{n+1}(A) \arrow[u, "\delta"] \arrow[r, "\partial"] & C^{n+1}(A,A^*) \arrow[u, "\delta"] \arrow[r, "\beta"] & C^{n+1}(A) \arrow[r] & {} \\
C^n(A) \arrow[u, "\delta"] \arrow[r, "\partial"] & C^{n-1}(A,A^*) \arrow[u, "\delta"] \arrow[r, "\beta"] & C^n(A) \arrow[u, "\delta"] \arrow[r] & {} \\
{} \arrow[u] & {} \arrow[u] & {} \arrow[u] & n\geq 1
\end{tikzcd}$$

Comes bicomplex

$(1\otimes\eta)\bar{I} : A[1]\otimes B \to B$

$(1\otimes\eta)\bar{I} = N$ cyclic summation $\Sigma\lambda^i$

This gives rise to δ.

β comes from $\bar{p}$

$\partial = N$ cyclic summing, $\quad \beta = \lambda - 1$

In this double complex the rows are exact and the image of ∂ = $\mathrm{Ker}\,\beta$ = $C_\lambda(A)$ cyclic cochain complex of A.

Comes bicomplex (chain version)

$$\begin{tikzcd}
 & & & {} \arrow[d, "-b'"] & {} \arrow[d, "b"] & \\
0 & A_\lambda^{\otimes 3} \arrow[l] & A^{\otimes 3} \arrow[l, dashed] \arrow[d] & & & \\
0 & A_\lambda^{\otimes 2} \arrow[l] & A^{\otimes 2} \arrow[l, dashed] \arrow[d, "b"'] & A^{\otimes 2} \arrow[l, "1-\lambda"'] \arrow[d, "-b'"] & A^{\otimes 2} \arrow[l, "N"'] \arrow[d, "b"] & {} \arrow[l, "1-\lambda"'] \\
0 & A \arrow[l] & A \arrow[l, dashed] & A \arrow[l, "1-\lambda"'] & A \arrow[l, "N"'] & {} \arrow[l, "1-\lambda"'] \\
 & & A[1]\otimes B \arrow[u, Rightarrow] & \bar{B} \arrow[u, Rightarrow] & &
\end{tikzcd}$$

$B = \mathrm{Bar}\,\bar{A}$

cyclic bar construction

(cochain version)

$C^n(A) = (A^{\otimes n})^*$

$C^n(A,A^*) = (A^{\otimes(n+1)})^*$

$$\begin{tikzcd}
{} & {} & {} & & \\
C^2_\lambda(A) \arrow[u] \arrow[r, hook] & C^2(A,A^*) \arrow[u] \arrow[r] & C^3(A) \arrow[u] & & \\
C^1_\lambda(A) \arrow[u] \arrow[r, hook] & C^1(A,A^*) \arrow[u] \arrow[r, "1-\lambda"] & C^2(A) \arrow[u, "-b'"] \arrow[r, "N"] & C^1(A,A^*) \arrow[u, "b"] & \\
C^0_\lambda(A) \arrow[u] \arrow[r, hook] & C^0(A,A^*) \arrow[u] \arrow[r, "1-\lambda"] & C^1(A) \arrow[u, "-b'"] \arrow[r, "N"] & C^0(A,A^*) \arrow[u, "b"] &
\end{tikzcd}$$

(take transpose to get dual)

$bf(a_1, \ldots, a_n) = f(b(a_1, \ldots, a_n))$ (transposes)

$HC^n(A) = H^n(C_\lambda(A))$

S operation on $HC^n(A)$

$$S : HC^n(A) \to HC^{n+2}(A)$$

is induced by the embedding of the double complex into itself given by moving two steps to the right. (periodicity)

Formula : given $\varphi \in C^n(A, A^*)$ $b\varphi = 0$ and $(1-\lambda)\varphi = 0$. Since the rows are exact we can choose $\psi \in C^{n+1}(A)$ $\psi' \in C^{n+1}(A, A^*)$

$N\psi = \varphi$ $b'\psi = (1-\lambda)\psi'$

Then $S[\varphi]$ is represented by $-b\psi'$

$S[\varphi] = -[b\psi']$

since $\varphi + b\psi' = d_{\text{total}}(\psi + \psi')$

A, L algebras $C^n(A, L) = \mathrm{Hom}_k(B_n, L)$

$B = \mathrm{Bar}(A)$ - DGA with

$$f \cdot g = \mu_L (f \otimes g)\Delta_B$$

Recall from yesterday

$$0 \to B \otimes A[1] \otimes B \xrightarrow{I} B^{\otimes 3} \to B^{\otimes 4}$$

Let $f, g, h \in C(A, L)$. Define $f\partial gh$ to be in $\mathrm{Hom}(B \otimes A[1] \otimes B, L)$

$$B \otimes A[1] \otimes B \xrightarrow{I} B^{\otimes 3} \xrightarrow{f \otimes g \otimes h} L^{\otimes 3} \xrightarrow{\mu} L$$

Clearly $\mathrm{Hom}(B \otimes A[1] \otimes B, L)$ is a bimodule over $C(A, L)$ and one has

$$f\partial gh = f \cdot \partial g \cdot h$$

in the sense of the bimodule where ∂g

$= \mu(\eta \otimes g \otimes h) I$ $\eta : B \to k$ is the counit

$$f\partial(gh)k = f\partial ghk + fg\partial hk$$

M

$\lambda \Rightarrow$

$M \boxtimes B$

$$\begin{tikzcd}
B \otimes A[1] \otimes B \arrow[r, "I"] & B^{\otimes 3} \\
A[1] \otimes B \arrow[u, "\sigma(1 \otimes \Delta)"] \arrow[r, "\bar{I}"] & B^{\otimes 2} \arrow[u, "\sigma(1 \otimes \Delta)"']
\end{tikzcd}$$

