Lectures courses by Daniel G Quillen

A. Topics in K Theory and Cyclic Co-Homology, Michaelmas Term 1989

69 pages of notes. The lecture course is concerned with the fundamental construction of cyclic cohomology, and covers the following topics. Ideals in a free algebra and cyclic cohomology of R/I. Cuntz's proof of the exact sequences. Operations on cochains. The doubly periodic complex and cyclic cochain complex. The bar construction. Curvature of a one cochain. Bianchi's identity. Characterising traces on RA. Cyclic cohomology of A as cohomology groups. Hochschild cohomology and it meaning in low dimensions. Connes definition of τ The bimodule of differential forms over A. Universal derivations. Traces on RA. Constructing the complex and double complex. Properties of RA, QA and ΩA. Superalgebras. $*$ product. Fredholm modules and QA. Karoubi's operators. Normalised Hochschild cochains. Applications to Fredholm modules. Homology for Fredholm modules and supertraces on Q. Inner superalgebras. Connes–Cuntz formula.

Editor's remark The lecture notes were taken during lectures at the Mathematical Institute on St Giles in Oxford. There have been subsequent corrections, by whitening out writing errors. The pages are numbered, but there is no general numbering system for theorems and definitions. For the most part, the results are in consecutive order, although in one course the lecturer interrupted the flow to present a self-contained lecture on a topic to be developed further in the subsequent lecture course. The note taker did not record dates of lectures, so it is likely that some lectures were missed in the sequence. The courses typicaly start with common material, then branch out into particular topics. Quillen seldom provided any references during lectures, and the lecture presentation seems simpler than some of the material in the papers.

- D. Quillen, Cyclic cohomology and algebra extensions, *K-Theory* **3**, 205–246.
- D. Quillen, Algebra cochains and cyclic cohomology, *Inst. Hautes Etudes Sci. Publ. Math.* **68** (1988), 139–174.
- J. Cuntz and D. Quillen, Cyclic homology and nonsingularity, *J. Amer. Math. Soc.* **8** (1995), 373–442.

Commonly used notation

k a field, usually of characteristic zero, often the complex numbers

A an associative unital algebra over k, possibly noncommutative

$\bar{A} = A/k$ the algebra reduced by the subspace of multiples of the identity

$\Omega^n A = A \otimes (\bar{A} \otimes \ldots \otimes \bar{A})$

$\omega = a_0 da_1 \ldots da_n$ an element of $\Omega^n A$

$\Omega A = \oplus_{n=0}^{\infty}\Omega^n A$ the universal algebra of abstract differential forms

e an idempotent in A

d the formal differential (on bar complex or tensor algebra)

b Hochschild differential

b', B differentials in the sense of Connes's noncommutative differential geometry

λ a cyclic permutation operator

K the Karoubi operator

$\circ$ the Fedosov product

G the Greens function of abstract Hodge theory

N averaging operator

P the projection in abstract Hodge theory

D an abstract Dirac operator

∇ a connection

I an ideal in A

V vector space

M manifold

E vector bundle over manifold

τ a trace

$T(A) = \oplus_{n=0}^{\infty} A^{\otimes n}$ the universal tensor algebra over A

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<u>D. Quillen</u>
Topics in K Theory and Cyclic CoHomology

$HC^*(A)$ $\quad HC^0(A) = \{\text{traces on } A\}$
$f(xy) = f(yx)$

Index theory of elliptic operators. Lie algebra cohomology of $gl_n(A)$

$0 \to I \to R \to A \to 0 \qquad I \triangleleft R$

<u>Theorem</u> If R is a free algebra then we have exact sequences

$$(R/[R,R])^* \longrightarrow (I^m/[I,I^{m-1}])^* \underset{\substack{\uparrow \\ \text{Connes}}}{\longrightarrow} HC^{2m-1}(A) \to 0$$

$$(\Omega^1 R/(\quad))^* \xrightarrow{d^t} HC^0(R/I^{m-1}) \longrightarrow H^{2m}(A) \to 0$$

$I^m \Omega^1 R + [R, \Omega^1 R]$

$\square$

J. Cuntz – easier proof of these exact sequences
Deal with the case of universal extension.

$RA = \bigoplus_n A^{\otimes n}$

$\begin{cases} \text{unital theory} & 1 \in A \\ \text{nonunital theory} & \end{cases}$

Usual in homological algebra to take $1 \in A$. But the tools used are based on nonunital techniques in working with $\tilde{A} = k \oplus A$.

(Category of nonunital algebras)
$$\cong$$
(category of unital algebras)

$$A \to \tilde{A}$$

Nonunital category: $RA = \bigoplus_{n \geq 1} A^{\otimes n}$

$0 \to IA \to RA \to A \to 0$ universal extension of A

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, "\rho"'] & RA \arrow[dl, dashed] \\ R & \end{tikzcd}$$

$\exists!$ algebra homomorphism which makes this diagram commute.

Cuntz' proved this by cyclic formalism for the case $R = RA$.

check $\neq 0$ Review of cyclic formalism: over field k

$\underline{A}$ nonunital algebra. Cochain of degree $\underline{n}$ on $\underline{A}$ is a multilinear map

$f(a_1, \dots, a_n)$ on A with values in a vector space.

Operations on cochains

(i) $(b'f)(a_1, \dots, a_{n+1}) = \sum_{i=1}^{n} (-1)^{i-1} f(a_1, \dots, a_i a_{i+1}, \dots, a_n)$

(ii) $(bf)(a_1, \dots, a_{n+1}) = (b'f)(a_1, \dots, a_{n+1}) + (-1)^n f(a_{n+1}a_1, a_2, \dots, a_n)$

(iii) $(\lambda f)(a_1, \dots, a_n) = (-1)^{n-1} f(a_n, a_1, \dots, a_{n-1})$

(iv) $Nf(a_1, \dots, a_n) = \sum_0^{n-1} \lambda^i f(a_1, \dots, a_n)$

Cochains with values in k

Doubly periodic

$$\begin{tikzcd}
 & \uparrow & \uparrow b' & \uparrow -b \\
0 \to CC^2(A) \arrow[r, "N"] & (A^{\otimes 3})^* \arrow[r, "\lambda - 1"] \arrow[u] & (A^{\otimes 3})^* \arrow[r, "N"] \arrow[u] & (A^{\otimes 3})^* \arrow[u] \\
0 \to CC^1(A) \arrow[r, "N"] & (A^{\otimes 2})^* \arrow[r, "\lambda - 1"] \arrow[u, "-b"] & (A^{\otimes 2})^* \arrow[r, "N"] \arrow[u, "b'"] & (A^{\otimes 2})^* \arrow[u, "-b"] \\
0 \to CC^0(A) \arrow[r, "N"'] & A^* \arrow[r, "\lambda - 1"] \arrow[u, "-b"] & A^* \arrow[r, "N"] \arrow[u, "b'"] & A^* \arrow[u, "-b"]
\end{tikzcd}$$

Key calculations: $\begin{cases} b^2 = (b')^2 = 0 \\ (\lambda-1)N = N(\lambda-1) = 0 \\ b'(1-\lambda) = (1-\lambda)b \\ bN = Nb' \end{cases}$

difficult one! →

Double complex anticommutes. Rows are exact since the characteristic of k is zero.

$CC^*(A)$ cyclic cochain complex with $HC^n(A) = H^n(CC(A))$ — resolved by the double complex.

For cochains on A with values in an algebra R define product and a differential δ

f p-cochain g q cochain

$$(fg)(a_1,\dots,a_{p+q}) = (-1)^{pq} f(a_1,\dots,a_p)\, g(a_{p+1},\dots,a_{p+q})$$

$$(\delta f) = (-1)^{p+1} b'f$$

Verify that the cochains with values in R form a differential graded algebra.

$$\mathrm{Hom}(A^{\otimes 0},R) \xrightarrow{\delta=0} \mathrm{Hom}(A,R) \xrightarrow{\delta} \mathrm{Hom}(A^{\otimes 2},R)$$

Bar construction $B(A)$ differential graded co-algebra

$\mathrm{Hom}(B(A),R)$

E.g. $\rho: A \to R$ linear map 1-cochain

$\omega \doteq \delta\rho + \rho^2$ the curvature of ρ

$$\omega(a_1,a_2) = (b'\rho)(a_1,a_2) + \rho^2(a_1,a_2)$$
$$= \rho(a_1a_2) - \rho(a_1)\rho(a_2)$$

$\therefore\ \omega = 0 \iff \rho$ is an algebra homomorphism

<u>Bianchi Identity</u>: $\delta\omega = \delta(\delta\rho + \rho^2) = \delta\rho\,\rho - \rho\,\delta\rho$
$= \omega\rho - \rho\omega$

$$\delta\omega = -b'\omega(a_1,a_2,a_3) = \omega\rho(a_1,a_2,a_3) - \rho\omega(a_1,a_2,a_3)$$

i.e. $-\omega(a_1a_2,a_3) + \omega(a_1,a_2a_3)$
$= \omega(a_1,a_2)\rho(a_3) - \rho(a_1)\omega(a_2,a_3)$

Let $\mathrm{ad}\rho$ the adjoint of ρ is
$\mathrm{ad}\rho(\alpha) = [\rho,\alpha] = \rho\alpha - (-1)^{|\rho||\alpha|}\alpha\rho$
where $|\cdot|$ denotes the degree.

$(\delta + \mathrm{ad}\rho)(\omega) = 0$

is the Bianchi identity

$\therefore\ (\delta + \mathrm{ad}\rho)(\omega^n) = 0$

since $\delta + \mathrm{ad}\rho$ is a derivation.

$\therefore\ \delta(\omega^n) + \rho\omega^n - \omega^n\rho = 0$

$\therefore\ b'\omega^n(a_1,\dots,a_{2n+1}) =$

$$= \rho(a_1)\omega^n(a_2, \ldots, a_{2n+1}) - \omega^n(a_1, \ldots, a_{2n})\rho(a_{2n+1})$$

$$= \rho(a_1)\,\omega(a_2,a_3)\ldots\omega(a_{2n},a_{2n+1}) - \omega(a_1,a_2)\ldots\omega(a_{2n-1},a_{2n})\rho(a_{2n+1})$$

Take $R = RA$ and ρ the canonical inclusion $A \hookrightarrow RA$. Consider the cochains $\omega = \delta\rho + \rho^2$

$$\omega^n(a_1, \ldots, a_{2n}) = \omega(a_1,a_2) \ldots \omega(a_{2n-1})\omega(a_{2n})$$

$$\rho\omega^n(a_0, \ldots, a_{2n}) = \rho(a_0)\omega^n(a_1, \ldots, a_{2n})$$

<u>Lemma</u> One has an isomorphism

$$\oplus_{p\geq 1} A^{\otimes p} \xrightarrow{\sim} RA$$

where the components are given by $\rho\omega^{n-1}$, ω^n $n \geq 1$.

$$\rho\omega,\ \rho\omega,\ \omega^2, \ldots$$

<u>Corollary</u> A linear functional τ on RA is equivalent to ~~the~~ inhomogeneous cochain $f = \sum_{p\geq 1} f_p$ where here

$$f_{2n} = \tau(\omega^n)$$

$$f_{2n+1} = \tau(\rho\omega^n)$$

<u>Theorem</u> A linear functional τ on RA is a trace if and only if the associated cochain f is a cocycle

$$\begin{cases} bf_{2n-1} = \frac{1}{n} N f_{2n}, \\ b' f_{2n} = (1-\lambda) f_{2n+1}. \end{cases}$$

<u>Proposition</u>: One has

$$\{b'(\omega^n) - (1-\lambda)(\rho\omega^n)\}(a_1, \ldots, a_{2n}) = -[\omega^n(a_1, \ldots, a_{2n}), \rho(a_{2n+1})]$$

$$\{b(\rho\omega^n) - (1+\lambda)\omega^{n+1}\}(a_0, \ldots, a_{2n+1}) = [\rho(a_0)\omega^n(a_1, \ldots, a_{2n}), \rho(a_{2n+1})]$$

Yesterday we proved that

$$(\delta + \mathrm{ad}\rho)\,\omega^n = \delta\omega^n + \rho\omega^n - \omega^n\rho = 0$$

i.e.

$$(b'\omega^n)(a_1, \ldots, a_{2n+1}) = \rho(a_1)\omega^n(a_2, \ldots, a_{2n+1}) - \omega^n(a_1, \ldots, a_{2n})\rho(a_{2n+1}).$$

$$= \rho(a_1)\omega^n(a_2, \ldots, a_{2n+1}) - \rho(a_{2n+1})\omega^n(a_2, \ldots, a_{2n}) + [\omega^n(a_1, \ldots, a_{2n}), \rho(a_{2n+1})]$$

which proves the first identity.
For the second formula

$$b\rho(\omega^n)(a_0, a_1, \ldots, a_{2n+1}) = \rho(a_0a_1)\omega^n(a_2, \ldots, a_{2n+1}) - \rho(a_0)b'\omega^n(a_1, \ldots, a_{2n+1}) - \rho(a_{2n+1}a_0)\omega^n(a_1, \ldots, a_{2n})$$

$$= (\rho(a_0a_1) - \rho(a_0)\rho(a_1))\,\omega^n(a_2,\dots,a_{2n+1})$$
$$- \rho(a_0)\,\omega^n(a_1,\dots,a_{2n})\,\rho(a_{2n+1})$$
$$- \rho(a_{2n+1}a_0)\,\omega^n(a_1,\dots,a_{2n})$$

$$= [\rho(a_0)\,\omega^n(a_1,\dots,a_{2n}), \rho(a_{2n+1})]$$
$$+ \omega(a_0a_1)\,\omega^n(a_2,\dots,a_{2n+1})$$
$$- \omega(a_{2n+1}, a_0)\,\omega^n(a_1,\dots,a_{2n})$$

which gives us the second identity.

Remark: These identities hold for any linear map $\rho: A \to R$ because such an ρ induces

$$\begin{tikzcd} \rho: A \arrow[r] \arrow[d] & R \\ RA \arrow[ur] & \end{tikzcd}$$

<u>Theorem</u> Let $R = RA$, $I = IA = \mathrm{Ker}\{RA \to A\}$.

1) Let τ be a linear functional on R, let
$f_{2n} = \tau(\omega^n)$, $f_{2n+1} = \tau(\rho\omega^n)$.
Then τ is a trace if and only if

a) $b' f_{2n} = (1-\lambda) f_{2n+1}$ $\quad n \geq 1$

b) $b f_{2n+1} = \frac{1}{n+1} N f_{2n+2}$

c) $\lambda^2 f_{2n} = f_{2n}$

Definition: Let M be a bimodule over an algebra R, e.g. an ideal I in R. Then a trace on M is a linear functional $\tau: M \to k$ such that $\tau(rm) = \tau(mr)$. We write this as $\tau([R,M]) = 0$, where $[R,M]$ is the space spanned by $rm - mr$ $(r \in R, m \in M)$.

Exercise: Show that if I is an ideal in R, I^p is the p^{th} power of I, then $[R, I^n] \subset [I^i, I^{n-i}] \subset [I, I^{n-1}]$.

<u>Theorem</u> (cont.)

2) Let τ be a linear functional on I^m where $m \geq 1$. Then τ vanishes on $[R, I^m]$ (resp. $[I, I^{m-1}]$) if and only if f_n satisfies a), b) for $n \geq m$ and
$$\lambda^2 f_{2n} = f_{2n} \quad \text{for } n > m$$
(resp $n \geq m$).

Here $f_{2n} = \tau(\omega^n)$
$f_{2n+1} = \tau(\rho\omega^n)$
are defined for $n \geq m$.

Proof: 1) ($\Rightarrow$) Assume τ a trace. Apply τ to the above identities. It will kill off the brackets so that

$$b'\tau(\omega^n) = (1-\lambda)\tau(\rho\omega^n)$$
$$b\tau(\rho\omega^n) = (1+\lambda)\tau(\omega^{n+1})$$
$$f_{2n}(a_1,\dots,a_{2n}) = \tau(\omega(a_1,a_2)\dots\omega(a_{2n-1},a_{2n}))$$
$\because$ τ is a trace $\lambda^2 f_{2n} = f_{2n}$

$\therefore$ $$Nf_{2n} = \sum_{i=0}^{2n-1}\lambda^i f_{2n} = n(1+\lambda)f_{2n}$$

($\Leftarrow$) Reverse the argument. We obtain
$$\tau[\omega^n(a_1,\dots,a_{2n}),\rho(a_{2n+1})] = 0$$
$$\tau[\rho(a_0)\omega^n(a_1,\dots,a_{2n}),\rho(a_{2n+1})] = 0$$
Conclude $\tau([R,\rho(A)]) = 0$

i.e. $\tau(x\rho(a)) = \tau(\rho(a)x)$ $(a\in A, x\in R)$

But $\rho(A)$ generates R. Hence τ is a trace.

2/ Similar to the proof of (1), especially in the nonbracketed case.

It is hard to show that if $\tau([R, I^m])=0$ and $f_{2m}=\tau(\omega^n)$ is λ^2 invariant, then
$$\tau[I, I^{m-1}] = 0$$
Proof by identity

$$b(1+\lambda)\omega^n(a_0,\dots,a_{2n})$$
$$= [\rho(a_0)\omega^{n-1}(a_1,\dots,a_{2n-2}), \omega(a_{2n-1},a_{2n})]$$
$$- [\omega^n(a_0,\dots,a_{2n-1}),\rho(a_{2n})]$$
$$- [\omega^n(a_{2n},a_0,\dots,a_{2n-2}),\rho(a_{2n-1})].$$
Reduces question to showing that

$$b(1+\lambda)f_{2n} \overset{?}{=} 0$$ assuming that $\lambda^2 f_{2n} = f_{2n}$

But
$$\frac{1}{m}Nf_{2m} = \frac{1}{m}bNf_{2m+1} = \cancel{\frac{1}{m}b'Nf}$$
$$= \frac{1}{m}Nb'f_{2m} = \frac{1}{m}N(1-\lambda)f_{2m+1} = 0$$

Cyclic cohomology of A $\quad$ $HC^n(A)$ is the n^{th} cohomology group of the complex of cyclic cochains
$$C^n_\lambda(A) = \{f(a_0,\dots,a_n)\in (A^{\otimes n+1})^* \mid \lambda f = f\}$$
Differential is b.

This is well-defined since the diagram below commutes.

$$\begin{tikzcd} {} \arrow[r, "1-\lambda"] & {} \\ {} \arrow[u, "b"] \arrow[r, "1-\lambda"'] & {} \arrow[u, "b'"'] \end{tikzcd}$$

$HC^0(A) = \{\text{traces on } A\}$.

Hochschild cohomology $H^n(A,M)$ $\quad$ M bimodule over A is the cohomology of the $C^n(A,M) = \{f: A^{\otimes n}\to M\}$ $\quad$ $C^0(A,M) = M$, with the differential
$$(\delta f)(a_1,\dots,a_{n+1}) = a_1 f(a_2,\dots,a_{n+1})$$
$$+ \sum_{i=1}^{n} f(\dots,a_i a_{i+1},\dots)(-1)^i$$
$$+ (-1)^{n+1}f(a_1,\dots,a_n)a_{n+1}$$

Examples: $H^0(A,M) = \{m\in M: am=ma\}$

$H^1(A,M) = \{\text{Derivations } D: A \to M\} / \text{Inner derivations}$

where inner derivations $a \mapsto [a,m]$

$H^2(A,M) = \{$ isomorphism classes of extensions

$$0 \to M \to R \to A \to 0$$

with $M.M = 0\}$.

1/ $M = k$ with zero multiplication by elements of A.

$C^\cdot(A,k)$ = complex of k-valued cochains with the differential $(-b')$. (extra k of degree zero).

2/ $M = A^*$ dual $\begin{cases} (af)(a_1) = f(a_1 a) \\ (fa)(a_1) = f(a a_1) \end{cases}$

$$C^n(A,A^*) \simeq (A^{\otimes n+1})^*$$

$$f(a_1,\dots,a_n) \longmapsto \varphi(a_0,\dots,a_n) = f(a_1,\dots,a_n)(a_0)$$

$$(\delta f)(a_1,\dots,a_{n+1}) = a_1 f(a_2,\dots,a_n) + \sum_1^n (-1)^i f(\dots,a_i a_{i+1},\dots) + (-1)^{n+1} f(a_1,\dots,a_n)\, a_{n+1}$$

$$(a_1 f(a_2,\dots,a_n))(a_0) = \varphi(a_0 a_1, a_2,\dots,a_n)$$

$$(f(a_1,\dots,a_n)\,a_{n+1})(a_0) = (-1)^{n+1} \varphi(a_{n+1} a_0, a_1,\dots,a_n)$$

$\therefore$ $C^\cdot(A,\tilde{A}^*)$ = complex of k-valued cochains of degrees ≥ 1 with differential b.

3/ $\tilde{A} = k \oplus A$ algebra obtained by augmenting an identity 1.

Take $M = (\tilde{A})^*$. One has the exact sequence of A bimodules

$$0 \to A \to \tilde{A} \to k \to 0$$

Dualising gives

$$0 \leftarrow A^* \leftarrow \tilde{A}^* \leftarrow k \leftarrow 0$$

This gives an exact sequence of complexes

$$0 \to C^\cdot(A,k) \to C^\cdot(A,\tilde{A}^*) \to C^\cdot(A,A^*) \to 0$$

$-b'$ complex $\qquad\qquad\qquad\qquad$ b complex

Claim that $C^\cdot(A,\tilde{A}^*)$ is the complex

(*)
$$\begin{tikzcd}
 & & & & \\
0 \arrow[r] & C^1_\lambda \arrow[r] \arrow[u, "b"] & (A^{\otimes 2})^* \arrow[r, "1-\lambda"] \arrow[u, "b"] & (A^{\otimes 2})^* \arrow[r, "N"] \arrow[u, "b'"] & C^1_\lambda \arrow[r] & 0 \quad \text{degree } 1 \\
0 \arrow[r] & C^0_\lambda \arrow[r] \arrow[u, "b"] & A^* \arrow[r, "1-\lambda"] \arrow[u, "b"] \arrow[urr, "b"] & A^* \arrow[r, "N"] \arrow[u, "b'"] & C^0_\lambda \arrow[r] \arrow[u, "b"] & 0 \\
 & & & k \arrow[u, "0"] & \text{degree } 0 &
\end{tikzcd}$$

Define

$$C^\cdot(A,A^*) \to C^\cdot(A,\tilde{A}^*)$$
$$\psi(a_0,\dots,a_n) \longmapsto \tilde{\psi}(\tilde{a}_0, a_1,\dots,a_n)$$

$$\tilde{\psi}(\tilde{a}_0,\dots,a_n) = \begin{cases} \psi(a_0,\dots,a_n) & \text{if } a_0 \in A \\ 0 & a_0 = 1 \end{cases}$$

$$(b\tilde\varphi)(\tilde a_0,\dots,a_{n+1}) = \tilde\varphi(\tilde a_0 a_1,\dots,a_{n+1}) + \sum_i (-1)^i \tilde\varphi(\tilde a_0,\dots,a_i a_{i+1},\dots) + (-1)^{n+1}\tilde\varphi(a_{n+1}\tilde a_0, a_1,\dots,a_n)$$

If $\tilde a_0$ is restricted to a, we recover $b\varphi$. If $\tilde a_0 = 1$, then we have $(1-\lambda)\varphi$.
By diagram chasing one gets the following from $(*)$, the Connes exact sequence

$$HC^0(A) = HC^0(A,\tilde A) = \{\text{traces on } A\}$$

$$0 \to HC^1(A) \to H^1(A,\tilde A^*) \to HC^0(A) \xrightarrow{S} HC^2(A) \to H^2(A,\tilde A^*) \to HC^1(A) \xrightarrow{S} HC^3(A) \to H^3(A,\tilde A^*) \to HC^2(A)$$

Theorem Take $R = RA$, $I = IA$ the kernel of the canonical map of R onto A.

$$\begin{tikzcd} (R/[R,R])^* \arrow[r] & (I^m/[I,I^{m-1}])^* \arrow[d] \\ & HC^{2m-1}(A) \arrow[d] \\ & 0 \end{tikzcd}$$

Then the sequence here is exact.

Proof: Let us recall that
$$\tau \in (I^m/[R,I^m])^*$$
is equivalent to a cocycle $f_{\geq 2m} = (f_p)$ $(p \geq 2m)$ satisfying the symmetry condition $\lambda^2 f_{2n} = f_{2n}$ for $n > m$.

$\tau \in (I^m/[I,I^{m-1}]) \Leftrightarrow$ same as above but also $\lambda^2 f_{2m} = f_{2m}$.

Formulae $\begin{cases} f_{2n} = \tau(\omega^n)/n! \\ f_{2n+1} = \tau(\rho\,\omega^n) \end{cases}$

$$\begin{tikzcd} \arrow[u, "b'"] f_{2m+2} \arrow[r, "N"] & \arrow[u, "b"] f_{2m+1} \arrow[r, "1-\lambda"] & \arrow[u, "b'"] f_{2m} \end{tikzcd}$$

$$b' f_{2n} = (1-\lambda) f_{2n+1}$$
$$b f_{2n+1} = N f_{2n+2} \ (n \geq m)$$

Connes has constructed the map $\tau \in (I^m/[R,I^m])^*$ to the $N f_{2m}$ class in $HC^{2m}(A)$. Let $f_{\geq 2m}$ be the cocycle given by τ.

$$
\begin{array}{ccccc}
\uparrow b & & & & \\
f_{2m+1} & \xrightarrow{1-\lambda} & & 0 & \\
 & \uparrow b' & & \uparrow b & \\
 & f_{2m} & \xrightarrow{N} & Nf_{2m} & \xrightarrow{1-\lambda} 0
\end{array}
$$

$\therefore$ Nf_{2m} is a cyclic cocycle.

Assume $\tau \in (I^m/[I, I^{m-1}])^*$ comes from a trace on R. Then $f_{\geq 2m}$ extends to a cocycle $f_{\geq 1}$

$$
\begin{array}{llll}
f_{2m+1} \to & & & \\
\quad\uparrow & & & \\
\quad f_{2m} \to & & & \\
 & \quad\uparrow & & \\
 & f_{2m-1} \to & & \\
 & \qquad \uparrow & & \\
 & & \quad f_1 \to 0 &
\end{array}
$$

By diagram chasing we can modify the cocycle $f_{\geq 1}$ and suppose $f_1 = \ldots = f_{2m-2} = 0$.
Then f_{2m-1} is a cyclic cochain (degree $2m-2$) with $bf_{2m-1} = Nf_{2m}$.
$\therefore$ The cyclic cohomology class of Nf_{2m} is zero.
Similarly, if $\tau \in (I^m/[I, I^{m-1}])^*$ has its Connes class 0, then τ extends.