Let K be an ideal in L and $\tau : K/[L, K] \to k$ be a trace defined in K. If $\alpha \in \mathrm{Hom}(B \otimes A[1] \otimes B, K)$ then define

$$\tau_\#(\alpha) = \tau\alpha\bar{I} \in \mathrm{Hom}_k(A[1] \otimes B, k)$$

Thus if one has

$$\tau_\#(f\alpha) = (-1)^{\deg f \deg \alpha}\tau_\#(\alpha f)$$

If $f, g, h \in C(A, L)$ and one has values in the ideal then $f\partial gh \in \mathrm{Hom}(B \otimes A[1] \otimes B, K)$, and we can form

$$\tau_\#(f\partial gh) \in \mathrm{Hom}(A[1] \otimes B, k) = C(A, A^*)$$

Properties: 1) $\tau_\#(\partial(fg)h) = \tau_\#(\partial fgh) + (-1)^{\deg f(\deg g + \deg h + 1)}\tau_\#(\partial ghf)$

2) $\delta\tau_\#(\partial f \cdot g) = \tau_\#(\partial\delta fg) + (-1)^{\deg f}\tau_\#(\partial f\,\delta g)$

Identifications $C^n(A) = \mathrm{Hom}(B_n, k)$

$C^n(A, A^*) = \mathrm{Hom}((A[1] \otimes B)_{n+1}, k)$

$= (A \otimes A^{\otimes n})^*$

$C^n(A)$ $\quad \delta\varphi = (-1)^{\deg\varphi+1} b'\varphi$

$C^n(A,A^*)$ $\quad \delta\psi = (-1)^{\deg\psi+1} b\psi$ where $\deg\psi$ means # of arguments

$\partial : C^n(A) \to C^{n-1}(A,A^*)$ $\quad \partial = N$

$R \xrightarrow{\partial} \Omega^1_R \otimes_R \xrightarrow{b} R$ $\quad \beta(\partial x\, y) = -[x,y]$

$\beta : C^n(A,A^*) = C^{n+1}(A)$ $\quad \beta = \lambda - 1$

Now let $\rho : A \to L$ be a linear map such that it is a homomorphism mod K.

$\omega = \delta\rho + \rho^2$

$\omega(a_1,a_2) = \rho(a_1 a_2) - \rho(a_1)\rho(a_2) \in K$

$\omega \in C^2(A,K)$

$\omega^n \in C^n(A,K^n)$ $\quad$ Let τ be a trace on K^n

$\tau : K^m \to k$ vanish on $[K,K^{m-1}]$ $\quad (\Rightarrow \tau[K^i,K^j] = 0$ for $i+j \geqslant m)$

We can form $\tau(\omega^{2n}) \in C^{2n}(A)$

$\tau_\#(\partial\rho\, \omega^n) \in C^{2n}(A,A^*)$ $\quad n \geqslant m$

<u>Proposition:</u> $\quad \delta\,\tau(\omega^n) = \beta\,\tau_\#(\partial\rho\,\omega^n)$

$$\delta\tau_\#\left(\delta\rho \frac{\omega^n}{n!}\right) = \partial\tau\left(\omega^{n+1}/(n+1)!\right)$$

Proof: $\quad \delta\tau(\omega^n) = \tau(\delta\omega^n) = \tau(-\rho\omega^n + \omega^n\rho)$

$= \beta\tau_\#(\partial_\rho\,\omega^n)$

$$\begin{aligned}\delta\tau_\#(\partial\rho\,\omega^n) &= \tau_\#(\partial(\delta\rho)\omega^n - \partial\rho\,\delta(\omega^n)) \\ &= \tau_\#(\partial(\delta\rho)\omega^n + \partial\rho(\rho\omega^n - \omega^n\rho)) \\ &= \tau_\#((\partial(\delta\rho) + \rho\partial\rho + \partial\rho\,\rho)\omega^n) \\ &= \tau_\#(\partial\omega\,\omega^n)\end{aligned}$$

$$\partial\tau_\#(\omega^{n+1}) = \tau_\#\left(\sum_{i=0}^{n} \omega^{i}\partial\omega\,\omega^{n-i}\right)$$

$$= (n+1)\,\tau_\#(\partial\omega\,\omega^n)$$

This calculation means

$\varphi_{2n} = \tau(\omega^n)(a_1,\dots,a_{2n}) = \tau\,\omega(a_1,a_2)\dots\omega(a_{2n-1},a_{2n}))$

$\psi_{2n+1} = \tau_\#(\partial\rho\,\omega^n)(a_0,\dots,a_{2n}) = \tau(\rho(a_0)\,\omega(a_1,a_2)\dots\omega(a_{2n-1},a_{2n}))$

Then

$b'\varphi_{2n} = (1-\lambda)\psi_{2n+1}$

$b\psi_{2n+1} = \frac{1}{n+1} N\varphi_{2n+2}$

Verify these using Bianchi $\quad \omega(a_0 a_1, a_2) - \omega(a_0, a_1 a_2) = \rho(a_0)\omega(a_1,a_2) - \omega(a_0,a_1)\rho(a_2)$