Surjectivity of the Connes map: Let φ_{2m} be a cyclic $(2m-1)$ cocycle

$$
\begin{array}{ccccc}
f_{2m+1} \longrightarrow & \uparrow b' & & & \\
 & f_{2m} & \xrightarrow{N} & \varphi_{2m} & \xrightarrow{1-\lambda} 0
\end{array}
$$

$\therefore$ construct a cocycle $f_{\geq i}(f_\cdot)$ by diagram chase.
To check that it comes from a trace, need to check the symmetry conditions. Need $\lambda^2 f_{2m} = f_{2m}$
Can do this by taking
$$f_{2m} = \frac{1}{2m}\varphi_{2m}$$

<u>Theorem</u> (Even case)
Let $R = RA$, $I = IA = \mathrm{Ker}\{RA \to A\}$.

$$(\Omega^1 R/[R, \Omega^1 R] + I^m \Omega^1 R)^* \longrightarrow (R/[R,R] + I^{m+1})^* \longrightarrow HC^{2m}(A) \longrightarrow 0$$

$\Omega^1 R$ = bimodule of noncommutative differential over R.

There is a canonical derivation $d: R \to \Omega^1 R$ which is a universal derivation: given $D: R \to M$ D derivation, there is a unique bimodule map $\Omega^1 R \xrightarrow{u} M$ such that $ud = D$.

Facts 1) One has an exact sequence of R bimodules

$$\tilde R \otimes R \otimes R \otimes \tilde R \xrightarrow{b'} \tilde R \otimes R \otimes \tilde R \to \Omega^1 R \to 0$$

$$\tilde r_1 \otimes r \otimes \tilde r_2 \longmapsto \tilde r_1 \, dr \, r_2$$

2) There is an exact sequence – standard normalised resolution

$$\begin{tikzcd} \arrow[r, "b"] & \tilde R \otimes R \otimes R \otimes \tilde R \arrow[r, "b'"] & \tilde R \otimes R \otimes \tilde R \arrow[r, "b'"] \arrow[l, bend left, "s"] & \tilde R \otimes \tilde R \arrow[l, bend left, "s"] \end{tikzcd}$$

$$\hookrightarrow \tilde R \longrightarrow 0$$

One has exact sequence

$$0 \longrightarrow \Omega^1 R \xrightarrow{i} \tilde R \otimes \tilde R \xrightarrow{b'} \tilde R \to 0$$

$$i(dr) = b'(1 \otimes r \otimes 1) = r \otimes 1 - 1 \otimes r$$

3) $$\tilde R \otimes R \xrightarrow{\sim} \Omega^1 R$$
$$\tilde r_0 \otimes r_1 \longmapsto \tilde r_0 \, dr_1$$

4) Equivalence between:

(i) traces τ' on $\Omega^1 R$ on R bimodule

(ii) 1-cocycles in complex $C^{\cdot}(R, \tilde R^*)$

(iii) pair $\psi_2 \in (R^{\otimes 2})^*$, $\varphi_1 \in R^*$ satisfying

$$b' \varphi_1 = (1-\lambda)\psi_2$$
$$b \psi_2 = 0$$

$$\begin{tikzcd} 0 & \\ \psi_2 \arrow[u, "b"] \arrow[r, "1-\lambda"] & \cdot \\ & \arrow[u, "b'"] \end{tikzcd}$$

(i) $\Rightarrow$ (ii) - By (3), $\tau' \in (\Omega^1 R)^*$ is equivalent to a (pairing) bilinear map $\Psi(\tilde r_0, r_1)$ on $\tilde R \times R$ given by $\Psi(\tilde r_0, r_1) = \tau'(\tilde r_0 \, dr_1)$

τ' trace means

$$\tau'(\tilde r_0 \, dr_1 \, r_2) \overset{?}{=} \tau'(r_2 \tilde r_0 \, dr_1)$$
$$= \Psi(r_2 \tilde r_0, r_1)$$
$$= \tau'(\tilde r_0 \, d(r_1 r_2) - \tilde r_0 r_1 \, dr_2)$$
$$= \Psi(\tilde r_0, r_1 r_2) - \Psi(\tilde r_0 r_1, r_2)$$

$\therefore \ (b\Psi)(\tilde r_0, r_1, r_2) = 0$
$\iff \tau'$ is a trace on $\Omega^1 R$.

5) If $R = RA$, then $\Omega^1 R \cong \tilde R \otimes A \otimes \tilde R$

$$\Omega^1 R \cong \tilde R \otimes A \otimes \tilde R$$
$$\tilde r_1 \, d\rho(a) \, \tilde r_2 \longleftarrow\!\shortmid \tilde r_1 \otimes a \otimes \tilde r_2$$

(obvious by universal property of $\Omega^1 R$).

6) $$\Omega^1 R / [R, \Omega^1 R] \xleftarrow{\sim} \tilde R \otimes A$$
$$\tilde r \, d\rho(a) \longleftarrow\!\shortmid \tilde r \otimes a$$

Recall $R \xleftarrow{\sim} \bigoplus_{p \geq 1} A^{\otimes p}$

[From (5) onwards $R = RA$]

components ω^n $p = 2n$, $\rho\omega^n$ $p = 2n+1$.

$\therefore$ We get a linear isomorphism

$$\Omega^1 R / [R, \Omega^1 R] \xleftarrow{\sim} \bigoplus_{p \geq 1} A^{\otimes p}$$

with components $\begin{cases} \omega^{n-1} d\rho & p = 2n-1 \\ -\rho\omega^{n-1} d\rho & p = 2n \end{cases}$

$$\begin{aligned} (\omega^{n-1} d\rho)(a_1, a_2, \dots, a_{2n-1}) &= \\ &= \omega^{n-1}(a_1, a_2, \dots, a_{2n-2})\, d\rho(a_{2n-1}) - \\ -(\rho\omega^{n-1} d\rho)(a_0, a_1, \dots, a_{2n-1}) & \\ &= \rho(a_0)\, \omega^{n-1}(a_1, \dots, a_{2n-2})\, d\rho(a_{2n-1}) \end{aligned}$$

7) $\therefore$ Equivalence between traces τ' on $\Omega^1 R$ (where $R = RA$) and inhomogeneous cochains $g = \{g_p : p \geq 1\}$; given by $\begin{cases} g_{2n-1} = \tau'(\omega^{n-1} d\rho) \\ g_{2n} = \tau'(-\rho\omega^{n-1} d\rho) \end{cases}$

General definition (any R): If τ' is a trace on ~~A~~ $\Omega^1 R$, then $\tau' d$ is a trace on R. Such a trace will be called null cobordant.

A trace τ on R is null cobordant if and only if $\tau \in \operatorname{Im} B$ in the exact sequence ($\tau \in HC^0(R)$)

$$H^1(R, R^{2*}) \xrightarrow{B} HC^0(R) \xrightarrow{S} HC^2(R) \to \cdots$$

$\Psi = (\psi_2, \varphi_1)$

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \uparrow b & & & & \\ 0 & \to & C^1_\lambda & \to & \psi_2 & \xrightarrow{1-\lambda} & & \to C^1_\lambda & \to 0 \\ & & & & b' \uparrow & & & & \\ 0 & \to & C^0_\lambda & \to & \to \varphi_1 & \xrightarrow{N} & & C^0_\lambda & \to 0 \end{array}$$

B takes $[\Psi]$ to $N\varphi_1 = \varphi_1$

$(\, \varphi(v_1) = \tau'(dv_1),\ \psi_2(v_0 v_1) = \tau'(v_0\, dv_1))$

Theorem Let $R = RA$, τ' a trace on $\Omega^1 R$, $\tau = \tau' d$ corresponding null-cobordant trace on R, f is the cocycle of τ. Then f is the coboundary in the double complex.

In addition to the g-cochain associated to τ' we define the h-cochain by

$$\begin{cases} h_{2n-1} = \tau'(\mu_{2n-1}) \\ h_{2n} = \tau'(-\rho\,\mu_{2n-1}) \end{cases}$$

where $\mu_{2n-1} = \sum_1^n \omega^{i-1} d\rho\, \omega^{n-i}$

$$\begin{cases} g_{2n-1} = \tau'(\omega^{n-1} d\rho) \\ g_{2n} = \tau'(-\rho\omega^{n-1} d\rho) \end{cases}$$

Theorem (cont) f = coboundary of h:

$$f_{2n} = \tau' d(\omega^n) = b' h_{2n-1} + (\lambda - 1) h_{2n}$$

$$f_{2n+1} = \tau' d(\rho\omega^n) = -b h_{2n} + \frac{1}{n+1} N h_{2n+1}$$

$$\begin{array}{ccccc}
 & & \uparrow -b & & \\
\xrightarrow{N} & & f_{2n+1} & \xrightarrow{\lambda - 1} & \bullet \\
 & & \uparrow -b & & \\
h_{2n} & \xrightarrow{\lambda - 1} & f_{2n} & \xrightarrow{N} & \\
 & & \uparrow b' & &
\end{array}$$

Lemma: One has

$$h_{2n-1} = \sum_{i=1}^{n} \lambda^{2n-2i} g_{2n-1}$$

$$h_{2n} = \sum_{1 \le i \le n} \lambda^{2n-2i} (g_{2n} + g_{2n,i})$$

where

$$g_{2n,i} = \tau' \{ b'(\omega^{n-i})\, \omega^{i-1} d\rho \}$$

is a cochain depending only on g_{2n}.

Recall: Definition: A trace τ on R is null homotopic/cobordant if it is of the form

$$\tau = \tau' d$$

for some trace τ' on $\Omega' R$ as a R-bimodule

Now $\Omega'_R \cong \tilde{R} \otimes A \otimes \tilde{R}$

$$\Omega'_R / [R, \Omega' R] \cong \tilde{R} \otimes A \leftarrow \bigoplus_{p \ge 0} A^{\otimes p+1}$$

given by cochains $\omega^{n-1} d\rho$, $-\rho\omega^{n-1} d\rho$

A trace on Ω'_R is equivalent to a sequence of homogeneous cochains

$$\begin{cases} g_{2n-1} = \tau'(\omega^{n-1} d\rho) \\ g_{2n} = \tau'(-\rho\omega^{n-1} d\rho) \end{cases}$$

Call this the g cochain of τ'. Introduce the h-cochain of τ' defined by

$$h_{2n-1} = \tau'\Big(-\sum_1^n \omega^{i-1} d\rho\, \omega^{n-i}\Big)$$

$$h_{2n} = \tau'(-\rho\mu_{2n-1})$$

where

$$\mu_{2n-1} = \sum_1^n \omega^{i-1} d\rho\, \omega^{n-i}$$

$$\mu_{2n-1}(a_1, \dots, a_{2n-1}) = \sum_{i=1}^{n} \omega^{i-1}(a_1, \dots, a_{2i-2}) \cdot d\rho(a_{2i-1})\, \omega^{n-i}(a_{2i}, \dots, a_{2n-1})$$

$$= \sum_{i=1}^{n} \omega(a_1, a_2) \dots \omega(a_{2i-3}, a_{2i-2})\, d\rho(a_{2i-1}) \cdot \omega(a_{2i}, a_{2i+1}) \dots \omega(a_{2n-2}, a_{2n-1})$$

Theorem 2 If $\tau = \tau' d$ then the cocycle of τ is the coboundary of the h-cochain associated with τ'.

r.t.p. $\begin{cases} f_{2n} = \tau' d(\omega^n) = b' h_{2n-1} + (\lambda - 1) h_{2n} & \text{(i)} \\ f_{2n+1} = \tau' d(\rho\omega^n) = (-b) h_{2n} + N h_{2n+1} / n+1 & \text{(ii)} \end{cases}$

$$d\omega = \delta d\rho + d\rho\rho + \rho d\rho \qquad \omega = \delta\rho + \rho^2 : A^{\otimes} \to R$$
$$= (\delta + ad\rho)(d\rho)$$
$$(\delta + ad\rho)(\omega) = 0$$

$$d\omega^n = \sum_1^n \omega^{i-1} (d\omega) \omega^{n-i}$$
$$= \sum_1^n \omega^{i-1} (\delta + ad\rho)(d\rho) \omega^{n-i}$$
$$= (\delta + ad\rho) \sum_1^n \omega^{i-1} d\rho\, \omega^{n-i}$$
$$= (\delta + ad\rho)(\mu_{2n-1})$$

$$\therefore \delta \mu_{2n-1} = -\rho\mu_{2n-1} - \mu_{2n-1}\rho + d\omega^n$$

$$\therefore b' \mu_{2n-1}(a_1, \dots, a_{2n}) = \rho(a_1) \mu_{2n-1}(a_2, \dots, a_{2n}) + \mu_{2n-1}(a_1, \dots, a_{2n-1}) \rho(a_{2n}) + d(\omega^n)(a_1, \dots, a_{2n}) \qquad (*)$$

Apply τ' to this expression. Now τ' commutes with b'

$$b' h_{2n-1}(a_1, \dots, a_{2n}) = h_{2n}(a_1, \dots, a_{2n}) + h_{2n}(a_{2n}, a_1, \dots, a_{2n-1}) + \tau' d(\omega^n)(a_1, \dots, a_{2n})$$

which is formula (i).