Connes cyclic bicomplex

$$\begin{tikzcd}
A^{\otimes 3} \arrow[d, "b"] & A^{\otimes 2} \arrow[l] & \\
A^{\otimes 2} \arrow[d, "b"] & A^{\otimes 2} \arrow[l, "1-\lambda"'] \arrow[d, "b'"] & A^{\otimes 2} \arrow[l, "N"'] \arrow[d] \\
A & A \arrow[l, "1-\lambda"'] & A \arrow[l, "N"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\arrow[u, "b"] & & \\
(A^{\otimes 2})^* \arrow[u] \arrow[r, "1-\lambda"] & (A^{\otimes 2})^* \arrow[r, "N"] & {} \\
A^* \arrow[u, "b"] \arrow[r, "1-\lambda"] & A^* \arrow[u, "b'"'] \arrow[r, "N"] & {}
\end{tikzcd}$$

$\Uparrow$ cyclic bar construction $A[\iota] \otimes B$ with b'

$\Uparrow$ bar construction B of $\bar{A} = k \oplus A$ up to renaming k and shifting dimensions

Set $C(A) = \mathrm{Hom}(B_n, k)$. Put $B^\# =$ cyclic bar construction

$B^\#_n = A^{\otimes n}$ with differential b

$C_\#(A) = \mathrm{Hom}(B_\#, k)$

Then comes bicomplex

$$\begin{tikzcd}
{} & {} & {} \\
C^2_\#(A) \arrow[u, "\delta"] \arrow[r, "\beta"] & C^2(A) \arrow[u, "\delta"] \arrow[r, "\partial"] & C^2_\#(A) \arrow[u, "\delta"] \\
C^1_\#(A) \arrow[u, "\delta"] \arrow[r, "\beta=-\lambda+1"] & C^1(A) \arrow[u, "\delta"] \arrow[r, "\partial"] & C^1_\#(A) \\
0 \arrow[u] & C^0(A)=k \arrow[u] &
\end{tikzcd}$$

degree of a cochain equals the number of arguments.

$\delta\varphi = (-1)^{\deg\varphi+1} b'\varphi \qquad \varphi \in C(A)$

$\delta\psi = (-1)^{\deg\psi+1} b\psi \qquad \psi \in C_\#(A)$

$\beta = \lambda - 1$

$\partial = N$

Middle column is like R. Right column is like $\Omega^1_R \otimes_R$.

$$\Omega^1_R \otimes_R \xrightarrow{\beta} \bar{R} \xrightarrow{\partial} \Omega^1_R \otimes_R$$

$$x\,\partial y\, z \longmapsto -[y, zx]$$

Last time

$$A \xrightarrow{\rho} L \supset \overset{\text{ideal}}{I} \supset I^n \xrightarrow{\tau} k$$

ρ : homomorphism mod I $\qquad \tau([L, I^n]) = 0$

$\rho \in C^1(A, L) \qquad C(A,L) = \mathrm{Hom}(B, L)$ DGA

$\omega = \delta\rho + \rho^2 \in C^2(A, I) \qquad \omega^n \in C^{2n}(A, I^n)$

Can form $\tau(\omega^n) \in C^{2n}(A)$

$\tau_\#(\partial_\rho \omega^n) \in C^{2n+1}_\#(\underline{A})$

$$B_\# = A[1] \otimes B \xhookrightarrow{I} B \otimes B \xrightarrow{\rho \otimes \omega^n} L \otimes I^n \to I^n \xrightarrow{\tau} k$$

Can form this for $n \geq m$. Concrete formulae

$\tau(\omega^n)(a_1, \dots, a_{2n}) = \tau(\omega(a_1, a_2) \dots \omega(a_{2n-1}, a_{2n}))$

$\tau_\#(\partial\rho\, \omega^n) \in C^{2n+1}_\#(A)$

$\tau_\#(\partial\rho\,\omega^n)(a_0, \dots, a_{2n}) = \tau(\rho(a_0)\omega(a_1, a_2) \dots \omega(a_{2n-1}, a_{2n}))$

Proposition:

$$\begin{cases} \delta\, \tau(\omega^n) = \beta\, \tau_\#(\partial\rho\,\omega^n) \\ \delta\, \tau_\#(\partial\rho\,\omega^n) = \partial\, \tau(\omega^{n+1}/n+1) \end{cases}$$

$\iff$

$$\begin{cases} b'\,\tau(\omega^n) = (1-\lambda)\,\tau_\#(\partial\rho\,\omega^n) \\ b\, \tau_\#(\partial\rho\,\omega^n) = \frac{1}{n+1} N\, \tau(\omega^{n+1}) \end{cases}$$

$$d_{\text{total}} = d_{\text{horizontal}} + (-1)^{\text{degree}} d_{\text{vertical}}$$

Corollary: For $n \geq m$ $\quad N\tau(\omega^n)$ is a cyclic cocycle of degree $2n-1$ and so it determines a cyclic cocycle class $[N\tau(\omega^n)] \in HC^{2n-1}(A)$

One has

$$S[N\tau(\omega^n/n!)] = [N\tau(\omega^{n+1}/(n+1)!)]$$

Suppose $\tau[I, I^{n-1}] = 0$. $\quad$ (holds if $n > m$)

then

$N\tau(\omega^n)(a_1, a_2, \dots, a_{2n}) = N\tau(\omega^n)((1+\lambda)(a_1, \dots, a_{2n})$

$= n\,\tau(\omega(a_1, a_2), \dots \omega(a_{2n-1}, a_{2n}))$

$\quad - n\,\tau(\omega(a_{2n}, a_1) \dots \omega(a_{2n-2}, a_{2n-1}))$

$\therefore\ \lambda^2\tau(\omega^n) = \tau(\omega^n)$

<u>Homotopy</u>: Suppose we have $\rho : A \to \mathbb{C}[t]\otimes L$

Work in $C(A,\ \mathbb{C}[t,dt]\otimes L)$

$$\left(\mathbb{C}[t] \xrightarrow{dt\partial} dt\,\mathbb{C}[t]\right) = \mathbb{C}[t,dt]$$

$$= \mathrm{Hom}(B,\ \mathbb{C}[t,dt]\otimes L)$$

is a bigraded DA

$$C^{p,q} = \mathrm{Hom}(B_q,\ (\mathbb{C}[t,dt]\otimes L)_p)$$

$\xi \in C^{p,q}$ $\quad \eta \in C^{p',q'}$ $\quad \xi.\eta = \mu(\xi\otimes\eta)\Delta$

$$(\xi.\eta)(a_1,\dots,a_{q+q'}) = (-1)^{q(p+q')}\xi(a_1,\dots,a_q)\,\eta(a_{q+1},\dots,a_{q+q'})$$

$$d_{total}\,\xi = d_0\xi - (-1)^{\deg\xi}\xi.d$$

If $\xi \in C^{p,q}$ then $d_{total}\xi = dt\,\partial_t\xi + \delta\xi$

$\delta\xi = (-1)^{p+q+1}\xi b'$

$\rho \in C^{0,1}$ $\quad$ Put $\tilde\omega = (dt\,\partial_t + \delta)\rho + \rho^2$

$$= dt\,\dot\rho + \omega$$

$$\tilde\omega^n = (dt\,\dot\rho + \omega)^n = \omega^n + \sum \omega^{i-1}dt\,\dot\rho\,\omega^{n-i}$$

$$= \omega^n + dt\sum_{i=1}^n \omega^{i-1}\dot\rho\,\omega^{n-i}$$

$$= \omega^n + dt\,\mu_n$$

$(dt)^2 = 0$ $\qquad \dot\rho \in C^1(A, L[t]) = C^{0,1}$

$\mu_n \in C^{0,2n-1}$

$\tau : I^n \to k$ extends to $\mathbb{C}[t,dt]\otimes I^n \to \mathbb{C}[t,dt]$

and then to $C(A, \mathbb{C}[t,dt]\otimes I^n) \to C(A,\mathbb{C}[t,dt])$

– this is a map of complexes.

$$\tau(\tilde\omega^n) = \tau(\omega^n) + dt\,\tau(\mu_n)$$

$$\tau_\#(\partial\rho\,\tilde\omega^n) = \tau_\#(\partial\rho\,\omega^n) - dt\,\tau_\#(\partial\rho\,\mu_n)$$

$\tau(\mu_n) \in C^{2n-1}$

<u>Proposition</u>: $\tau(\omega^n)^\bullet = \delta\tau(\mu_n) + (-\beta)\tau_\#(\partial\rho\,\mu_n)$

$$\tau_\#(\partial\rho\,\omega^n)^\bullet = (-\delta)\tau_\#(\partial\rho\,\mu_n) + \partial\tau(\mu_{n+1}/n+1)$$

Proof: First show

$$(dt\,\partial_t + \delta)\tau(\tilde\omega^n) = \beta\,\tau_\#(\partial\rho\,\omega^n)$$

$$(dt\,\partial_t + \delta)\tau_\#(\partial\rho\,\tilde\omega^n) = \partial\tau(\tilde\omega^{n+1}/n+1)$$

– by the same argument as above.

Then find coefficient of dt.

Applications

(1) If $\dot\rho \in C^1(A, \mathbb{C}[t]\otimes I)$ (this implies that $\rho : A \to (L/I)[t]$ is constant)

Then $\mu_n \in C^{2n-1}(A, I^n)$ so $\tau(\mu_n)$, $\tau_\#(\partial\rho\,\mu_n)$ are defined for $n \geq m$.

So $N\tau(\omega^n)^\bullet = \delta N\tau(\mu_n)$ so $[N\tau(\omega^n)]$ $\in HC^{2n-1}(A)[t]$ or more precisely

$$\mathrm{Hom}(HC_{2n-1}(A), \mathbb{C}[t])$$

is independent of t i.e. is constant. $\forall n \geq m$

This shows that $[N\tau(\omega_t^n)]$, the cyclic class attached to $A \to L \supset I \supset I^n \xrightarrow{\tau} k$ depend only on the homomorphism $\rho : A \to L/I$

Also shows that if $\dot\rho \in C^1(A, \mathbb{C}[t]\otimes L)$ then $\mu_n \in C^{2n-1}(A, \mathbb{C}[t]\otimes I^{n-1})$ so $\tau(\mu_n)$, $\tau_\#(\partial\rho\,\mu_n)$

are defined for $n \geq m$, so the cyclic cohomology classes ~~are~~ for $n > m$ are homotopy invariant.

Same thing works for the vector-valued traces via the universal trace $\tau : I^n \to I^m/[I^m, L]$

② $A \xrightarrow{\rho} L \twoheadrightarrow L/I^{m+1} \xrightarrow[\text{trace}]{\tau} k$

ρ homomorphism mod I

$\delta\rho + \rho^2 \in C^2(A, I)$

If τ is defined on L itself then all these classes are all zero. $[N\tau(\omega^n)] = 0$. Take $I = L$ in the above and use the homotopy.

Chern-Simons cyclic cocycles

$\rho_t = t\rho$ deforms ρ to 0

$\omega_t = \delta\rho_t + \rho_t^2 = t\delta\rho + t^2\rho^2$

$\mu_{n,t} = \sum_{i=1}^n \omega_t^{i-1}\dot{\rho}_t\,\omega_t^{n-i} = \sum_{i=1}^n \omega_t^{i-1}\rho\,\omega_t^{n-i}$

For $t = 1$ $\rho_1 = \rho$ is a homomorphism mod I

$\omega_1^n \in C^{2n}(A, I^n)$

$\tau(\omega_t^n)^{\cdot} = \delta\tau(\mu_{n,t}) - \beta\tau_\#(\partial\rho_t\,\omega_t^n)$

$\tau_\#(\partial\rho_t\,\omega_t^n)^{\cdot} = -\delta\tau_\#(\partial\rho_t\,\omega_t^n) + \partial\tau\left(\frac{\mu_{n+1,t}}{n+1}\right)$

Integrate from $t=0$ to $t=1$ w.r.t. t.