For the equation (ii) we calculate

$$h_{2n} = -\tau'(\rho \mu_{2n-1})$$

$$(b h_{2n})(a_0, \dots, a_{2n}) = \tau'(\rho(a_0 a_1) \mu_{2n-1}(a_2, \dots, a_{2n})) - \tau'(\rho(a_0)\, b' \mu_{2n-1}(a_1, \dots, a_{2n})) + \tau'(\rho(a_{2n} a_0) \mu_{2n-1}(a_1, \dots, a_{2n-1}))$$

$(*)$ gives

$$= \tau' \{ (\rho(a_0 a_1) - \rho(a_0)\rho(a_1)) \mu_{2n-1}(a_2, \dots, a_{2n})$$
$$(\rho(a_{2n} a_0) - \rho(a_{2n})\rho(a_0)) \mu_{2n-1}(a_1, \dots, a_{2n})$$
$$- \rho(a_0)\, d(\omega^n)(a_1, \dots, a_{2n}) \}$$

$$= \tau' \Big(\sum_1^n \omega^i (a_0, \dots, a_{2i-1})\, d\rho(a_{2i})\, \omega^{n-i}(a_{2i+1}, \dots, a_{2n})$$
$$+ \sum_1^n \omega^i (a_{2n}, a_0, \dots, a_{2i-1})\, d\rho(a_{2i-1})\, \omega^{n-i}(a_{2i}, \dots, a_{2n-1})$$
$$- \rho(a_0)\, d(\omega^n)(a_1, \dots, a_{2n}) \Big)$$

Move $d\rho$ to the front using trace property.

$$= \tau'\{ \textstyle\sum_1^n d\rho(a_{2i})\, \omega^{n*}(a_{2i+1}, \dots, a_{2n}, a_0, \dots, a_{2i-1})$$

$$+ \textstyle\sum_1^n d\rho(a_{2i-1})\, \omega^{n*}(a_{2i}, \dots, a_{2i-2})$$

$$+ \rho(a_0)\, d(\omega^n)(a_1, \dots, a_{2n}) + d\rho(a_0)\, \omega^n(a_1, \dots, a_{2n}) - d\rho(a_0)\, \omega^n(a_1, \dots, a_{2n}) \}$$

$$= \sum_{j=0}^{2n} \tau'\big(d\rho(a_j)\, \omega^n(a_{j+1}, \dots, a_{j-1})\big)$$

$$- \tau'\big((d\rho\, \omega^n)(a_0, \dots, a_{2n})\big)$$

$$N\tau'\Big(\textstyle\sum_0^n \omega^i d\rho\, \omega^{n-i}\Big) = N\tau'\Big(\textstyle\sum_0^n \omega^n d\rho\Big)$$

and (ii) follows.

<u>$HC^{ev}(A)$ and traces on RA</u>

<u>Theorem</u>

$$\Big(\Omega^1 R / ([R, \Omega^1 R] + I^m \Omega^1 R)\Big)^* \xrightarrow{d^t} \Big(R/(I^{m+1} + [R,R])\Big)^* \to HC^{ev}(A) \to 0$$

$$\Big(R/(I^{m+1} + [R,R])\Big)^* = \text{traces on } R/I^{m+1}$$

is an exact sequence

Given a trace τ on R, vanishing on I^{m+1}, its cocycle f is such that $f_p = 0$ for $p \geq 2m+2$

$$f_{2m+1} = \tau(\rho\omega^m)$$

$$\begin{tikzcd}
0 & \\
f_{2m+1} \arrow[u] \arrow[r, "1-\lambda"] & 0 \\
f_{2m} \arrow[r, "N"] & 0 \arrow[u, "b'"] \\
f_3 \arrow[r, "\lambda-1"] & 0 \arrow[u, "-b"] \\
f_2 \arrow[r] & \cdot \arrow[u] \\
f_1 \arrow[r] & 0
\end{tikzcd}$$

Rows are exact so one can find a cochain $h_1, h_2, \dots, h_{2m}$ such that f − coboundary of h is a single cochain – only the degree $2m+1$ component different from zero.

This component is a cyclic $2m$ cocycle. Its class in $HC^{2m}(A)$ is ~~dep~~ independent of the choices in the choice.

Proof: $CC^{2m-1}(A) \xrightarrow{b} Z^{2m}(A) \longrightarrow HC^{2m}(A) \to 0$

cyclic cochains $\quad$ cyclic cocycles

$$\begin{tikzcd}
CC^{2m-1}(A) \arrow[r, "b"] \arrow[d, "u"] & ZC^{2m}(A) \arrow[r] \arrow[d, "v"] & HC^{2m}(A) \arrow[r] \arrow[d, dashed] & 0 \\
(\Omega^1 R/[R,\Omega^1 R] + I^n\Omega^1 R)^* \arrow[r] & (R/[R,R] + I^{m+1})^* \arrow[r] & \mathrm{Coker}\, d^t &
\end{tikzcd}$$

Define v: given a cyclic $(2m)$ cocycle $\psi_{2m+1}(a_0, \dots, a_{2m})$

$$(1-\lambda)\psi_{2m} = 0 = b\psi_{2m}$$

$$\begin{tikzcd}
0 & 0 & \\
& \psi_{2m+1} \arrow[u, "-b"] \arrow[r, "\lambda-1"] & 0 \\
& 0 &
\end{tikzcd}$$

Thus there is a unique trace τ on R ~~such that~~ whose cocycle is just ψ_{2m+1} and all other components are zero.

i.e. $\tau(\omega^{n-1}) = 0 \quad n \geq 1$

$$\tau(p\omega^n) = \begin{cases} 0 & n \neq m \\ \psi_{2m+1} & n = m \end{cases}$$

Let $v(\psi_{2m+1}) = \tau$ (Note that $\tau(I^{m+1}) = 0$ since I^n is generated by $\omega^n, p\omega^n$).

To define u, given a cyclic $(2m-1)$ cochain ψ_{2m}

Let τ' be the trace on $\Omega^1 R$ with

$$g_{2n-1} = \tau'(\omega^{n-1}dp) = 0$$

$$g_{2n} = \tau'(-p\omega^{n-1}dp) = \begin{cases} 0 & n \neq m \\ \psi_{2m}(-1/m) & n = m \end{cases}$$

Note that $\tau'(I^n\Omega^1 R) = 0$ since $I^n\Omega^1 R$ is the image of $\omega^{n-1}dp$ and $-p\omega^{n-1}dp$.

Define $u(\psi_{2m}) = \tau'$.

Need to check also that the square commutes.
This comes down to checking that

$${}^t d u \psi_{2m} = v b \psi_{2m}$$

${}^t d u \; : \; \tau' d$

The cocycle is the coboundary of the h-cochain of τ'. Recall the formulae

$$h_{2n-1} = \tau'\left(\sum_1^n \omega^{i-1} dp\, \omega^{n-i}\right)$$

$$= \sum_{i=1}^n \lambda^{2n-2i} \underbrace{\tau'(\omega^{n-1}dp)}_{g_{2n-1} \text{ cochain}}$$

$$h_{2n} = \tau'\left(-p\sum_1^n \omega^{i-1} dp\, \omega^{n-i}\right)$$

$$= \sum_1^n \lambda^{2n-2i} \tau'(-\omega^{n-i} p \cancel{dp}\, \omega^{i-1} dp)$$

$$= \sum_1^n \lambda^{2n-2i} \left(\tau'(-p\omega^{n-1}dp) + \tau'(-\delta(\omega^{n-i}) . \omega^{i-1}dp)\right)$$

Lemma: $h_{2n-1} = \sum_1^n \lambda^{2n-2i} g_{2n-1}$

$$h_{2n} = \sum_1^n \lambda^{2n-2i}\left(g_{2n} + \text{term depending only on } g_{2n-1}\right)$$

Now $g_{2m} = -1/m\, \psi_{2m}$ and all other g_p are zero
then only $h_{2m} = \sum_i^m \lambda^{2m+2i} g_{2m}$
$= -\psi_{2m}$

$$\begin{array}{ccc} 0 & \longrightarrow & b\psi_{2m} \\ & & \uparrow -b \\ & & -\psi_{2m} \longmapsto \quad \uparrow \\ & & \parallel \\ & & h_{2m} \quad \alpha \quad 0 \end{array}$$

$\therefore$ The trace $\tau' d$ has the cochain $b\psi_{2m}$ and other components zero.

$\therefore \tau' d = \nu(b\psi_{2m})$

Let ω be the map induced by ν. To prove that ω injective: Start with a cyclic $2m$ cocycle ψ_{2m+1} such that $\nu\psi_{2m+1} = \check{d}\tau' = \tau' d$ for some τ'. Must show that $\psi_{2m+1} = b$ (cyclic cochain).

$$\begin{array}{cccc} & \psi_{2m}\,0 & & \\ 0 & \uparrow -b & & \\ 0 & \psi_{2m+1} & \xrightarrow{\lambda-1} & 0 \\ & h_{2m} & & 0 \\ & & h_{2m-1} & \end{array}$$

Then the coboundary of the h-cochain of τ' is ψ_{2m+1} in degree $(2m+1)$.
Diagram chasing forces ψ_{2m+1} to be a cyclic coboundary.

To prove that ω is surjective, we start with τ a trace on R/I^{2m+1}. Let f be the cocycle of τ.

$$\begin{array}{c} 0 \\ f_{2m+1} \\ f_{2m} \\ \ddots \\ f_2 \end{array}$$

We know that, by diagram chasing, that there is a cochain $h = h_1 + \dots + h_{2m}$ such that $f -$ coboundary of h $= \psi_{2m-1}$ a cyclic $2m$ cocycle, other components zero.

The problem is to show that h can be chosen to be the h cochain of a τ'! Use induction, looking at the smallest p such that $f_p \neq 0$.

Suppose $p = 2m-1$

$$\begin{array}{ccccc} h_{2n} & & \uparrow b' & & \\ & h_{2n-1} & \xrightarrow{N} f_{2n-1} & \xrightarrow{1-\lambda} & \\ & & & \uparrow -b & \\ & & 0 & 0 & \end{array}$$

~~Take~~ Let τ' have $g_{2n-1} = f_{2n-1}/(2n-1)$ with other components zero. Only h_{2n} and h_{2n-1} are possibly non-zero.

$h_{2n-1} = \left(\sum_i^n \lambda^{2n-2i} g_{2n-1}\right) = \frac{n}{2n-1} f_{2n-1}$

Removing $\tau' d$ from τ we raise the order of τ.

Suppose $p = 2n$

$f_{2n-1} \xrightarrow{\lambda - 1}$.

$h_{2n} \xrightarrow{\lambda-1} f_{2n} \longrightarrow 0$ ($\lambda b'$)

This time we must find a g

Take τ' to have only $g_{2n} \neq 0$. Then we have

$h_p = 0 \quad p \neq 2n$

$h_{2n} = (\sum \lambda^{2n-2i} g_{2n})$

It suffices to show that g_{2n} can be chosen so that when we apply $\{(1-\lambda) \sum_i^n \lambda^{2n-2i}\} g_{2n} = f_{2n}$

$\sum_j^{2n-1} (-1)^j$ Reminder $\Leftarrow\Rightarrow$ $-\lambda f_{2n} = f_{2n}$

But this follows from $N f_{2n} = \sum_j^{2n-1} \lambda^j f_{2n} = 0$

and

$\lambda^2 f_{2n} = f_{2n}$ ($\because f$ arose from here τ)

Properties of RA, QA, ΩA

United algebra category $1 \in A$, $\bar{A} = A/k1$

ΩA is the algebra of noncommutative differential forms over A can be defined as the universal differential graded algebra generated by A in degree - zero.

$A = \Omega^0 A \to \Omega^1 A \to \Omega^2 A \to$

Universal means that any homomorphism $A \to$ degree zero subalgebra of a DGA S extends uniquely to a hom. $\Omega A \to S$ of DG Algebras

$d1 = d(1.1) = d1.1 + 1.d1 = 2d1$

$\therefore da$ depends only on a in $\bar{A}$

Prop^n $A \otimes \bar{A}^{\otimes n} \longrightarrow \Omega^n A$

$(a_0, a_1, \dots, a_n) \longmapsto a_0 da_1 \dots da_n$

gives a vector space isomorphism for all n.

Proof. Surjectivity: let $L^n \subset \Omega A$. be the subspace spanned by $a_0 da_1 \dots da_n$ where the a_i range through A; $L = \bigoplus_{n \geq 0} L^n \subset A$.