$n > 0$

$\tau\left(\frac{\omega_1^n}{n!}\right) = \delta\int_0^1\tau\left(\frac{\mu_n}{n!}\right)dt - \beta\left\{\int_0^1\tau_\#\left(\partial\rho_t\frac{\omega_t^n}{n!}\right)dt\right\}$

$\tau_\#\left(\partial\rho_1\frac{\omega_1^n}{n!}\right) = -\delta\left\{\int_0^1\tau_\#\left(\partial\rho_t\frac{\omega_t^n}{n!}\right)dt\right\} + \partial\left\{\int_0^1\tau\left(\frac{\mu_{n+1,t}}{(n+1)!}\right)dt\right\}$

Suppose that $n > m$ $\tau(I^{m+1}) = 0$

Then $\tau(\omega_1^n/n!) = 0$, $\tau_\#(\partial\rho_t\,\omega_1^n/n!) = 0$

$\psi_{2n+1} = \int_0^1 \tau_\#(\partial\rho_t\,\omega_t^n/n!)\,dt \in C_\#^{2n+1}(A)$

$\varphi = \left\{\int_0^1\tau(\mu_{n+1,t}/(n+1)!)\,dt\right\} \in C^{2n+1}(A)$

$\varphi_{2n-1} = \left\{\int_0^1\tau\left(\frac{\mu_{n,t}}{n!}\right)dt\right) \in C^{2n-1}$

The family $\{\varphi_{2n-1}, \psi_{2n}\}$ is a cocycle in the Connes double complex

$\varphi_{2n+1} \to \uparrow\delta$

$\varphi_{2n} \xrightarrow{\beta} \uparrow\delta$

φ_{2n+1}

Proposition: $n \geq m$ - $N\varphi_{2n+1}$ is a cyclic $2n$ cocycle

its class $S[N\varphi_{2n+1}] = [N\varphi_{2n+3}]$

$N\varphi_{2n+1} = (\text{const})\int_0^1 N\tau\left(\rho(t\delta\rho + t^2\rho^2)^n\right)dt$

$\longleftarrow$

M manifold E vector bundle over M with a connection ∇.

$A = \Gamma(M, \mathrm{End}\,E) = \Omega^0(M, \mathrm{End}\,E) \subset \Omega(M, \mathrm{End}\,E)$
$= \Gamma(M, \Lambda T^* \otimes \mathrm{End}\,E)$

operates on $\Omega(M, E)$

The connection ∇ gives rise to a derivation $\mathrm{ad}\,\nabla$

$\Omega^1(M, \mathrm{End}\,E)$ $\quad (\mathrm{ad}\,\nabla)(\xi) = \nabla\xi + (-1)^{\deg\xi}\xi\nabla$

as operators on $\Omega(M, E)$

One has $(\mathrm{ad}\,\nabla)^2 = \mathrm{ad}(\nabla^2)$ where $\nabla^2 \in \Omega^2(M, \mathrm{End}\,E)$ is the curvature.

Consider $C(A, \Omega(\mathrm{End}\,E)) = \mathrm{Hom}(B, \Omega(\mathrm{End}\,E))$ which is a bigraded algebra.

$C^{p,q}(A, \Omega(\mathrm{End}\,E)) = \mathrm{Hom}(B_q, \Omega^p(\mathrm{End}\,E))$

= multiplication $\mu * \otimes y \Delta$

On $C(A, \Omega(\mathrm{End}\,E))$ we have derivations δ, $\mathrm{ad}\,\nabla$

$(\delta\xi)(a_0, \dots, a_q) = (-1)^{p+q+1}\xi b'(a_0, \dots, a_q)$
$\quad pq$

$(\mathrm{ad}\,\nabla)(\xi)(a_1, \dots, a_p) = [\nabla, \xi(a_1, \dots, a_p)]$

δ, $\mathrm{ad}\,\nabla$ anticommute

Canonical element $\Theta \in C^1(A, \Omega(\mathrm{End}\,E))$

$\delta\Theta + \Theta^2 = 0$ (homomorphism)

$K = \nabla^2 + [\nabla, \Theta] \in C^{2,0} + C^{1,1}$

(formally $(\delta + \nabla + \Theta)^2 = \nabla^2 + [\nabla, \Theta] = K$)

Verify $(\delta + \mathrm{ad}\,\nabla + \mathrm{ad}\,\Theta)(K) = 0$
$\Rightarrow (\delta + \mathrm{ad}\,\nabla + \mathrm{ad}\,\Theta)e^K = 0$

Prop. $\tau = \mathrm{tr}_E : \Omega(\mathrm{End}\,E) \to \Omega(M)$
$\tau(e^K) \in C(A, \Omega(M))$
$\tau_\#(\partial\Theta e^K) \in C_\#(A, \Omega(M))$
$(\delta + d)\,\tau(e^K) = \beta\,\tau_\#(\delta\Theta e^K)$
$(\beta + d)\,\tau_\#(\delta\Theta e^K) = \underset{\partial}{\delta}\,\tau(e^K)$