We show that L is a left ideal in ΩA, hence all of ΩA (since $1 \in L$

Since ΩA is generated by the elements a, da for $a \in A$ we note

$a(a_0 da_1 \dots da_n) \in L$

$da . a_0 da_1 \dots da_n$

$= 1 . d(aa_0) da_1 \dots da_n - a da_0 \dots da_n$

$\in L$

To prove injectivity, put $\Omega^n = A \otimes \bar{A}^{\otimes n}$ and define d on Ω^n by the rule

$d(a_0, a_1, \dots, a_n) = (1, a_0, a_1, \dots, a_n)$

This defines a complex $(\Omega^\cdot, d)$. Use the fact that the space $\mathrm{Hom}_k(\Omega^\cdot, \Omega^\cdot)$ is a DG algebra with the differential $du = d\circ u - (-1)^{|u|} u \circ d$

We have a homomorphism $A \to \mathrm{Hom}_0(\Omega, \Omega)$
$$a \longmapsto \left((a_0, \dots, a_n) \longmapsto (aa_0, \dots, a_n) \right)$$
This induces a homomorphism of DG algebras
$$\Omega A \longrightarrow \mathrm{Hom}_k(\Omega, \Omega)$$
$$a \longmapsto (\text{left multiplication by})\, a$$
$$da \longmapsto [d,a] = d\circ a - a\circ d$$
This makes Ω into a left DG module over ΩA.
$$a_0\, da_1 \dots da_n \longmapsto a_0 [d,a_1] \dots [d,a_n]$$
We have on applying this to $1 \in \Omega^0$
$$(a_0 [d,a_1] \dots [d,a_n])(1)$$
$$[d,a_i] \dots [d,a_n] = (1, a_i, \dots, a_n)$$
$$\because\ d(a_n 1) = da_n = (1, a_n)$$
$$\&\ -a_n d1 = a_n (1,1) = 0$$
$$\therefore\ (a_0 [d,a_1] \dots [d,a_n]) 1 = (a_0, a_1, \dots, a_n)$$
This produces a map $\Omega^n A \to \Omega^n = A \otimes \bar{A}^{\otimes n}$ inverse to the map in the proposition. $\square$

Exercise: $A = k + ke \qquad e^2 = e$

Work out ΩA

The Cuntz Algebra $QA = A * A$ the free product of A with itself.

Thus there are two canonical homomorphisms
$$A \underset{\bar{i}}{\overset{i}{\rightrightarrows}} QA$$
which are a universal pair. – evident universal property.

Consequences

1)
$$\begin{tikzcd} A \arrow[r, shift left, "i"] \arrow[r, shift right, "\bar{i}"'] \arrow[dr, "id"'] & QA \arrow[d, "\text{fold}\ \phi"] \\ & A \end{tikzcd}$$
$\phi i = \phi \bar{i} = id$

$\therefore\ i\ \ \bar{i}$ injective

2) There is an automorphism of order 2 on QA
$$x \longmapsto x^F$$
such that $\quad i^F = \bar{i}, \quad (\bar{i})^F = i.$

$\therefore\ QA$ is a superalgebra
$$QA = (QA)^+ \oplus (QA)^-$$
$$(x^F = x) \qquad (x^F = -x)$$
odd-even decomposition

Put $\quad a^\pm = $ even (odd) components of ia
$$a^+ = \frac{ia + \bar{i}a}{2}$$
$$a^- = \frac{ia - \bar{i}a}{2}$$

Quillen

$$\begin{cases}(a_1 a_2)^+ = a_1^+ a_2^+ + a_1^- a_2^- \\ (a_1 a_2)^- = a_1^+ a_2^- + a_1^- a_2^+\end{cases}$$

Verify:

1) The linear maps $a \mapsto a^+$, $a \mapsto a^-$ from A to Q are universal pair satisfying the above relations.

2) QA is the superalgebra generated by A in the sense that

$$\mathrm{Hom}_{\text{superalg}}(QA, S) = \mathrm{Hom}_{\text{alg}}(A, S)$$

$1 \in A$ $\quad 1^- = \frac{1}{2}(\iota 1 - \bar\iota 1) = 0$

$1^+ = 1$

Define maps $A \otimes \bar{A}^{\otimes n} \to QA$

$(a_0, \ldots, a_n) \longmapsto a_0^+ a_1^- \ldots a_n^-$

Prop^n: 1) The sum of these maps is an vector space isomorphism.

$$\bigoplus_{n \geq 0} A \otimes \bar{A}^{\otimes n} \xrightarrow{\sim} QA.$$

2) There is a vector space isomorphism

$$QA \xrightarrow{\simeq} \Omega A$$

$$a_0^+ a_1^- \ldots a_n^- \longleftrightarrow a_0 da_1 \ldots da_n$$

3) W.r.t. respect to this isomorphism the algebra structure in QA corresponds to the following product on ΩA.

$$\omega * \eta = \omega\eta - (-1)^{|\omega|} d\omega d\eta$$

Proof of 1) Surjectivity: Let L be the span of $a_0^+ a_1^- \ldots a_n^-$. Show that it is a left ideal. Use the fact that Q is ~~alg~~ generated by a^+, a^- $(a \in A)$.

$$a^+ a_0^+ a_1^- \ldots a_n^- = (aa_0)^+ a_1^- \ldots a_n^- - 1 a^- a_0^- \ldots a_n^- \in L$$

'a^-' gives similar argument. Note that $1 \in L$, so $L = QA$.

Injectivity: We define an action of QA on ΩA (i.e. a left module structure), and use the action on the element $1 \in \Omega^0 A$ to obtain a map $QA \to \Omega A$, which is inverse to the map $\Omega A \to QA$ $\quad a_0 da_1 \ldots da_n \mapsto a_0^+ a_1^- \ldots a_n^-$.

Let us define the map, using the universal property of QA. Algebra homomorphism

$$QA \to \mathrm{Hom}_k(\Omega A, \Omega A)$$

$$\iota a \longmapsto (1+d)a(1-d)$$

$$\bar\iota a \longmapsto (1-d)a(1+d)$$

$$(1+d)a(1-d) = a + da - (da)d$$

$$(a : b \mapsto ab \text{ etc})$$

$$(1-d)a(1+d) = a - da - (da)d$$

$$a^+ = a - (da)d$$

$$a^- = da$$

What happens to $a_0^+ a_1^- \ldots a_n^-$?

$$\eta \mapsto ((a_0 - (da_0)d)\, da_1 \ldots da_n)(\eta)$$

is the operator which it becomes.

$= a_0 da_1 \ldots da_n \eta - (-1)^n da_0 \ldots da_n d\eta$

Take $\eta = 1$. This gives the inverse map

$a_0^+ a_1^- \ldots a_n^- \longmapsto a_0 da_1 \ldots da_n$

This proves (2)

$a_0^+ a_1^- \ldots a_n^- (\eta) = (a_0 da_1 \ldots da_n) * \eta$

$\Omega A \simeq QA = A * A$

$a_0 da_1 \ldots da_n \longmapsto a_0^+ a_1^- \ldots a_n^-$

product in QA corresponds to

$$\omega * \eta = \omega\eta - (-1)^{|\omega|} d\omega d\eta$$

Complements:

Let $J = \mathrm{Ker}(A * A \xrightarrow{\text{fold}} A)$

= ideal in QA generated by

$a^- = \frac{1}{2}(a - \gamma a) \quad a \in A.$

J-adic filtration

$QA \supset J \supset J^2 \supset \ldots$

associated graded algebra $gr^J Q = \bigoplus J^n / J^{n+1}$ is canonically isomorphic to ΩA as graded algebras

Also $J^n \simeq \Omega^{\geq n} A$ linear isomorphism.

In the unital category, define

$RA = T(A) /$ ideal generated by $1_{T(A)} - 1_A$

$= \bigoplus_{n \geq 0} A^{\otimes n} /$ ideal

If one chooses a splitting of the exact sequence

$0 \to k \cdot 1 \to A \leftrightarrows \bar{A} \to 0$

then $RA \simeq T(\bar{A})$ $\quad R(\tilde{\mathbb{C}}) = \widetilde{R(\mathbb{C})}$

Let ρ be the map $A \to RA$, namely

$A = A^{\otimes 1} \subseteq T(A) \twoheadrightarrow RA$

Now ρ is a universal linear map from A to an unital algebra such that $\rho(1) = 1$.

3) Claim $RA = (QA)^+$

Proof: Define $\rho: A \to (QA)^+$ by $\rho(a) = a^+$ $\quad a \mapsto a^+$

This extends to a homomorphism $u: RA \to (QA)^+$

~~Because~~ Surjectivity: First show that RA is spanned by elements of the form

$\rho(a_0)\omega(a_1, a_2) \ldots \omega(a_{2n-1}, a_{2n})$

for $n \geq 0$

$\omega(a_1, a_2) = (\delta\rho \cdot \rho^1)(a_1, a_2) = \rho(a_1 a_2) - \rho(a_1)\rho(a_2)$

$\longmapsto (a_1 a_2)^+ - a_1^+ a_2^+ = a_1^- a_2^-$

$\therefore u$ carries

$\rho(a_0)\omega(a_1 a_2) \ldots \omega(a_{2n-1}, a_{2n})$

to $a_0^+ a_1^- \ldots a_{2n}^-$

First check that $\{\rho(a_0)\omega \ldots \omega(a_{2n-1}, a_{2n}) ; n \geq 0\}$ span RA, by the left ideal argument as before.

Now define a map from $(QA)^+$ back to RA by $a_0^+ a_1^- \ldots a_{2n}^- \longmapsto \rho(a_0)\omega(a_1, a_2) \ldots \omega(a_{2n-1}, a_{2n})$

well defined because $QA \simeq \Omega A$.
How canonical is the isomorphism?

$$QA \simeq \Omega A$$

ΩA is graded. As a graded algebra, ΩA has the automorphisms $\omega \mapsto t^{|\omega|}\omega$

$$QA \longrightarrow \mathrm{Hom}_k(\Omega A, \Omega A)$$
$$\iota a \mapsto (1+td)a(1-td) = a + tda - t^2 da d$$

However, the filtration J-adic of QA has a natural splitting as vector spaces.

$$A^+ = \{a^+ : a \in A\}$$
$$A^- = \{a^- : a \in A\}$$
$$A^+A^- = A^-A^+ \supseteq A^-$$
$$A^+(A^-)^n = \mathrm{span}\{a_0^+ a_1^- \dots a_n^-\}$$
$$= (A^-)^i A^+ (A^-)^{n-i}$$

The subspace $A^+(A^-)^n$ is ~~isomorphic canonic~~

$$J^n = A^+(A^-)^n \oplus J^{n+1}$$

$\therefore$ One gets a canonical basic isomorphism of QA with $\mathrm{gr}^J QA = \Omega A$.

Key example of where QA was discovered
Fredholm modules

Def^n: A Fredholm module over $k = \mathbb{C}$ with A a $*$-algebra is given by a $*$ representation of A

$$A \to \mathcal{L}(H)$$

together with an involution $F \in \mathcal{L}(H)$

$$F^2 = I, \quad F^* = F$$

such that $[F, a]$ is compact $\forall a \in A$.

$\{A \to \mathcal{L}(H) \ni F\}$ is called p-summable if $[F, a] \in \mathcal{L}^p(H)$ ideal such that $(\mathcal{L}^p(H))^n$ is trace class for $n \geq p$.

Given such a (p-summable) Fredholm module, we have a homomorphism

$$QA \longrightarrow \mathcal{L}(H)$$
$$\iota a \longmapsto a$$
$$\bar{\iota} a \longmapsto FaF$$

$\therefore a^+ \mapsto \frac{1}{2}(a + FaF)$

$a^- \longmapsto \frac{1}{2}(a - FaF) = \frac{1}{2} F[F, a]$

$\therefore a^-$ is compact

Assume p-summable. Then $a^- \in \mathcal{L}^p(H)$

$a_0^+ a_1^- \dots a_n^- \in (\mathcal{L}^p(H))^n \subset \mathcal{L}^1(H)$

$\mathcal{L}^1(H)$ is the space of trace-class operators.

Thus for $n \geq p$ we have that $a_0^+ a_1^- \dots a_n^- \in \mathcal{L}^1(H)$ so $\mathrm{tr}(B a_0^+ a_1^- \dots a_n^-)$ is defined for $B \in \mathcal{L}(H)$

$\therefore J^n \to \mathcal{L}^1(H)$ for $n \geq p$.

Put $\tau(x) = \mathrm{tr}(Fx)$ $\quad (x \in J^n, n \geq p)$.

Then τ is an even supertrace on J^n, in the following sense. Recall that QA is a superalgebra

$$\tau(xy) = (-1)^{|x||y|} \tau(yx)$$

is the supertrace identity.

Even means that $\tau(FxF) = \tau(x)$

odd means that $\tau(FxF) = -\tau(x)$.

Obviously an even trace. To verify the supertrace identity, can suppose x, y are of the same parity.

Then $\tau(xy) = tr(Fxy) = tr(yFx)$

$$= tr(FFyFx)$$
$$= (-1)^{|y|} tr(Fyx)$$
$$= (-1)^{|y|} \tau(yx)$$
$$= (-1)^{|x||y|} \tau(yx) \quad \text{Done.}$$

<u>Proposition:</u> Let τ be an even supertrace on QA

Then put $\varphi_{2n}(a_1, a_2, \dots, a_{2n}) = \tau(a_1^- \dots a_{2n}^-)$

$\psi_{2n+1}(a_0, \dots, a_{2n}) = \tau(a_0^+ a_1^- \dots a_{2n}^-)$

Then $\{\psi_1, \varphi_2, \psi_3, \varphi_4, \dots\}$ satisfies

1) $b'\varphi_{2n} = (1-N)\psi_{2n+1}$

2) $b\psi_{2n+1} = \frac{1}{n+1} N\varphi_{2n+2}$

2) $N\varphi_{2n} = \varphi_{2n}$

& Conversely ??

Astérisque 1987 §2.12 Karoubi's Operator K on $\Omega^n A = A \otimes \bar{A}^{\otimes n} \hookrightarrow QA$

$$K(a_0\, da_1 \dots da_n) = (-1)^{n-1} da_n\, a_0\, da_1 \dots da_{n-1} (-1)^{n-1}$$
$$= (-1)^n a_n\, da_0\, da_1 \dots da_{n-1}$$
$$+ (-1)^{n-1} d(a_n a_0)\, da_1 \dots da_{n-1}$$

$$K(a_0, \dots, a_n) = (-1)^n (a_n, a_0, \dots, a_{n-1})$$
$$+ (-1)^{n-1}(1, a_n a_0, a_1, \dots, a_{n-1})$$

$bK = Kb$

$s(a_0, \dots, a_n) = (1, a_0, \dots, a_n)$

$s = d : a_0\, da_1 \dots da_n \mapsto da_0\, da_1 \dots da_n$

Formulae:

1) $bs + sb = 1 - K$

Proof: by calculation. Lift using the surjection $A \otimes A^{\otimes n} \twoheadrightarrow A \otimes \bar{A}^{\otimes n}$

b makes sense on $A \otimes \bar{A}^{\otimes n}$, but b' does not. Both b and b' make sense on $A^{\otimes n+1}$.

$b's + sb' = 1$ is a basic calculation.

$\therefore (bs + sb)(a_0, a_1, \dots, a_n)$

$= (b's + sb')(a_0, \dots, a_n) + $ [illegible] $(1, a_0, \dots, a_n)$

$+ s$ [illegible] $(a_0, a_1, \dots, a_n)$

$= (a_0, \dots, a_n) + (-1)^{n+1}(a_n, a_0, \dots, a_{n-1})$

$+ (-1)(1, a_n a_0, \dots, a_{n-1})$

2) $bK = Kb$, $sK = Ks = s$?

Proof: $bs + sb = 1 - K$ — apply b

$b(bs+sb) = b^2 s + bsb = (bs+sb)b \quad \because b^2 = 0$

$s^2 = 0$ so $sK = Ks$ similarly

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \bar{A}^{\otimes n} & \xrightarrow{s} & A\otimes\bar{A}^{\otimes n} & \xrightarrow[\text{since}]{\pi} & \bar{A}^{\otimes n+1} & \longrightarrow & 0 \\
 & & \downarrow \lambda_n & & \downarrow K & & \downarrow \lambda_{n+1} & & \\
0 & \longrightarrow & \bar{A}^{\otimes n} & \xrightarrow{s} & A\otimes\bar{A}^{\otimes n} & \xrightarrow{\pi} & \bar{A}^{\otimes n+1} & \longrightarrow & 0
\end{array}
$$

Hence K is an automorphism (of infinite order)

Defn $B = sN$:

$$A\otimes\bar{A}^{\otimes n} \longrightarrow \bar{A}^{\otimes n+1} \xrightarrow{N} \bar{A}^{\otimes(n+1)} \xrightarrow{s} A\otimes\bar{A}^{\otimes(n+1)}$$

Proposition: $\begin{cases} bB + Bb = 0 \\ B^2 = 0 \end{cases}$

Proof of the first identity

$$bs + sb = 1 - K$$

Apply K^i to give

$$K^i bs + K^i sb = K^i - K^{i+1}$$

$$\therefore\ bs\lambda^i + s\lambda^i b = K^i - K^{i+1}$$

Sum from $i=0$ to $i=n-1$

$$bs(N - \lambda^n_{n+1}) + sNb = 1 - K^n$$

$$\therefore\ bB + Bb - bs\lambda^n_{n+1} = 1 - K^n$$

$\therefore$ to verify that

$$-bs\lambda^n_{n+1} = 1 - K^n$$

$$K^i(a_0\,da_1\dots da_n) = (-1)^i da_{n-i+1}\dots da_n\, a_0\, da_1 \dots da_{n-i}$$

$$\therefore\ K^n(a_0\,da_1\dots da_n) = da_1\dots da_n\, a_0$$

etc.

$$Q = QA = A * A \underset{\tau}{\overset{\iota}{\leftleftarrows}} A$$

$\circlearrowleft$ $n \mapsto n^F$ makes QA a superalgebra

$$\iota a = a^+ + a^- \qquad (a_1 a_2)^+ = a_1^+ a_2^+ + a_1^- a_2^-$$

$$\tau a = a^+ - a^- \qquad (a_1 a_2)^- = a_1^+ a_2^- + a_1^- a_2^+$$

$$QA = \bigoplus_{n\geq 0} (A\otimes\bar{A}^{\otimes n})$$

$$a_0^+ a_1^- \dots a_n^- \longleftrightarrow (a_0, a_1, \dots, a_n)$$

$\therefore$ a linear functional τ on QA is the same as an inhomogeneous cochain $f = \Sigma_{n\geq 0} f_n$

$$f_n \in (A\otimes\bar{A}^{\otimes n})^*$$

normalized Hochschild cochain

$$J = \operatorname{Ker}\left(QA \xrightarrow{\text{fold}} A\right)$$

$$J^m \simeq \oplus_{n\geq m} A\otimes\bar{A}^{\otimes n}$$

$\tau \in (J^m)^*$ equivalent to $\Sigma_{n\geq m} f_n = f_{\geq m}$

$$f_n(a_0, \dots, a_n) = \tau(a_0^+ \dots a_n^-)$$

Proposition: τ vanishes on $[Q, J^{m+1}]$ if and only if

$$b f_n = \frac{2}{n+2} B f_{n+2} \qquad n \geq m$$

$$K f_n = f_n \qquad n \geq m$$

$[\cdot,\cdot]$ here denotes the super bracket

τ vanishes on $[J, J^{m+1}]$

if and only if

$$b f_n = \frac{2}{n+2} B f_{n+2} \quad n \geq m$$
$$K f_n = f_n \quad n \geq m$$

The conventional version of these equations introduces the splitting of f_n.

$$\text{initial } \tau(a_0^+ a_1^- \dots a_n^-) \longrightarrow \begin{cases} \tau(a_0^+ \dots a_n^-) & \varphi_{n+1} \\ \tau(a_1^- \dots a_n^-) & \varphi_n \end{cases}$$

and the above equations become

$$\begin{cases} b' \varphi_n = (1-\lambda) \varphi_{n+1} \\ b \varphi_{n+2} = \frac{2}{n+2} N \varphi_{n+2} \\ \lambda \varphi_n = \varphi_n \end{cases}$$

Identities:

$$(b f_n - (1+\lambda) s f_{n+2})(a_0, \dots, a_{n+1}) = \tau[a_0^+ a_1^- \dots a_n^-, a_{n+1}^+]$$

$$(1-K) f_n (a_0, \dots, a_n) = \tau[a_0^+ a_1^- \dots a_{n-1}^-, a_n^-]$$

Proof of the Proposition: Assume that τ is a supertrace.

$RHS = 0$

$\therefore K f_n = f_n$

(cochains!) $\therefore s f_n = s K f_n = K s f_n = s \lambda f_n\ \lambda s f_n$

$\therefore (1+\lambda) s f_{n+2} = 2 s f_{n+2} = \frac{2}{n+2} N s f_{n+2} = \frac{2}{n+2} B f_{n+2}$

Converse is basically the same.

Verification of the Identities:

Cochains with values in a superalgebra e.g. $\mathbb{Q}$

$\mathrm{Hom}(B(A), Q)$

$B(A)$ bar construction $= A^{\otimes n}$ in degree n with differential b'. This coalgebra with the diagonal $\Delta(a_1, \dots, a_n) = \sum_{i=0}^{n} (a_1, \dots, a_i) \otimes (a_{i+1}, \dots, a_n)$

$\mathrm{Hom}(B(A), Q) = \{$all inhomogeneous cochains on A with ~~..~~ values in $Q\}$

Introduce a $\mathbb{Z}/2$ grading on $\mathrm{Hom}(B(A), Q)$:

$$\mathrm{Hom}(B(A), Q)^+ = \mathrm{Hom}(B(A)^+, Q^+) \oplus \mathrm{Hom}(B(A)^-, Q^-)$$

$$(\quad)^- = \quad + \quad - \quad\quad - \quad +$$

Algebra structure defined via

$$fg = m_Q (f \otimes g) \Delta$$

$$\rightarrow fg(a_1, \dots, a_n) = \sum m_Q (f \otimes g)[(a_1, \dots, a_i) \otimes (a_{i+1}, \dots, a_n)]$$
$$= \sum_{i=0}^{n} f(a_1, \dots, a_i)\, g(a_{i+1}, \dots, a_n) (-1)^{i|g|}$$

$$\delta f = -(-1)^{|f|} f \circ b'$$

$Q = QA \quad (I \in A)$

$\cong \bigoplus_{n \geq 0} A \otimes \bar{A}^{\otimes n}$

$a_0^+ a_1^- \ldots a_n^- \Longleftrightarrow (a_0, \ldots, a_n)$

[illegible]

This gives a differential graded algebra.

Put

$p : A \to Q^+ \qquad \rho(a) = a^+$

$q : A \to Q^- \qquad q(a) = a^-$

ρ odd parity and q is even.

Claim $\begin{cases} \delta q + \rho q - q\rho = (\delta + ad\rho)(q) = 0 \\ \delta\rho + \rho^2 = q^2 \end{cases}$

$(\delta\rho + \rho^2)(a_1, a_2) = \cancel{\rho b'(a_1, a_2)}\ \rho(a_1 a_2) - \rho(a_1)\rho(a_2)$

$= (a_1 a_2)^+ - a_1^+ a_2^+$

$= a_1^- a_2^- = q^2(a_1, a_2)$

$(\delta q + \rho q - q\rho)(a_1, a_2) = -q(a_1 a_2) + \rho(a_1) q(a_2) + q(a_1)\rho(a_2)$

$= -(a_1 a_2)^- + a_1^+ a_2^- + a_1^- a_2^+ = 0$

$(\delta + ad_\rho)(q^n) = \sum_1^n q^{i-1}((\delta + ad_\rho)(q)) q^{n-i} = 0$

$\therefore + \delta q^n + \rho q^n - q^n \rho = 0$

$\therefore + b'(q^n)(a_1, \ldots, a_n) = a_1^+ a_2^- \ldots a_{n+1}^- - (-1)^n a_1^- \ldots a_n^- a_{n+1}^+$

To prove ~~that~~ $(b(\rho q^n) - (1+\lambda)(q^{n+1}))(a_0, \ldots, a_{n+1})$

$= (-1)^n [a_0^+ a_1^- \ldots a_n^-, a_{n+1}^+]$

$(1-K)$

$b(\rho q^n)(a_0, \ldots, a_{n+1}) = (a_0 a_1)^+ a_2^- \ldots a_{n+1}^-$

$\cancel{- a_0^+ (a_1 a_2)^- \ldots a_{n+1}^-}$

$- a_0^+ b'(q^n)(a_1, \ldots, a_{n+1})$

$+ (-1)^{n+1} (a_{n+1} a_0)^+ a_1^- \ldots a_n^-$

$= (a_0^- a_1) a_2^- \ldots a_{n+1}^- + (-1)^n a_0^+ a_1^- \ldots a_n^- a_{n+1}^+$

$+ (-1)^{n+1} (a_{n+1} a_0)^+ a_1^- \ldots a_n^-$

$= a_0^- \ldots a_{n+1}^- + (-1)^{n+1} a_{n+1}^- a_0^- \ldots a_n^- +$

$(-1)^n [a_0^+ a_1^- \ldots a_n^-, a_{n+1}^+]$

$= (1+\lambda) q^{n+2}(a_0, \ldots, a_{n+1}) + (-1)^n [a_0^+ a_1^- \ldots a_n^-, a_{n+1}^+]$

To prove

$(1-K)(\rho q^n)(a_0, \ldots, a_n)$

Recall K is defined on $A \otimes \bar{A}^{\otimes n} \subseteq Q$

$= a_0^+ a_1^- \ldots a_n^- - K(a_0^+ a_1^- \ldots a_n^-)$

$= a_0^+ a_1^- \ldots a_n^- - (-1)^n a_n^- a_0^+ a_1^- \ldots a_{n-1}^-$

$= [a_0^+ a_1^- \ldots a_{n-1}^-, a_n^-]$

More generally, $(1-K)^i (\rho q^n)(a_0, \ldots, a_n) =$

$= [a_0^+ a_1^- \ldots a_{n-i}, a_{n-i+1}^- \ldots a_n^-]$

This completes the proof of

Prop: $\tau \in (J^m)^*$ $f_n = \tau(\rho q^n)$

$f_n(a_0, a_1, \ldots, a_n) = \tau(a_0^+ a_1^- \ldots a_n^-)$ $n \geq m$

Then $\tau[J, J^{m-1}] = 0$ (resp $\tau[Q, J^m] = 0$)

if and only if

$b f_n + (\ldots) B f_{n+2} = 0$ $n \geq m$.