$C^q(A, \Omega^p(\mathrm{End}\,E)) = \mathrm{Hom}(B_q, \Omega^p(\mathrm{End}\,E))$

bigraded algebra

Two derivations δ degree $(0,1)$

$\delta\xi = (-1)^{|\xi|+1}\xi \circ b'$ $\quad |\xi|$ = total degree of ξ

$\mathrm{ad}\,\nabla$ = deg$(1,0)$

These anticommute and $(\mathrm{ad}\,\nabla)^2 = \mathrm{ad}(\nabla^2)$

where $\nabla^2 \in \Omega^2(\mathrm{End}\,E)$ $\quad \Theta \in C^{0,1} = \mathrm{Hom}(A, \Omega^0(\mathrm{End}\,E))$

Put $K = \nabla^2 + [\nabla, \Theta]$ $\quad \Theta$ identity of A
$\in C^{2,0} \oplus C^{1,1}$

Proposition $(\delta + \mathrm{ad}\,\Theta + \mathrm{ad}\,\nabla)K = 0$
$(\delta + \mathrm{ad}\,\Theta + \mathrm{ad}\,\nabla)(K^n) = 0 \quad \forall n \geq 1$

– also e^K

Proof: By calculation $(\delta + \mathrm{ad}\,\Theta + \mathrm{ad}\,\nabla)K$
$= \delta K + [\Theta, K] + [\nabla, K]$

$\delta K = \delta(\nabla^2 + [\nabla, \Theta])$
$= -[\nabla, \delta\Theta] = [\nabla, \Theta^2] = [\nabla, \Theta]\Theta - \Theta[\nabla, \Theta]$

$[\Theta, K] = [\Theta, \nabla^2 + [\nabla, \Theta]]$
$= [\Theta, \nabla^2] + \Theta[\nabla, \Theta] - [\nabla, \Theta]\Theta$

$[\nabla, K] = [\nabla, \nabla^2 + [\nabla, \Theta]] = [\nabla, [\nabla, \Theta]]$
$= [\nabla^2, \Theta]$

$tr_E : \Omega(\operatorname{End} E) \to \Omega(M)$

$tr_E([\xi,\eta]) = 0$

Let $\tau = tr_e$

$\tau : C(A, \Omega(\operatorname{End} E)) \to C(A, \Omega(M))$

$\tau \circ \operatorname{ad} \nabla = d \circ \tau \qquad d\, tr_E(\xi) = tr_E[\nabla, \xi]$

$\tau : \xi \longmapsto tr_E \circ \xi$

$\delta\tau(\xi) = \tau\delta\xi$

$d\tau(\xi) = \tau([\nabla,\xi])$

Similarly τ induces a map

$$\operatorname{Hom}(B \otimes A[1] \otimes B, \Omega(\operatorname{End} E)) \xrightarrow{\tau_\#} \operatorname{Hom}(A[1] \otimes B, \Omega(M))$$

(this is complex of Hochschild cochains with values in the complex Ω) $= C_\#(A, \Omega(M))$

Such that

$\delta\tau_\# = \tau_\# \delta$

$d\tau_\# = \tau_\# \operatorname{ad} \nabla$

<u>Theorem</u> The elements

$\tau(e^K) \in C(A, \Omega(M))$

$\tau_\#(\partial\theta e^K) \in C_\#(A, \Omega(M))$

satisfy

$(\delta + d)\tau(e^K) = \beta\tau_\#(\partial\theta e^K)$

$(\delta + d)\tau_\#(\partial\theta e^K) = \partial\tau(e^K)$

If γ is a <u>closed</u> current in M of dimension r then

$\int_\gamma : \Omega(M) \to k[r]$

and $\qquad \varphi = \int_\gamma \tau(e^K) \in C(A), \ \psi = \int_\gamma \tau_\#(\partial\theta e^K) \in C_\#(A)$

Then $\qquad \delta\varphi = \beta\psi, \qquad \delta \cancel{\beta}\psi = \partial\varphi$

Proof: $\quad (\delta + d)\tau(e^K) = \tau((\delta + \operatorname{ad}\nabla)e^K)$

$= \tau((-\operatorname{ad}\theta)e^K) \quad$ (by Bianchi)

$= \beta\tau_\#(\partial\theta e^K)$

$(\delta + d)\tau_\#(\partial\theta e^K) = \tau_\#((\delta + \operatorname{ad}\nabla)(\partial\theta e^K))$

$= \tau_\#(\partial(\delta\theta + [\nabla,\theta])e^K - \partial\theta(\delta + \operatorname{ad}\nabla)e^K)$

$= \tau_\#((\partial(-\theta^2) + \partial[\nabla,\theta])e^K + \partial\theta(\theta e^K - e^K\theta))$

$= \tau_\#(-\theta\partial\theta e^K + \partial[\nabla,\theta]e^K - \partial\theta e^K\theta)$

$= \tau_\#(\partial[\nabla,\theta]e^K)$

$\partial\tau(e^K) = \tau_\#(\partial K e^K)$

Why?

Proof 1: $\quad \partial\tau(K^n) = \tau_\#(\partial K^n) = \tau_\#\left(\sum_{i=1}^{n} K^{i-1}\partial K K^{n-i}\right)$

$= n\tau_\#(\partial K K^{n-1})$

Proof 2: Duhamel's Principle

$\partial\tau(e^K) = \tau_\#(\partial e^K) = \tau_\#\left(\int_0^1 e^{(1-t)K}\partial K e^{tK} dt\right)$