$k f_n = f_n$ $n \geq m$ (resp $n \ldots m$)

Application to Fredholm Modules

$A \longrightarrow \mathcal{L}(H) \ni F$

$[F, a] \in \mathcal{L}^m(H)$ m-summable

We obtain from the universal property of $QA = A * A$ a homomorphism

$QA \to \mathcal{L}(H)$

$\iota a \mapsto a$

$\bar{\iota} a \mapsto FaF$

$a^+ \mapsto \frac{1}{2}(a + FaF)$

$a^- \mapsto \frac{1}{2}(a - FaF) = \frac{1}{2}F[F, a] \in \mathcal{L}^m$

which carries J into $\mathcal{L}^m$

$\therefore J^m$ is carried into $\mathcal{L}^1$, the trace class operators.

Two cases to consider: Ungraded case: let

$\tau(X) = tr(FX)$ $(X \in J^m)$

usual operator trace tr on $\mathcal{L}^1(H)$.

Then τ is an even supertrace (strong) on J^m.

$\tau(X^F) = \tau(X)$ (X^F is the superalgebra grading $X \mapsto X^F$ automorphism of Q)

i.e. τ "supported" on the even elements.

$\tau(xy) = (-1)^{|x||y|}\tau(yx)$ $(x \in J^i, y \in J^{m-i})$

Equivalent for an even supertrace to

$\tau(xy) = \tau(yx^F) = (-1)^{|x||y|}\tau(yx)$

$\tau(xy) = tr(Fxy) = tr(yFx)$

$= tr(FyFxF)$

Graded case: Suppose that given γ ($\mathbb{Z}/2$) grading of H $\gamma^2 = 1$, $\gamma = \gamma^*$

$H = H_+ \oplus H_-$

such that $\begin{cases} \gamma a = a\gamma \\ \gamma F = -F\gamma \end{cases}$

Then $\tau(x) = tr(\gamma x)$ $x \in J^m$

is an odd strong supertrace on J^m

In general odd traces = odd supertraces

"odd" $\tau(x^F) = -\tau(x)$

In the ungraded case we obtain then an even

cocycle $f_n(a_0,\dots,a_{2n}) = tr(F a_0^+ a_1^- \dots a_{2n}^-)$
$= \text{const}\; tr(F(a^+[F,a_1]\dots[F,a_{2n}]))$

In the graded case we obtain an odd cocycle
$f_{2n-1}(a_0,\dots,a_{2n-1}) = tr(\gamma a_0^+ a_1^- \dots a_{2n-1}^-)$
$= (\text{const})\; tr(\gamma \cancel{a_0^{\pm}} a_0 [F,a_1]\dots[F,a_{2n-1}])$

These cocycles are K-invariant (normalized Hochschild cochains).

Cocycle condition
$$b f_n = \frac{2}{n+2} B f_{n+2}, \quad Kf = f$$

Goal: To obtain a homotopy formula, saying that if F is deformed then the corresponding cocycles are cohomologous. (cocycle changes by a coboundary)"

Consider a one-parameter smooth family
$$F = \{F_t : t\in\mathbb{R}\}$$
of involutions. $F^2 = I \quad F = F^*$

<u>Lemma:</u> There is a family of unitary operators $U = \{U_t : t \in \mathbb{R}\}$ forming a smooth one-parameter family such that $U_0 = I$ and UFU^{-1} is the constant involution family with value F_0.
$$U_t F_t U_t^{-1} = F_0$$
and $$\dot U_t U_t^{-1} F_t + F_t \dot U_t U_t^{-1} = 0.$$

Proof: Solve the differential equation
$$\begin{cases} \dot u = u\left(\frac12 F\dot F\right) \\ u_0 = I \end{cases}$$
$0 = (F^2)^\cdot = F\dot F + \dot F F$

$\Rightarrow$ $F\dot F$ is skew adjoint.

$\Rightarrow$ u_t is a unitary operator.

$$(uFu^{-1})^\cdot = \dot u F u^{-1} + u\dot F u^{-1} - uFu^{-1}\dot u u^{-1}$$
$$= u\left(\tfrac12 F\dot F F\right)u^{-1} + u\dot F u^{-1} - \tfrac12 uFu^{-1}uF\dot F u^{-1}$$
$$= u\left(\dot F - [F, \tfrac12 F\dot F]\right)u^{-1}$$
$$= 0$$

$\therefore$ uFu^{-1} is constant.

Since F anticommutes with $u^{-1}\dot u = \frac12 F\dot F$, then $F_0 = uFu^{-1}$ anticommutes with $u u^{-1}\dot u u^{-1} = \dot u u^{-1}$

Replace $A \to \mathcal{L}(H) \ni F = F_t$ varying
by $A \to \mathcal{L}(H) \ni uFu^{-1} = F_0$ fixed
$a \mapsto uau^{-1}$

$\therefore$ have constant involution and varying ι
$\iota: A \to \mathcal{L}(H) \quad \iota a \mapsto uau^{-1}$
$\tilde\iota a \mapsto F_0 uau^{-1} F_0$

Put $L = \dot u u^{-1}$ (anticommutes with F_0)
$$(\iota a)^\cdot = (uau^{-1})^\cdot = [L, \iota a]$$
$$(\tilde\iota a)^\cdot = F_0[L, \iota a]F_0$$
$$= -[L, \tilde\iota a]$$

$$(a^+)^\bullet = [L, a^-]$$
$$(a^-)^\bullet = [L, a^+]$$

This motivates the following situation: super

super/ Let QLQ denote the free/ Q-bimodule with one generator L of odd degree $QLQ \simeq Q\otimes Q$
even

even/ Lemma: There is a unique/ degree zero derivation D
$D : Q \to QLQ$ such that
$$D(a^+) = \cancel{[L, a^-]}\ La^- - a^-L$$
$$D(a^-) = La^+ - a^+L$$

Proof: Verify the relations: check consistency: the $a^\pm$ generate algebra. $(a_1 a_2)^\pm = \ldots$

Recall $\mathrm{Hom}(B(A), Q)$ is a differential superalgebra
We can consider $\mathrm{Hom}(B(A), QLQ)$ as a differential super bimodule

$$\rho(a) = a^+ \qquad \text{odd}$$
$$q(a) = a^- \qquad \text{even}$$
$$(\rho(q)^n)(a_0, \ldots, a_n) = a_0^+ a_1^- \ldots a_n^-$$

$\delta L = 0$
$$Dq = L\rho + \rho L = (\delta + ad\rho)(L)$$
$$D\rho = Lq + qL \quad \text{is not a superbracket}$$

Put $\mu_n = \sum_{i=0}^{n} q^i L q^{n-i}$ odd

$$\mu_n(a_1, \ldots, a_n) = \sum_{i=0}^{n} (-1)^i a_1^- \ldots a_i^- L a_{i+1}^- \ldots a_n^-$$

(before
$$d(\omega^{n+1}) = \sum_{i=0}^{n+1} \omega^{i-1}(d\omega)\omega^{n-i}$$
$$= \sum_{i=0}^{n} \omega^{i-1}((\delta\, ad\rho)\, dq)\, \omega^{n-i}$$
$$= (\delta\, ad\rho)\, \mu_n)$$

Now
$$D(q^{n+1}) = \sum_0^n q^i (Dq) q^{n-i}$$
$$= \sum_0^n q^i ((\delta\, ad\rho)L) q^{n-i} = (\delta\, ad\rho)\mu_n$$

Recall $\left\{\begin{array}{l}(\delta\, ad\rho)q = 0 \\ (\delta + \rho^2)^2 = q^2\end{array}\right\} \iff (a_1 a_2)^\pm = \ldots$

Theorem Let τ' be a supertrace on QLQ
(More generally a supertrace defined on $\sum_{i+j=n-1} J^i(QLQ)J^j$)

$$\tau' D(\rho q^n) = (-b)\tau'(-\rho\,\mu_{n-1}) + \frac{2}{n+2} B\,\tau'(-\rho\mu_{n+1})$$

($\tau' D(\rho q^n)$ cocycle associated to the supertrace $\tau' D$ on Q. rhs is the coboundary of the cochain $\tau'(-\rho\mu_n)$)

$-\rho\mu_n = \sum_0^n -\rho q^i L q^{n-i}$ Apply τ'

$\tau'(-\rho q^i L q^{n-i})(a_0, \ldots, a_n) =$

$(-1)^i \tau'(a_0^+ a_1^- \ldots a_i^- L a_{i+1}^- \ldots a_n^-) =$

$(-1)^i (-1)^{(n-i)i} \tau'(L a_{i+1}^- \ldots a_n^- a_0^+ a_1^- \ldots a_i^-) =$

$(-1)^{in+i} \tau'$

$K^{n-i}(a_0^+ a_1^- \ldots a_n^-) = (-1)^{n-i\,i} a_{i+1}^- \ldots a_n^- a_0^+ a_1^- \ldots a_i^-$

$\therefore \tau'(-\rho q^i L q^{n-i}) = K^{n-i} \tau'(L\rho q^n)$

$\tau'(-\rho\mu_n) = \sum_{i=0}^n K^{n-i} \tau'(L\rho q^n)$

<u>Homotopy for Fredholm modules and supertraces on Q</u>

$f_n(a_0, \ldots, a_n) = tr(F\rho a_0 q a_1 \ldots q a_n)$ $(n \geq m)$

$\rho(a) = \frac{1}{2}(a + FaF)$ $q(a) = \frac{1}{2}(a - FaF)$

$f_n = 0$ for n odd

$\therefore$ get an even cochain

In the graded case we get $f_n = 0$ for all n because operators anticommute with γ. Instead we consider the cochain

$$f_n(a_0, \ldots, a_n) = tr(\gamma\rho a_0 q a_1 \ldots q a_n)$$

$(n \geq m)$.

Now $f_{2n} = 0$, as the operator we are taking the trace of anticommutes with F. Hence we get an odd cochain. $\{f_{2k-1}\}_{2k-1 \geq m}$

Prop: The above cochains are cocycles which are K invariant.

Homotopy: Consider a homotopy of Fredholm modules where the homomorphism $A \to \mathcal{L}(H)$ is fixed but $F = (F_t)$ varies smoothly.

To show that

$\dot{f} = \partial_t f$ is a coboundary

Put $L = \frac{1}{2}F\dot{F}$ anticommutes with F.

Derivation $Dv = \dot{v} + Lv - vL$

if $v = (v_t)$ is a family of operators.

$D(F) = \dot{F} + \frac{1}{2}F\dot{F}F - \frac{1}{2}FF\dot{F}$

$= \dot{F} - \frac{1}{2}\dot{F} - \frac{1}{2}\dot{F} = 0$

$Da = \dot{a} + La - aL$

0

$$D(FaF) = F(Da)F = FLaF - FaLF$$
$$= -LFaF + (FaF)L$$
$$\therefore 2D(\rho(a)) = La - aL + (-LFaF) + FaFL$$

$$\therefore D(\rho a) = Lqa - qaL$$
$$Dqa = L\rho a - \rho a L$$

$$\partial_t f_n(a_0, a_1, \ldots, a_n) = \mathrm{tr}\{(\partial_t + \mathrm{ad} L)(F\rho a_0 q a_1 \ldots q a_n)\}$$
$$= \mathrm{tr}\,(D(F\rho a_0 q a_1 \ldots q a_n))$$
$$= \mathrm{tr}\{F(D(\rho a_0 q a_1 \ldots q a_n))\}$$

Consider the universal case of this sort of algebra

QLQ = free bimodule generated by an odd element L

D derivation $Q \to QLQ$ such that

$$Da^+ = La^+ - a^+L$$
$$Da^- = La^+ - a^+L$$

τ' is a super trace on QLQ (more generally one considers τ defined on $\bigoplus_{i+j=n-1} J^i(QLQ)J^j$

To prove a homotopy formula for the trace $\tau' D$ super trace on Q.

Homotopy formula

$$(\tau' D)(\rho q^n) = \underline{(-b)\,\tau'(-\rho\mu_n)} + \frac{2}{n+2} B\tau'(-\rho\mu_{n+2})$$

$$R = RA = T(A)/(I_{T(A)} - I_A) = Q^+$$

$$\begin{tikzcd} R \arrow[r, hook] \arrow[d, "d"] & Q \arrow[d, "D"] \\ R \otimes \bar{A} \otimes R = \Omega^1 R \arrow[r, "2"] & QLQ \arrow[d] \\ & QLQ/[Q, QLQ] \end{tikzcd}$$

$\rho a \mapsto a^+$

$\rho(a_1 a_2) - \rho(a_1)\rho(a_2)$

$\mapsto a_1^- a_2^-$

$\Omega^1 R/[R, \Omega^1]$ $QLQ/[Q, QLQ]$

$\| \qquad\qquad \|$

$R \otimes \bar{A} \qquad Q$

$R/[R,R] \twoheadrightarrow (Q/[Q,Q])^+$

$\circlearrowleft_{K}$

K is induced by $a_0^+ q_1^- \ldots q_{2n}^- \mapsto -a_{2n}^- a_0^+ q_1^- \ldots q_{2n-1}^-$

Quotient of this $\mathbb{Z}/2$ action

__Lemma:__ $2: \Omega^1_R \to QLQ$ is injective

Proof: $Q = R \oplus RA^- \simeq R \otimes (k \oplus \bar{A})$

$a_0^+ q_1^- \ldots a_n^- \simeq A^- R \otimes R = (k \oplus \bar{A}) \otimes R$

Free as left or as right R-module

$$QLQ \simeq (R \otimes ((k \oplus \bar{A}) L (k \oplus \bar{A}) \otimes R)$$

$$kL \oplus LA^- \oplus A^- L \oplus A^- L A^-$$

2 sends da^+ to $Da^+ = La^- - a^- L$

2 is a map of free R-bimodules induced by $da^+ \mapsto La^- - a^-L$ injective

$\therefore$ 2 is injective

Recall that the homotopy formulas for traces on R over the cochain

$$\mu_{2n+1} = \sum_0^n \omega^i \, d\rho \, \omega^{n-i}$$

$$d(\omega^{n+1}) = (\text{?}\, e\, ad\rho)(\mu_{2n+1})$$

Under this goes to $\omega = q^2$

$\therefore$ ℓ odd

$$\sum_0^n q^{2i}(Lq - eqL)q^{2n-2i}$$

$$= \sum_0^{2n+1} q^i L q^{2n-i+1}$$

projection back $f \mapsto \left(\frac{1+K}{2}\right) f$

$$(R/[R,R])^* \underset{\longleftarrow}{\xrightarrow{(1+K)/2}} \left((Q/[Q,Q])^+\right)^*$$

Question: What about traces τ on R of the such that $K\tau = -\tau$ even

(same as ordinary traces vanishing on A^+)

To show such a τ is uninteresting from the cyclic cohomology viewpoint.