$= \tau_\#(\partial K e^K)$

Put $K = \nabla^2 + [\nabla, \theta]$

$\partial[\nabla^2] = 0$

$$\varphi = \int_\gamma \tau(e^{K}) = \int_\gamma \tau(e^{\nabla^2 + [\nabla, \theta]})$$

$$e^{A+B} = e^{A} + \int_0^1 e^{(1-t_1)A} B e^{(1-t_1)A} dt_1$$

$$+ \iint_{t_1+t_2 \le 1} e^{(1-t_1-t_2)A} B e^{\overset{t_1 A}{\cancel{(1-t_1-t_2)A}}} e^{t_2 A} \, dt_1 dt_2$$

$$+ \dots + \iint \dots \int_{t_1 + \dots + t_n \le 1} e^{(1-t_1-\dots-t_n)A} B e^{(1-t_1-\dots-t_{n-1})A} \dots B e^{t_n A} dt_1 \dots dt_n$$

- to see this take the Laplace transform.

$$e^{u(A+B)} \longleftrightarrow 1/\lambda - A - B$$

and expand as geometric series.

$$\varphi = \Sigma_n \int_{t_1 + \dots + t_n \le 1} \dots \int e^{t_0 \nabla^2} [\nabla, \theta] \dots e^{t_n \nabla^2} dt_1 \dots dt_n$$

$$t_0 = 1 - t_1 - \dots + t_n$$

$$= \Sigma_i \text{ const. } (\nabla^2)^{i_0} [\nabla, \theta] (\nabla^2)^{i_1} \dots [\nabla, \theta] (\nabla^2)^{i_n}$$

Simplest example E trivial bundle $\mathbb{C}_1$ $\dim \gamma = n$

$A = C^\infty(M)$ $\nabla = d$ $\nabla^2 = 0$, $\tau = id : \Omega(M) \circlearrowleft$

$$\varphi = \int_\gamma \tau(e^{d\theta}) = \int_\gamma \frac{d\theta^n}{n!} = \int_\gamma d\left(\theta \frac{d\theta^{n-1}}{n!}\right) = 0$$

$$\psi = \int_\gamma \tau_\#(\partial\theta e^{d\theta}) = \int_\gamma id_\#\left(\partial\theta \frac{d\theta^n}{n!}\right)$$

is a Hochschild n-cochain

$$\psi(a_0, \dots a_n) = \int_\gamma a_0 da_1 \dots da_n$$

The Theorem says that $\beta\psi = \delta\varphi = 0$

$\delta\psi = \partial\varphi = 0$

$\therefore$ ψ is a cyclic cochain and a Hochschild coboundary

$\therefore$ ψ is a cyclic cocycle.

$$\begin{tikzcd}
\varphi & & & & \\
{} \arrow[u] \arrow[r, "N"] & \psi \arrow[r, "1-\lambda"] & {} & & \\
{} \arrow[u, "b'"] \arrow[r, "N"'] & {} \arrow[u, "b"] \arrow[r, "1-\lambda", "\beta"'] & \varphi \arrow[u, "b'"] \arrow[r, "1-\lambda"] & {} & \\
& & {} \arrow[u] & {} \arrow[u, "\psi"'] &
\end{tikzcd}$$

<u>Superconnections</u> & <u>Jaffe</u>, <u>Lesniewski</u>, <u>Osterwalder</u> <u>cocycle</u>

Connes Entire Cyclic Cohomology

$$\begin{tikzcd}
(A^{\otimes 3})^* \arrow[r, bend left] & {} \arrow[l, bend left] \\
(A^{\otimes 2})^* \arrow[u, "b"] \arrow[r, bend left] & {} \arrow[l, bend left] \arrow[u, "b'"] \\
(A)^* \arrow[u] \arrow[r, bend left] & {} \arrow[l, bend left] \arrow[u, "b'"]
\end{tikzcd}$$

defined using sequences of cochains $\{\varphi_n\}$ subject to a growth condition

$$\sum_n z^n \| \varphi_n \| < \infty \quad \forall z$$

H Hilbert space X Skew adjoint operator on H (unbounded)

A * algebra over $\mathbb{C}$ operates on H

Assume that for any $a \in A$ the operator $[X,a]$ is densely defined and extends to a bounded operator.

E.g. $A = C^\infty(S^1)$ $H = L^2(S^1)$

$X = d/dx$ $S^1 = \mathbb{R}/2\pi\mathbb{Z}$

e^{tX^2} is trace class for $t > 0$.

$C(A, \mathcal{L}(H))$ - algebra for working in DGA

Contains e^{tX^2} 0-cochain θ 1-cochain

the homomorphism $A \to \mathcal{L}(H)$.

$$\delta\theta + \theta^2 = 0$$

$[X, \theta]$ is also a 1-cochain

e^{tX^2} 0-cochain in $I = \mathcal{L}^1(H)$.

Digression on Superconnections:

M connection E, ∇

$d\,tr(\nabla^{2n}) = 0$

for superconnections also have an odd operator

M, E, ∇ + X a skew adjoint operator on E

E has inner product preserved by ∇

∇, X operate on $\Omega(M,E)$, also the space $\Omega(End E)$

Form the algebra $\Omega(End\, E)[\sigma]$ $\sigma^2 = 1$

σ anticommutes with forms $\sigma\omega = (-1)^{\deg\omega}\omega\sigma$

$$\Omega(End\, E) \oplus \sigma\Omega(End\, E) = \Omega(End\, E)[\sigma]$$

$\Omega(End\, E)[\sigma]$ operates also on E-valued forms $\Omega(E)$

$\sigma\xi = (-1)^{\deg\xi}\xi$ $\xi \in \Omega(E)$.

Consider $\nabla + \sigma X$ operator on $\Omega(E)$.

$$(\nabla + \sigma X)^2 = \nabla^2 + \nabla\sigma X + \sigma X\nabla + \sigma X\sigma X$$
$$= \nabla^2 + (\nabla X - X\nabla)\sigma + X^2$$
$$= \nabla^2 + [\nabla, X]\sigma + X^2$$

called the curvature of the superconnection.