Let τ be any trace on R. Consider $(1-K)\tau$.

$f_{2n} = \tau\rho(q^{2n}) : (a_0, \ldots, a_{2n}) \mapsto \tau(a_0^+ a_1^- \ldots a_{2n}^-)$

$bf_{2n} = \frac{2}{2n+2} B f_{2n+2}$ $\qquad K^2 f_{2n} = f_{2n}$

Look at $(1-K)f = (bs+sb)f = bsf$

$\therefore$ $sB = Bs = 0$

Thus the trace $(1-K)\tau$ has a cocycle of a very special sort, namely that it consists of Hochschild coboundaries.

Claim that $(1-K)f = (b+B)$ coboundary of $(-sf)$

– true because $Bsf = 0$.

(To see this: $f_{2n} = (\varphi_{2n+1}, \varphi_{2n})$ in the canonical picture

$$\begin{array}{ccc}
\varphi_{2n+2} \xrightarrow{b'} & & \\
\varphi_{2n+1} & \xrightarrow{-1+\lambda} & 0 \\
{\scriptstyle -b}\uparrow & & \uparrow {\scriptstyle b'} \\
-\varphi_{2n} & \xrightarrow{-1+\lambda} & \varphi_{2n}
\end{array}$$

$-sf_{2n} = (-\varphi_{2n}, 0)$

coboundary of $-\varphi_{2n}$ consists of

$(+b\varphi_{2n}, (1-\lambda)(-\varphi_{2n})) \Longleftrightarrow (1-K)f_{2n}$

$$\begin{array}{ccc}
R = Q^+ & \overset{i}{\subset} & Q \\
\downarrow d & & \downarrow D \\
\Omega^1 R & \overset{i}{\longrightarrow} & QLQ
\end{array}$$

$Da^+ = La^- - a^-L$

$Da^- = La^+ - a^+L$

i' induced R-bimodule map with $Di = i'd$. Hence seen already that i' is injective.

$$\begin{array}{ccccc} A^{\otimes 2n-1} \xrightarrow{\mu_{2n-1}} & \Omega' R & \xhookrightarrow{i'} & QLQ & = A^{\otimes 2n-1} \\ & \downarrow \natural & & \downarrow \natural & A^{\otimes 2i} \\ R \otimes A & \Omega' R / [R, \Omega' R] & \xrightarrow{i''} & QLQ / [Q, QLQ] & \simeq LQ \end{array}$$

$$i' \mu_{2n-1} = i' \sum_{i=1}^{n} \omega^{i-1} dp\, \omega^{n-i}$$

$$= \mu_{2n-1} = \sum_{i=1}^{n} q^{i} L q^{2n-1-i}$$

Consistency between the homotopy formulae at the level of $\Omega' R \to QLQ$: $\omega = q^2$ and $dp = Lq + qL$.

On the level of the commutator quotient spaces we have

$$i'' \natural (\rho(a_0)\, \omega^n (a_1, \dots, a_{2n})\, dp\, (a_{2n+1}))$$
$$= i'' \natural a_0^+ a_1^- \dots a_{2n}^- \, d a_{2n+1}^+$$

$$= \natural \{ a_0^+ a_1^- \dots a_{2n}^- (L a_{2n+1}^- - a_{2n+1}^- L) \}$$

$$= \natural \{ L a_{2n+1}^- a_0^+ a_1^- \dots a_{2n}^- + L a_0^+ a_1^- \dots a_{2n+1}^- \}$$

Conclude that i'' can be identified with the operation $1+K$ on $A \otimes \bar{A}^{\otimes 2n+1}$

consequence is that i'' is neither injective nor surjective.

even

Leven Question: Start with τ' even supertrace on QLQ. Then we get a supertrace $\tau' D$ on Q whose cocycle$_{2n+1}$ is the coboundary of

$$h'_{2n+2} = \left(\sum_0^{2n+1} K^j\right) \tau'(L \rho q^{2n})$$

by the homotopy formulae for supertraces on Q.

The cocycle of this trace $\tau' D$ is the cocycle of the trace $\tau' D i = \tau' i' d$ on R, which by the homotopy formula for traces on R is the coboundary of

$$h''_{2n+1} = \left(\sum_0^n K^{2i}\right) \tau' i' (-\rho \omega^n dp)$$

What is the relation between h' and h''?

Answer: $h' = h''$ because we have shown

$$\natural i'(-\rho \omega^{n} dp) = (1+K) \natural (L \rho q^{2n})$$

$$\therefore \tau' i'(-\rho \omega^n dp) = (1+K)\, \tau' (L \rho q^{2n})$$

and

$$\sum_0^n K^{2i} (1+K) = \sum_0^{2n+1} K^j$$

Ex $\Omega(QA) = Q(\Omega A)$ universal properties

$\Omega^n A \cong A \otimes (\bar{A})^{\otimes n}$ $\quad F = (-1)^n, b, s = d, \kappa, B$

$a_0 da_1 \dots da_n \longleftrightarrow (a_0, a_1, \dots, a_n)$

On ΩA we have left multiplication by ΩA, also the action of QA given by

$\iota a \mapsto (1+d) a (1-d) = a - da$ ~~dad~~

$\bar{\iota} a \mapsto (1-d) a (1+d) = a - da -$ ~~dad~~

$a^+ \mapsto a -$ ~~dad~~

$a^- \mapsto da$

$k[d] = k \oplus kd$ with $d^2 = 0$

$k[F] = k \oplus kF$ with $F^2 = 1$

Formulae: A is unital $\quad \Omega = \Omega A, Q = QA$

$A * k[d] \cong \Omega + \Omega d = \Omega \tilde{\otimes} k[d]$

where $d * \omega = d\omega + (-1)^{|\omega|} \omega d$

in the cross product

$A * k[F] \cong Q \oplus QF = Q \tilde{\otimes} k[F]$

where $\cancel{x} F n = (-1)^{|n|} n F$

S.1 In $\mathrm{Hom}_k(\Omega, \Omega)$ consider the subalgebra generated by A (left multiplication), by d and F.

$F(\omega) = (-1)^{|\omega|} \omega$

$[d, a] =$ left multiplication by da

$\Omega = \Omega A \subseteq S$ as left multiplication.

$\Omega \oplus \Omega d \oplus \Omega F \oplus \Omega dF = \Omega^{\oplus 4}$

$\Omega^{\oplus 4} \longrightarrow S \subset \mathrm{Hom}(\Omega, \Omega)$

Claim: this map is injective and an isomorphism.

Proof: $\Omega + \Omega d + \Omega F + \Omega dF$ is a subalgebra of $\mathrm{Hom}(\Omega, \Omega)$

$d \cdot \Omega \subseteq \Omega + \Omega d$

$F \cdot \Omega \subseteq \Omega F$

Hence the map is surjective.

Injective: If $(\omega_0 + \omega_1 d + \omega_2 F + \omega_3 dF)(\eta) = 0$ for all η then

$\eta = 1 \qquad \omega_0 + \omega_2 = 0$

$\eta = da \qquad (\omega_0 - \omega_2) da = 0$

If $A \neq k$ then right multiplication by da is injective on Ω. Hence $\omega_0 = \omega_2 = 0$.

$\eta = a \qquad \omega_1 da + \omega_3 da = 0 \quad \therefore \omega_1 + \omega_3 = 0$

$\eta = ada \qquad \omega_1 dada - \omega_3 dada = 0 \therefore \omega_1 - \omega_3 = 0$

$\therefore \omega_1 = \omega_3 = 0$.

$\therefore \Omega + \Omega d + \Omega F + \Omega dF \subseteq \mathrm{Hom}(\Omega, \Omega)$.

Proposition: $k[d] = k \oplus kd \quad d^2 = 0$

$k[F] = k \oplus kF \quad F^2 = 1$

$A * k[d] = \Omega \tilde{\otimes} k[d]$

$A * k[F] = Q \tilde{\otimes} k[F]$

$A * k[d]$ is the universal $\mathbb{Z}$ graded algebra with A in degree 0 and an element d of degree 1 with $d^2 = 0$

1. ($\mathbb{Z}$-graded algebras with degree d ... $d^2 = 0$)

2. DG algebras with inner differential d of degree 1 such that $d^2 = 0$ $d\omega = [d, \omega]$

These categories are the same.
ΩA = universal DG algebra with A in degree 0.
Adjoin an element d
$$\Omega A \tilde{\otimes} k[d] = \Omega A + \Omega A d$$
is the universal inner DG algebra with A in degree zero.

$$\begin{tikzcd} A * k[d] & A \arrow[l] \\ & \Omega A \tilde{\otimes} k[d] \arrow[ul, dashed] \end{tikzcd}$$

categories

1. Algebras together with an element F in S such that $F^2 = 1$

2. Inner superalgebras — superalgebra S (algebra

with automorphism $\pi \mapsto \bar{\pi}$ of order two) together with an element $F \in S$ such that $F \pi F = \bar{\pi}$.

$A * k[F]$ universal object in category 1 generated by A.
$QA \tilde{\otimes} k[F]$ universal object in category 2 generated by A

$$\begin{tikzcd} A * k[F] & A \arrow[l] \\ & QA \tilde{\otimes} k[F] \arrow[ul, dashed] \end{tikzcd}$$

Formula:
$$\Omega + \Omega d + \Omega F + \Omega dF \subseteq \mathrm{Hom}(\Omega, \Omega)$$
$$= (\Omega \tilde{\otimes} k[d]) \tilde{\otimes} k[F]$$
To identify $Q \subseteq \mathrm{Hom}(\Omega, \Omega)$ in this subalgebra
Recall that
$$\iota a \mapsto (1+d)a(1-d)$$
$$\bar{\iota} a \mapsto (1-d)a(1+d)$$
$$a^+ \mapsto a - dad$$
$$a^+ \mapsto da$$
$$\therefore QA \subseteq \Omega A \tilde{\otimes} k[d]$$
$$QA \tilde{\otimes} k[F] \subseteq (\Omega A \tilde{\otimes} k[d]) \tilde{\otimes} k[F].$$

QA = / (2×2) matrices over ΩA
subalgebra of the algebra of

$Q \subseteq \Omega + \Omega d = \Omega + d\cdot\Omega$ right Ω-modules

$a^+ \mapsto a - dad$

$(a - dad)(\omega + d\eta)$

$= a\omega - dad\omega + ad\eta + dad\,d\eta$ $\quad 0$

$= a\omega - da\,d\omega + da\eta + d.a\eta$

$$(a - dad)\begin{pmatrix}\omega \\ \eta\end{pmatrix} = \begin{pmatrix}(a-dad)\omega - da\eta \\ a\eta\end{pmatrix}.$$

$$= \begin{pmatrix} a - dad & -da \\ a & 0 \end{pmatrix}$$

$\therefore Q \subseteq M_2(\Omega)$.

Conversely, we have an embedding $\Omega \subseteq M_2(Q)$.

Formula (Connes-Cuntz Positivity _)

$$a \longmapsto \begin{pmatrix} \iota a & 0 \\ 0 & \tau a \end{pmatrix}$$

$$da \longmapsto \begin{pmatrix} 0 & -a^- \\ a^- & 0 \end{pmatrix}$$

Obtained as follows: $A \to \mathcal{L}(H) \ni F$

$a^- \longmapsto \frac{1}{i} F[F, a]$

Define via $\Omega \hookrightarrow Q \tilde{\otimes} h[F]$

$a \longmapsto \iota a$

$da \longmapsto Fa^- = [F/2, a]$

From this we get ΩQ represented as acting on $Q + QF$ $\quad \Omega \subseteq M_2(Q)$.

Proposition: $\Omega(QA) \cong Q(\Omega A)$

Proof: $Q(\Omega A) = \Omega A * \Omega A$ as algebras

Claim that the differentials on the factors ΩA induce a differential on the free product.

$m^2 = 0$ Derivations (R, M) $\quad M$ is an R bimodule

$=$ algebra homomorphisms $R \to R \oplus M$ which are congruent to Identity modulo M

Conclude that $Q(\Omega A)$ with $A * A = QA$ in degree 0. Now use the universal property of ΩQA get a map.

$\Omega(QA) \longrightarrow Q(\Omega A)$