$e^{X^2 + [\nabla,X]\sigma + \nabla^2} \in \Omega(End\, E)[\sigma]$

Need a trace - two cases in the theory

1) ungraded $\tau : \Omega(End\, E) \to \Omega(M)$

$$\tau(\alpha + \beta\sigma) = tr_E(\beta)$$

$\tau(e^{X^2 + [\nabla,X]\sigma + \nabla^2})$ is an odd form.

It is closed.

2) graded case. Assume E $\mathbb{Z}(2)$ graded bundle

$E = E^+ \oplus E^-$ grading compatible with ∇

i.e. stable under covariant differentiation.

$\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $[\nabla, \varepsilon] = 0$

$\varepsilon X = -X\varepsilon$ $X = \begin{pmatrix} 0 & -T^* \\ T & 0 \end{pmatrix}$

Here one has a trace $\tau(\alpha + \beta\sigma) = tr_E(\varepsilon\alpha)$ defined on

$\Rightarrow$ $\tau(e^{X^2 + [\nabla,X]\sigma + \nabla^2}) \in \Omega^{ev}(M)$

Theorem $\tau(e^{X^2 + [\nabla,X]\sigma + \nabla^2})$ is closed.

Proof:

Let $K = X^2 + [\nabla, X]\sigma + \nabla^2$

$$d\tau(e^K) = \tau([\nabla, e^K]) = \tau([\nabla + X\sigma, e^K]) = 0$$

$\because \; K = (\nabla + X\sigma)^2$

[redacted]

Return to the earlier set-up.
Here we let K be the curvature of the "Superconnection operator"

$$(\delta + \theta + X\sigma)$$

Form $C(A, \mathcal{L}(H))[\sigma]$

To be precise

$$K = (\delta + \theta + X\sigma)^2$$
$$= (X\sigma)^2 + (\delta + \theta)^2 + [\delta + \theta, X\sigma]$$
$$= \cancel{X^2} \; X^2 + [\theta, X\sigma]$$

Set $K = X^2 + [\theta, X\sigma]$

$$e^K = \sum_{n=0}^{\infty} \int_{t_1 + \dots + t_n \leq 1} \cdots \int e^{t_0 X^2} [\theta, X\sigma] e^{t_1 X^2} \cdots e^{t_n X^2} dt_1 \dots dt_n$$

$$t_0 = 1 - (t_1 + t_2 + \dots + t_n)$$
$$\in \prod_n C^n(A, \mathcal{L}(H))[\sigma]$$

- to be correct one needs to check the growth conditions for entire cyclic cochains.

$\tau(e^K)$ Let $\tau : \mathcal{L}^1(H)[\sigma] \to \mathbb{C}$ be defined as the trace on $\mathcal{L}^1(H)$ extended according to the graded or ungraded case.

Then we have $\varphi = \tau(e^K) \in C(A)$

$\psi = \tau_\#(\partial \theta e^K) \in C_\#(A)$

Theorem These form a cocycle in the double complex

i.e.
$$\delta \tau(e^K) = \beta \tau_\# (\delta \theta e^K)$$
$$\delta \tau_\# (\delta \theta e^K) = \partial \tau(e^K)$$

Proof:
$$D e^K = \int_0^1 e^{(1-s)K} D(K) e^{sK} ds$$
for derivations D

$$(\delta + \mathrm{ad}(\theta + X\sigma)) e^K = 0$$

because $(\delta + \mathrm{ad}(\theta + X\sigma)) K = 0$ 'Bianchi'

Hence $\delta \tau(e^K) = \tau((\delta + \mathrm{ad}(X\sigma)) e^K)$

because $\tau(\mathrm{ad}(X\sigma) e^K) = \tau([X\sigma, e^K])$
$= 0$

$(\because e^K$ is trace class and τ is a trace$)$.

$$= \tau(-[\theta + X\sigma, e^K]) = \beta \tau_\# (\partial \theta e^K)$$

$$\delta \tau_\# (\partial \theta e^K) = \tau_\# ((\partial \theta^2) e^K - \partial \theta \delta e^K)$$
$$= \tau_\# ((-\partial \theta \theta - \theta \partial \theta) e^K \cancel{+ \partial \theta} + \partial \theta [\theta + X\sigma, e^K])$$
$$= \tau_\# (\partial \theta X\sigma e^K - e^K X\sigma)$$
$$= \tau_\# (\partial [\theta, X\sigma] e^K)$$

$$\partial \tau(e^K) = \tau_\# \left(\int_0^1 e^{(1-s)K} \partial K e^{sK} ds \right)$$
$$= \tau_\# (\partial K e^K)$$
$$= \tau_\# (\partial (X^2 + [\theta, X\sigma]), e^K)$$

$$\tau(e^K)(a_1, \dots, a_n) = \pm \int_{t_1 + \dots + t_n \leq 1} \int e^{t_0 X^2} [X, a_1] e^{t_1 X^2} [X, a_2] \dots [X, a_n] e^{t_n X^2} dt_1 \dots dt_n$$

$\tau_\#(e^{K})(a_0, \ldots, a_n) = \iiint a_0 \, e^{t_0 X^2} \cdots [X, a_n] e^{t_n X^2}$

$\tau(e^{K}) = \tau\left(e^{\hbar^2 X^2 + [\not{D}, \hbar X \sigma]}\right) \xrightarrow[\hbar \to 0]{} \int_M \hat{A}(M) \, \mathrm{tr}_E \, e^{(\nabla^2 + [\nabla, \theta])}$

$X = \not{D}$ Let

by Getzler calculus. 'Index Theorem'.